MORE ON ALMOST SOUSLIN KUREPA TREES

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Abstract. It is consistent that there exists a Souslin tree $T$ such that after forcing with it, $T$ becomes an almost Souslin Kurepa tree. This answers a question of Zakrzewski [6].

1. Introduction

In this paper we continue our study of $\omega_1$-trees started in [3] and prove another consistency result concerning them. Let $T$ be a normal $\omega_1$-tree. Let’s recall that:

- $T$ is a Kurepa tree if it has at least $\omega_2$-many branches.
- $T$ is a Souslin tree if it has no uncountable antichains (and hence no branches).
- $T$ is an almost Souslin tree if for any antichain $X \subseteq T$, the set $S_X = \{ht(x) : x \in X\}$ is not stationary (see [1], [6]).

We refer to [3] and [4] for historical information and more details on trees.

In [6], Zakrzewski asked some questions concerning the existence of almost Souslin Kurepa trees. In [3] we answered two of these questions but one of them remained open:

**Question 1.1.** Does there exist a Souslin tree $T$ such that for each $G$ which is $T$-generic over $V$, $T$ is an almost Souslin Kurepa tree in $V[G]$?

In this paper we give an affirmative answer to this question.

**Theorem 1.2.** It is consistent that there exists a Souslin tree $T$ such that for each $G$ which is $T$-generic over $V$, $T$ is an almost Souslin Kurepa tree in $V[G]$.

The rest of this paper is devoted to the proof of this theorem. Our proof is motivated by [2] and [3].

This research was in part supported by a grant from IPM (No. 91030417). The author also would like to thank the referee of the paper for some useful remarks and comments.
2. Proof of Theorem 1.2

Let $V$ be a model of $ZFC + GCH$. Working in $V$ we define a forcing notion which adds a Souslin tree which is almost Kurepa, in the sense that $T$ becomes a Kurepa tree in its generic extension. The forcing notion is essentially the forcing notion introduced in [2] and we will recall it here for our later usage. Conditions $p$ in $S$ are of the form $(t, \langle \pi_\alpha : \alpha \in I \rangle)$, where we write $t = t_p, I = I_p$ and $\langle \pi_\alpha : \alpha \in I \rangle = \vec{\pi}^p$ such that:

1. $t$ is a normal $\omega$-splitting tree of countable height $\eta$, where $\eta$ is either a limit of limit ordinals or the successor of a limit ordinal. We denote $\eta$ by $\eta_p$.
2. $I$ is a countable subset of $\omega_2$.
3. Every $\pi_\alpha$ is an automorphism of $t \upharpoonright Lim$, where $Lim$ is the set of countable limit ordinals and $t \upharpoonright Lim$ is obtained from $t$ by restricting its levels to $Lim$.

The ordering is the natural one: $(s, \vec{\sigma}) \leq (t, \vec{\pi})$ iff $s$ end extends $t, dom(\vec{\sigma}) \supseteq dom(\vec{\pi})$ and for all $\alpha \in dom(\vec{\pi}), \sigma_\alpha \upharpoonright t = \pi_\alpha$.

Remark 2.1. In [2], the conditions in $S$ must satisfy an additional requirement that we do not impose here. This is needed in [2] to ensure the generic $T$ is rigid. Its exclusion does not affect our proof, and in fact simplifies several details.

Let $P = \{p \in S : \text{for some } \alpha_p, \eta_p = \alpha_p + 1\}$.

It is easily seen that $P$ is dense in $S$. Let $G$ be $P$-generic over $V$. Let $T = \bigcup\{t_p : p \in G\}$ and for each $\alpha < \omega_2$ set $\pi_\alpha = \bigcup\{\sigma_i : \exists u = \langle t, \vec{\sigma} \rangle \in G, i \in I_u\}$.

Then (see [2], Lemmas 2.3, 2.7, 2.9 and 2.14):

Lemma 2.2. (a) $P$ is $\omega_1$-closed and satisfies the $\omega_2$-c.c.,
(b) $T = \langle \omega_1, <_T \rangle$ is a Souslin tree.
(c) Each $\pi_\alpha$ is an automorphism of $T \upharpoonright Lim$.
(d) If $b$ is a branch of $T$, which is $T$-generic over $V[G]$, and if $b_i = \pi_i '' b, i < \omega_2$, then the $b_i$’s are distinct branches of $T$. In particular $T$ is almost Kurepa.
Let $S = \{\alpha_p : p \in G, \alpha_p = \bigcup_{}\{\alpha_q : q \in G, \alpha_q < \alpha_p\} \text{ and } I_p = \bigcup_{}\{I_q : q \in G, \alpha_q < \alpha_p\}\}$. Then as in [3], Lemma 2.4, we can prove the following:

**Lemma 2.3.** $S$ is a stationary subset of $\omega_1$.

Working in $V[G]$ let $Q$ be the usual forcing notion for adding a club subset of $S$ using countable conditions and let $H$ be $Q$-generic over $V[G]$. Then (see [5] Theorem 23.8):

**Lemma 2.4.**

(a) $Q$ is $\omega_1$-distributive and satisfies the $\omega_2$-c.c.,

(b) $C = \bigcup H \subseteq S$ is a club subset of $\omega_1$.

Let

$$\mathbb{R} = \{\langle p, \check{c} \rangle : p \in P, p\parallel\check{c} \in Q \text{ and } \max(c) \leq \alpha_p\}.$$ 

Since $P$ is $\omega_1$-closed, $Q \subseteq V$ and hence we can easily show that $\mathbb{R}$ is dense in $P \ast \check{Q}$.

**Lemma 2.5.** $T$ remains a Souslin tree in $V[G][H]$.

**Proof.** We work with $\mathbb{R}$ instead of $P \ast \check{Q}$. Let $A$ be an $\mathbb{R}$-name, $r_0 \in \mathbb{R}$ and $r_0\parallel\check{A}$ is a maximal antichain in $T'$. Let $f$ be a name for a function that maps each countable ordinal $\alpha$ to the smallest ordinal in $A[G \ast H]$ compatible with $\alpha$. Then as in [2] we can define a decreasing sequence $\langle r_n : n < \omega \rangle$ of conditions in $\mathbb{R}$ such that

- $r_0$ is as defined above,
- $r_n = \langle p_n, \check{c}_n \rangle = \langle \langle t_n, \check{\pi}^n \rangle, \check{c}_n \rangle$,
- $\alpha_{p_n} < \alpha_{p_{n+1}}$,
- $r_{n+1}$ decides $f \restriction t_n$, say it forces “$f \restriction t_n = f_n$”,
- $r_{n+1}\parallel\check{\mathcal{C}} \cap (\alpha_{p_n}, \alpha_{p_{n+1}}) \neq \emptyset$,

Let $p = \langle t, \check{\pi} \rangle$ where $t = \bigcup_{n<\omega} t_n, \text{dom}(\check{\pi}) = \bigcup_{n<\omega} \text{dom}(\check{\pi}^n)$ and for $i \in \text{dom}(\check{\pi}), \pi_i = \bigcup_{n<\omega} \pi_i^n$. Let $c = \bigcup_{n<\omega} c_n \cup \{\alpha_p\}$, where $\alpha_p = \text{sup}_{n<\omega} \alpha_{p_n}$. Then $p \in S$, but it is not clear that $p\parallel\check{c} \in \check{Q}$.

Let $f = \bigcup_{n<\omega} f_n$ and set $a = \text{ran}(f \restriction t)$. As in [2], Lemma 2.9, we can define a condition $s = \langle q, \check{c} \rangle$ such that

- $s \in \mathbb{R}$,
- $\eta_q = \alpha_p + 1$, (and hence $\alpha_q = \alpha_p$),
• $s \parallel \mathcal{A}$ is a maximal antichain in $\mathcal{A}$.

• Every new node (i.e. every node at the $\alpha_p$-th level) of the tree part of $s$ is above a condition in $a$.

It is now clear that $s \parallel \mathcal{A} = \mathcal{A}$, and hence $s \parallel \mathcal{A}$ is countable”. The lemma follows.

From now on we work in $V^* = V[G][H]$. Thus in $V^*$ we have a Souslin tree $T$. We claim that $T$ is as required. To see this force with $T$ over $V^*$ and let $b$ be a branch of $T$ which is $T$-generic over $V^*$.

Lemma 2.6. In $V^*[b], T$ is an almost Souslin Kurepa tree.

Proof. Work in $V^*[b]$. By Lemma 2.2(d) $T$ is a Kurepa tree. We now show that $T$ is almost Souslin. We may suppose that $T$ is obtained using the branches $b$ and $b_i$, $i < \omega_2$, in the sense that for each $\alpha < \omega_1, T_\alpha$, the $\alpha$-th level of $T$, is equal to $\{b(\alpha)\} \cup \{b_i(\alpha) : i < \omega_2\}$ where $b(\alpha)$ ($b_i(\alpha)$) is the unique node in $b \cap T_\alpha$ ($b_i \cap T_\alpha$). We further suppose that $b = b_0$.

Now let $\alpha \in C$, and let $p \in G$ be such that $\alpha = \alpha_p$. We define a function $g_\alpha$ on $T_\alpha$ as follows. Note that $T_\alpha = \{b_i(\alpha) : i \in I_p\}$. Let

$$g_\alpha(b_i(\alpha)) = b_i(\alpha_q)$$

where $q \in G$ is such that $\alpha_q < \alpha$ is the least such that $i \in I_q$ (such a $q$ exists using the fact that $C \subseteq S$). It is easily seen that $g_\alpha$ is well-defined (it does not depend on the choice of $p$), and that for each $x \in T_\alpha, g_\alpha(x) <_T x$. The rest of the proof of the fact that $T$ is almost Souslin is essentially the same as in [3], Lemma 2.6.

This concludes the proof of Theorem 1.2.

References


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