MORE ON ALMOST SOUSLIN KUREPA TREES

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Abstract. It is consistent that there exists a Souslin tree T such that after forcing with it, T becomes an almost Souslin Kurepa tree. This answers a question of Zakrzewski [6].

1. Introduction

In this paper we continue our study of ω_1-trees started in [3] and prove another consistency result concerning them. Let T be a normal ω_1-tree. Let's recall that:

- T is a Kurepa tree if it has at least ω_2-many branches.
- T is a Souslin tree if it has no uncountable antichains (and hence no branches).
- T is an almost Souslin tree if for any antichain $X \subseteq T$, the set $S_X = \{ht(x) : x \in X\}$ is not stationary (see [1], [6]).

We refer to [3] and [4] for historical information and more details on trees.

In [6], Zakrzewski asked some questions concerning the existence of almost Souslin Kurepa trees. In [3] we answered two of these questions but one of them remained open:

Question 1.1. *Does there exist a Souslin tree T such that for each G which is T-generic over V, T is an almost Souslin Kurepa tree in $V[G]$?*

In this paper we give an affirmative answer to this question.

Theorem 1.2. *It is consistent that there exists a Souslin tree T such that for each G which is T-generic over V, T is an almost Souslin Kurepa tree in $V[G]$.*

The rest of this paper is devoted to the proof of this theorem. Our proof is motivated by [2] and [3].

This research was in part supported by a grant from IPM (No. 91030417). The author also would like to thank the referee of the paper for some useful remarks and comments.

2. Proof of Theorem 1.2

Let V be a model of $ZFC + GCH$. Working in V we define a forcing notion which adds a Souslin tree which is almost Kurepa, in the sense that T becomes a Kurepa tree in its generic extension. The forcing notion is essentially the forcing notion introduced in [2] and we will recall it here for our later usage. Conditions p in $\mathbb{S}$ are of the form $\langle t, \langle \pi_\alpha : \alpha \in I \rangle \rangle$, where we write $t = t_p, I = I_p$ and $\langle \pi_\alpha : \alpha \in I \rangle = \vec{\pi}^p$ such that:

(1) t is a normal ω-splitting tree of countable height η, where η is either a limit of limit ordinals or the successor of a limit ordinal. We denote η by η_p.

(2) I is a countable subset of ω_2.

(3) Every π_α is an automorphism of $t \restriction Lim$, where Lim is the set of countable limit ordinals and $t \restriction Lim$ is obtained from t by restricting its levels to Lim.

The ordering is the natural one: $\langle s, \vec{\sigma} \rangle \leq \langle t, \vec{\pi} \rangle$ iff s end extends $t, dom(\vec{\sigma}) \supseteq dom(\vec{\pi})$ and for all $\alpha \in dom(\vec{\pi}), \sigma_\alpha \restriction t = \pi_\alpha$.

Remark 2.1. *In* [2]*, the conditions in* $\mathbb{S}$ *must satisfy an additional requirement that we do not impose here. This is needed in* [2] *to ensure the generic* T *is rigid. Its exclusion does not affect our proof, and in fact simplifies several details.*

Let

$$\mathbb{P} = \{p \in \mathbb{S} : \text{for some } \alpha_p, \eta_p = \alpha_p + 1\}.$$

It is easily seen that $\mathbb{P}$ is dense in $\mathbb{S}$. Let G be $\mathbb{P}$-generic over V. Let

$$T = \bigcup \{t_p : p \in G\}$$

and for each $\alpha < \omega_2$ set

$$\pi_i = \bigcup \{\sigma_i : \exists u = \langle t, \vec{\sigma} \rangle \in G, i \in I_u\}.$$

Then (see [2], Lemmas 2.3, 2.7, 2.9 and 2.14):

Lemma 2.2. (a) $\mathbb{P}$ *is* ω_1*-closed and satisfies the* ω_2*-c.c.*,

(b) $T = \langle \omega_1, <_T \rangle$ *is a Souslin tree.*

(c) *Each* π_i *is an automorphism of* $T \restriction Lim$.

(d) *If* b *is a branch of* T*, which is* T*-generic over* $V[G]$*, and if* $b_i = \pi_i``b, i < \omega_2$*, then the* b_i*'s are distinct branches of* T*. In particular* T *is almost Kurepa.*

Let $S = \{\alpha_p : p \in G, \alpha_p = \bigcup\{\alpha_q : q \in G, \alpha_q < \alpha_p\}$ and $I_p = \bigcup\{I_q : q \in G, \alpha_q < \alpha_p\}\}$. Then as in [3], Lemma 2.4, we can prove the following:

Lemma 2.3. *S is a stationary subset of ω_1.*

Working in $V[G]$ let $\mathbb{Q}$ be the usual forcing notion for adding a club subset of S using countable conditions and let H be $\mathbb{Q}$-generic over $V[G]$. Then (see [5] Theorem 23.8):

Lemma 2.4. *(a) $\mathbb{Q}$ is ω_1-distributive and satisfies the ω_2-c.c.,*

(b) $C = \bigcup H \subseteq S$ is a club subset of ω_1.

Let

$$\mathbb{R} = \{\langle p, \check{c}\rangle : p \in \mathbb{P}, p \Vdash \check{c} \in \underset{\sim}{\mathbb{Q}} \text{ and } \max(c) \leq \alpha_p\}.$$

Since $\mathbb{P}$ is ω_1-closed, $\mathbb{Q} \subseteq V$ and hence we can easily show that $\mathbb{R}$ is dense in $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$.

Lemma 2.5. *T remains a Souslin tree in $V[G][H]$.*

Proof. We work with $\mathbb{R}$ instead of $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$. Let $\underset{\sim}{A}$ be an $\mathbb{R}$-name, $r_0 \in \mathbb{R}$ and $r_0 \Vdash$ "$\underset{\sim}{A}$ is a maximal antichain in $\underset{\sim}{T}$". Let $\underset{\sim}{f}$ be a name for a function that maps each countable ordinal α to the smallest ordinal in $\underset{\sim}{A}[G * H]$ compatible with α. Then as in [2] we can define a decreasing sequence $\langle r_n : n < \omega\rangle$ of conditions in $\mathbb{R}$ such that

- r_0 is as defined above,
- $r_n = \langle p_n, \check{c}_n\rangle = \langle\langle t_n, \vec{\pi}^n\rangle, \check{c}_n\rangle$,
- $\alpha_{p_n} < \alpha_{p_{n+1}}$,
- r_{n+1} decides $\underset{\sim}{f} \restriction t_n$, say it forces "$\underset{\sim}{f} \restriction t_n = \check{f}_n$",
- $r_{n+1} \Vdash \underset{\sim}{C} \cap (\alpha_{p_n}, \alpha_{p_{n+1}}) \neq \emptyset$,

Let $p = \langle t, \vec{\pi}\rangle$ where $t = \bigcup_{n<\omega} t_n, \mathrm{dom}(\vec{\pi}) = \bigcup_{n<\omega} \mathrm{dom}(\vec{\pi}^n)$ and for $i \in \mathrm{dom}(\vec{\pi}), \pi_i = \bigcup_{n<\omega} \pi_i^n$. Let $c = \bigcup_{n<\omega} c_n \cup \{\alpha_p\}$, where $\alpha_p = \sup_{n<\omega} \alpha_{p_n}$. Then $p \in \mathbb{S}$, but it is not clear that $p \Vdash$ "$\check{c} \in \underset{\sim}{\mathbb{Q}}$".

Let $f = \bigcup_{n<\omega} f_n$ and set $a = ran(f \restriction t)$. As in [2], Lemma 2.9, we can define a condition $s = \langle q, \check{c}\rangle$ such that

- $s \in \mathbb{R}$,
- $\eta_q = \alpha_p + 1$, (and hence $\alpha_q = \alpha_p$),

- $s \Vdash$ "$\underset{\sim}{A} \cap \check{t}$ is a maximal antichain in $\check{t}$",
- Every new node (i.e. every node at the α_p-th level) of the tree part of s is above a condition in a.

It is now clear that $s \Vdash \underset{\sim}{A} = \check{a}$, and hence $s \Vdash$ "$\underset{\sim}{A}$ is countable". The lemma follows. □

From now on we work in $V^* = V[G][H]$. Thus in V^* we have a Souslin tree T. We claim that T is as required. To see this force with T over V^* and let b be a branch of T which is T-generic over V^*.

Lemma 2.6. *In $V^*[b], T$ is an almost Souslin Kurepa tree.*

Proof. Work in $V^*[b]$. By Lemma 2.2(d) T is a Kurepa tree. We now show that T is almost Souslin. We may suppose that T is obtained using the branches b and $b_i, i < \omega_2$, in the sense that for each $\alpha < \omega_1, T_\alpha$, the α-th level of T, is equal to $\{b(\alpha)\} \cup \{b_i(\alpha) : i < \omega_2\}$ where $b(\alpha)$ ($b_i(\alpha)$) is the unique node in $b \cap T_\alpha$ ($b_i \cap T_\alpha$). We further suppose that $b = b_0$.

Now let $\alpha \in C$, and let $p \in G$ be such that $\alpha = \alpha_p$. We define a function g_α on T_α as follows. Note that $T_\alpha = \{b_i(\alpha) : i \in I_p\}$. Let

$$g_\alpha(b_i(\alpha)) = b_i(\alpha_q)$$

where $q \in G$ is such that $\alpha_q < \alpha$ is the least such that $i \in I_q$ (such a q exists using the fact that $C \subseteq S$). It is easily seen that g_α is well-defined (it does not depend on the choice of p), and that for each $x \in T_\alpha, g_\alpha(x) <_T x$. The rest of the proof of the fact that T is almost Souslin is essentially the same as in [3], Lemma 2.6. □

This concludes the proof of Theorem 1.2.

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