THE GENERALIZED KUREPA HYPOTHESIS AT SINGULAR CARDINALS

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Abstract. We discuss the generalized Kurepa hypothesis KH_λ at singular cardinals λ. In particular, we answer questions of Erdös-Hajnal [1] and Todorcevic [6], [7] by showing that GCH does not imply $\mathrm{KH}_{\aleph_\omega}$ nor the existence of a family $\mathcal{F} \subseteq [\aleph_\omega]^{\aleph_0}$ of size $\aleph_{\omega+1}$ such that $\mathcal{F} \restriction X$ has size $\aleph_0$ for every $X \subseteq S, |X| = \aleph_0$.

1. introduction

For an infinite cardinal λ let the generalized Kurepa hypothesis at λ, denoted KH_λ, be the assertion: there exists a family $\mathcal{F} \subseteq P(\lambda)$ such that $|\mathcal{F}| > \lambda$ but $|\mathcal{F} \restriction X| \leq |X|$ for every infinite $X \subseteq \lambda, |X| < \lambda$, where $\mathcal{F} \restriction X = \{t \cap X : t \in \mathcal{F}\}$.

By a theorem of Erdös-Hajnal-Milner [2], if λ is a singular cardinal of uncountable cofinality, $\theta^{\mathrm{cf}(\lambda)} < \lambda$ for all $\theta < \lambda$ and if $\mathcal{F} \subseteq P(\lambda)$ is such that the set $\{\alpha < \lambda : |\mathcal{F} \restriction \alpha| \leq |\alpha|\}$ is stationary in λ, then $|\mathcal{F}| \leq \lambda$. In particular, GCH implies KH_λ fails for all singular cardinals λ of uncountable cofinality. On the other hand, by an unpublished result of Prikry [5], KH_λ holds in L, the Gödel's constructible universe, for singular cardinals of countable cofinality (see [7]). Later, Todorcevic [6], [7] improved Prikry's theorem by showing that if λ is a singular cardinal of countable cofinality, then $\square_\lambda$ implies KH_λ. The following question is asked in [6] and [7].

Question 1.1. *Does* GCH *imply* $\mathrm{KH}_{\aleph_\omega}$.

The question is also related to the following question of Erdös-Hajnal [1] (question 19/E)

Question 1.2. *Assume* GCH. *Let $|S| = \aleph_\omega$. Does there exist a family $\mathcal{F}, |\mathcal{F}| = \aleph_{\omega+1}, \mathcal{F} \subseteq [S]^{\aleph_0}$ such that $\mathcal{F} \restriction X$ has size $\aleph_0$ for every $X \subseteq S, |X| = \aleph_0$.*

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We show that, relative to the existence of large cardinals, both of the above questions can consistently be false, and so they are independent of ZFC.

2. KH_λ fails above a supercompact cardinal

In this section we prove the following.

Theorem 2.1. *Suppose κ is a supercompact cardinal and $\lambda \geq \kappa$. Then KH_λ fails.*

Proof. Let $\mathcal{F} \subseteq P(\lambda)$ be of size $\geq \lambda^+$. Let $j : V \to M$ be a λ^+-supercompactness embedding with $crit(j) = \kappa$. Also let U be the normal measure on $P_\kappa(\lambda)$ derived from j, .i.e.,

$$U = \{X \subseteq P_\kappa(\lambda) : j[\lambda] \in j(X)\}.$$

We have

- $M \models$"$j(\mathcal{F}) \subseteq P(j(\lambda))$ is of size $\geq j(\lambda)^+$".
- $j''[\lambda] \in M$ and $M \models$"$|j''[\lambda]| = \lambda < j(\lambda)$".
- $\mathcal{F} \in M$
- $a \neq b \in \mathcal{F} \implies j(a) \cap j''[\lambda] \neq j(b) \cap j''[\lambda]$.

In particular,

$$M \models \text{"}|j(\mathcal{F}) \restriction j''[\lambda]| \geq |\mathcal{F}| \geq \lambda^+\text{"}.$$

This implies that

$$\{x \in P_\kappa(\lambda) : |\mathcal{F} \restriction x| \geq |x|^+\} \in U.$$

In particular, $\mathcal{F}$ is not a KH_λ-family. □

Remark 2.2. *The above result is optimal in the sense that we can not in general find a set $x \subseteq \lambda$ of size in the interval $[\kappa, \lambda)$ such that $|\mathcal{F} \restriction x| \geq |x|^+$. To see this assume κ is supercompact and Laver indestructible. Then one can easily define a κ-directed closed forcing notion which adds a family $\mathcal{F} \subseteq P(\lambda)$ such that $|\mathcal{F}| \geq \lambda^+$, but $|\mathcal{F} \restriction x| \leq |x|$ for any set x with $\kappa \leq |x| < \lambda$.*

3. The Chang's conjecture and $\mathrm{KH}_{\aleph_\omega}$

In this section we prove our main theorem by showing a consistent negative answer to the questions of Erdös-Hajnal and Todorcevic. Recall from [4] that "GCH+ the Chang's conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$" is consistent, relative to the existence of a 2-huge cardinal. See also [3], where the large cardinal assumption is reduced to the existence of a $(+\omega+1)$-subcompact cardinal κ.

Theorem 3.1. *Assume* GCH+*Chang's conjecture* $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$. *Then* $\mathrm{KH}_{\aleph_\omega}$ *fails. Also, there does not exists a family* $\mathcal{F} \subseteq [\aleph_\omega]^{\aleph_0}, |\mathcal{F}| \geq \aleph_{\omega+1}$ *such that* $\mathcal{F} \restriction X$ *has size* $\aleph_0$ *for every* $X \subseteq S, |X| = \aleph_0$.

Proof. Suppose towards contradiction that there exists a family $\mathcal{F}$ which witnesses $\mathrm{KH}_{\aleph_\omega}$. Fix a bijection $f : H_{\aleph_{\omega+1}} \leftrightarrow \mathcal{F}$. Consider the structure

$$\mathcal{A} = (H_{\aleph_{\omega+1}}, \in, \mathcal{F}, \aleph_\omega, f).$$

Let $\mathcal{B} = (B, \in, \mathcal{G}, A, g) \prec \mathcal{A}$ be such that $|B| = \aleph_1$ and $|A| = \aleph_0$.

Note that $\mathcal{A} \models$"$\forall t \in \mathcal{F}, t \subseteq \aleph_\omega$, and hence $\mathcal{B} \models$"$\forall t \in \mathcal{G}, t \subseteq A$, in particular, $\mathcal{G} \subseteq \mathcal{F} \restriction A$. On the other hand $g : B \leftrightarrow \mathcal{G}$ is a bijection, hence we have

$$|\mathcal{F} \restriction A| \geq |\mathcal{G}| = |B| = \aleph_1 > \aleph_0.$$

We get a contradiction and the result follows.

Similar argument shows that there can not be a family $\mathcal{F} \subseteq [\aleph_\omega]^{\aleph_0}$ as stated above. □

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