ON A THEOREM OF MAGIDOR

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Abstract. Assuming κ is a supercompact cardinal and λ is an inaccessible cardinal above it, we present an idea due to Magidor, to find a generic extension in which $\kappa = \aleph_\omega$ and $\lambda = \aleph_{\omega+1}$.

1. introduction

Suppose κ is a supercompact cardinal and λ is an inaccessible cardinal above it. There are several ways to find an extension in which κ becomes a singular cardinal and $\lambda = \kappa^+$, let us mention at least two:

- Suppose κ is Laver indestructible. Force with $\mathrm{Col}(\kappa, < \lambda)$. In the extension, κ remains supercompact and hence measurable, so one can change the cofinality of κ to ω using Prikry's forcing. One can even makes $\kappa = \aleph_\omega$, preserving cardinals above κ. In the final model, κ is a singular cardinal of countable cofinality and λ is its successor.
- Merimovich's supercompact extender based Prikry forcing [2] can be used as well to get an extension in which $cf(\kappa) = \omega$ and $\lambda = \kappa^+$.

In this paper we present an idea of Magidor to present another way of making $\kappa = \aleph_\omega$ and $\lambda = \aleph_{\omega+1}$, which is of independent interest.

Thus suppose that κ is a supercompact cardinal and let λ be the least inaccessible cardinal above it. Let $\mathcal{U}$ be a normal measure on $P_\kappa(\lambda) = \{P \subseteq \lambda : o.t(P) < \kappa, P \cap \kappa \in \kappa\}$.

Definition 1.1. *(1). Given $P \in P_\kappa(\lambda)$, let $\kappa_P = P \cap \kappa$ and $\lambda_P = o.t(P)$,*

(2). Given $P, Q \in P_\kappa(\lambda)$, we define $P \prec Q$ iff $P \subseteq Q$ and $\lambda_P < \kappa_Q$,

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Let $D = \{P \in P_\kappa(\lambda) : \lambda_P$ is the least inaccessible cardinal above $\kappa_P\}$. Then $D \in \mathcal{U}$. We now define the forcing notion $(\mathbb{P}, \leq, \leq^*)$ as follows:

Definition 1.2. *A condition in $\mathbb{P}$ is a finite sequence $\langle P_1, \dots, P_n, f_0, \dots, f_n, A, F\rangle$ where*

(1) $P_1 \prec ... \prec P_n$ *are in* D,

(2) $f_0 \in Col(\omega_1, < \lambda_{P_1})$,

(3) $f_i \in Col(\lambda^+_{P_i}, < \lambda_{P_{i+1}}), i = 1, \dots, n-1$,

(4) $f_n \in Col(\lambda^+_{P_n}, < \kappa)$,

(5) $A \in \mathcal{U}$ *and* $A \subseteq D$,

(6) $\forall P \in A, P_n \prec P$ *and* $\sup(ran(f_n)) < \kappa_P$,

(7) $dom(F) = A$,

(8) *For all* $P \in A$, $F(P) \in Col(\lambda^+_P, < \kappa)$,

(9) *If* $P \prec Q$ *are in* A, *then* $\sup(ran(F(P))) < \kappa_Q$.

Definition 1.3. *Suppose* $\pi = \langle P_1, \dots, P_n, f_0, \dots, f_n, A, F\rangle$ *and* $\pi' = \langle Q_1, \dots, Q_m, g_0, \dots, g_m, B, G\rangle$ *are two conditions in* $\mathbb{P}$. *Then*

(a) $\pi' \leq \pi$ *(π' is an extension of π) iff*

  (1) $m \geq n$,

  (2) $Q_i = P_i, i = 1, \dots, n$,

  (3) $Q_i \in A, i = n+1, \dots, m$,

  (4) $g_i \leq f_i, i = 0, \dots, n$,

  (5) $g_i \leq F(Q_i,), i = n+1, \dots, m$,

  (6) $B \subseteq A$,

  (7) *For all* $P \in B$, $G(P) \leq F(P)$.

(b) $\pi' \leq^* \pi$ *(π' is a direct or a Prikry extension of π) iff* $\pi' \leq \pi$ *and* $m = n$.

Let G be $\mathbb{P}-$generic over V. Then in $V[G]$ we obtain the following sequences in the natural way:

(1) A $\prec -$increasing sequence $\langle P_n : 0 < n < \omega\rangle$ of elements of $P_\kappa(\lambda)$ with $\lambda = \bigcup_{n<\omega} P_n$,

(2) A sequence $\langle F_n : n < \omega\rangle$ such that F_0 is $Col(\omega_1, < \lambda_{P_1})-$generic over V and for $n > 0, F_n$ is $Col(\lambda^+_{P_n}, < \lambda_{P_{n+1}})-$generic over V.

The next lemma can be proved as in [1].

Lemma 1.4. (a) $(\mathbb{P}, \leq, \leq^*)$ *satisfies the Prikry property.*

(b) *If* $X \in V[G]$ *is a bounded subset of* κ, *then* $X \in V[\langle F_i : i < n\rangle]$ *for some* $n < \omega$.

(c) $CARD^{V[G]} \cap \kappa = \{\omega, \omega_1, \lambda_{P_n}, \lambda^+_{P_n} : 0 < n < \omega\}$,

(d) $V[G] \models \kappa = \aleph_\omega$.

But note that in $V[G], \lambda$ is collapsed to κ. We now define an inner model of $V[G]$ in which λ is preserved. Let $T = \{(A, F) : A \in \mathcal{U}, F$ is a function with domain A and $\forall P \in A,\ F(P) \in Col(\lambda^+_P, < \kappa)\}$.

Define an equivalence relation $\equiv$ on T by $(A, F) \equiv (B, G)$ iff there is a set $C \in \mathcal{U}, C \subseteq A \cap B$ such that for all P in C we have $F(P) = G(P)$. Let $[(A, F)]_\equiv$ denote the equivalence class of (A, F) with respect to $\equiv$. Also let

$$G_\equiv = \{[(A, F)]_\equiv : \exists \pi \in G, \pi = \langle P_1, \ldots, P_n, f_0, \ldots, f_n, A, F\rangle\}.$$

Define V^* to be

$$V^* = V[\langle \kappa_{P_n} : 0 < n < \omega\rangle, \langle F_n : 0 < n < \omega\rangle, G_\equiv].$$

Our main theorem is as follows:

Theorem 1.5. (a) $V^* \models \kappa = \aleph_\omega$,

(b) λ *remains a cardinal in* V^*,

(c) *Let* $\mu \in (\kappa, \lambda)$. *Then* $\langle P_n \cap \mu : 0 < n < \omega\rangle \in V^*$,

(d) *All cardinals* $\mu \in (\kappa, \lambda)$ *are collapsed in* V^* *into* κ

Proof. (a) is trivial, and (b) can be proved as in [1] using permutation arguments. Also (d) follows from (c), so let's prove (c).

Suppose that the condition $\pi \in G$ forces "for some μ, with $\kappa < \mu < \lambda$, we have $\langle P_n \cap \mu : 0 < n < \omega\rangle \notin V^*$". For $P \in P_\kappa(\lambda)$ denote by μ_P the order type of $P \cap \mu$. Let $\pi = \langle P_1, \ldots, P_n, f_0, \ldots, f_n, A, F\rangle$.

Without loose of generality we can assume that for $P \in A, \mu \in P$, so $\mu_P < \lambda_P$.

Claim 1.6. *By extending* F *if necessary we can assume that for every* $P, P' \in A$ *with* $\lambda_P = \lambda_{P'}$ *and* $P \cap \mu \neq P' \cap \mu$ *we have* $F(P)$ *is incompatible with* $F(P')$.

Proof. Use the fact that for any Q in A with $P, P' \prec Q$, the forcing notion $Col(\lambda_P^+, <\lambda_Q)$ has the property that below any condition there are at least 2^{μ_Q} incompatible conditions. □

Note that G is generic over V^* with respect to a forcing notion which is a subset of $\mathbb{P}$ and is definable in V^*. Denote this set of conditions by $\mathbb{P}^*$. Then $G \subseteq \mathbb{P}^*$.

Claim 1.7. *If $\pi' \leq \pi, \pi' \in \mathbb{P}^*, \pi' = \langle Q_1, \ldots, Q_m, f_0, \ldots, f_m, B, G \rangle$, then for all $n < i \leq m, Q_i \cap \mu = P_i \cap \mu$.*

Proof. Using the permutation arguments as in [1], one can show that for any two conditions $\eta, \eta' \in \mathbb{P}^*$ there is some permutation σ of λ such that $\sigma(\eta)$ is compatible with η', and if η, η' are both extensions of π, we can pick σ such that $\sigma(\pi) = \pi$.

If the claim fails, then we can find two extensions η, η' of π such that if

$$\eta = \langle Q_1, \ldots, Q_m, f_0, \ldots, f_m, B, G \rangle$$

and

$$\eta' = \langle Q'_1, \ldots, Q'_m, f_0, \ldots, f_m, B, G \rangle,$$

then there is $n < i \leq m$ such that $Q_i \cap \mu \neq Q'_i \cap \mu$ and such that for some permutation σ we have $\sigma(\pi) = \pi$ and $\sigma(\eta) = \eta'$. We can extend η and η' by picking the right member to B to put on the top of both of them, so that we can assume that $Q_m = Q'_m$. Let $n < j < m$ be maximal such that $Q_j \cap \mu \neq Q'_j \cap \mu$. But then $Q = Q_{j+1} = Q'_{j+1}$ satisfies $Q_j \prec Q, Q'_j \prec Q$. F_j must be an extension of both $F(Q_j)$ and $F(Q'_j)$ (F is the function in the condition π). But by our assumption about $\pi, F(Q_j)$ and $F(Q'_j)$ are supposed to be incompatible elements of $Col(\lambda_{Q_j}^+, <\lambda_Q)$. Contradiction!. □

Given the above Claim, we can easily see that $\langle P_k \cap \mu : n < k < \omega \rangle$ is in V^*. But then trivially the sequence $\langle P_n \cap \mu : 0 < n < \omega \rangle$, which is the same sequence with the addition of a finite initial segment, is in V^* as well. □

One can use the model V^* to prove the following.

Theorem 1.8. *The cardinal $\aleph_{\omega+1}^{V^*} = \lambda$ is an inaccessible cardinal in HOD^{V^*}.*

References

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[2] Merimovich, Carmi, Supercompact extender based Prikry forcing. Arch. Math. Logic 50 (2011), no. 5–6, 591-602.

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