SINGULAR COFINALITY CONJECTURE AND A QUESTION OF GORELIC

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Abstract. We give an affirmative answer to a question of Gorelic [5], by showing it is consistent, relative to the existence of large cardinals, that there is a proper class of cardinals α with $cf(\alpha) = \omega_1$ and $\alpha^\omega > \alpha$.

1. introduction

Around 1980, Pouzet [8] proved the fundamental result that if $(\mathbb{P}, \leq)$ is a poset of singular cofinality, then it contains an infinite antichain. This lead to the formulation of a very natural conjecture, first appearing implicitly in [8], and then explicitly in a paper by Milner and Sauer [7]:

Conjecture. Suppose that $(\mathbb{P}, \leq)$ is a poset of singular cofinality λ. Then $(\mathbb{P}, \leq)$ has an antichain of size $cf(\lambda)$.

This is called the *Singular Cofinality Conjecture.*

Set $C = \{\alpha : \alpha \text{ is a cardinal}, cf(\alpha) = \omega_1, \alpha^\omega > \alpha\}$. In [5], Gorelic observed that if C is not a proper class, then the Singular Cofinality Conjecture holds ultimately (in ZFC) in the case of cofinality ω_1, and he asked if it is consistent that C is a proper class. In this paper we give an affirmative answer to this question, assuming the existence of large cardinals:

Theorem 1.1. *Assuming the existence of suitable large cardinals, it is consistent that* $C = \{\alpha : \alpha \textit{ is a cardinal}, cf(\alpha) = \omega_1, \alpha^\omega > \alpha\}$ *is a proper class.*

Remark 1.2. *We give three different proofs for the above theorem. The first proof uses a strong cardinal (in fact a $\kappa^{+\omega_1+2}-$strong cardinal κ) and is based on extender based Radin forcing. The second proof assumes the existence of a proper class of $\kappa^{+\omega_1+1}-$strong cardinals κ, and is based on iterated Prikry forcing. The third proof also assumes the existence of a proper class of $\kappa^{+\omega_1+1}-$strong cardinals κ, and is based on iteration of extender based Prikry*

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forcing. We also show that the large cardinal assumption in our second and third proofs is almost optimal.

2. PROOF OF THE MAIN THEOREM

2.1. **First proof.** In this subsection we give our first proof of the main Theorem 1.1., assuming the existence of a strong cardinal. Thus suppose that GCH holds and let κ be a strong cardinal. Let $j: V \rightarrow M$ be an elementary embedding of the universe into some inner model M with $crit(j)=\kappa$ and $M \supseteq V_{\kappa^{+\omega_1+2}}$. Using j construct, as in [6], an extender sequence system $\bar{E}$ of length κ^+ and of size $\kappa^{+\omega_1+1}$, and let $\mathbb{P}_{\bar{E}}$ be the corresponding extender based Radin frocing as is defined in [6]. Also let G be $\mathbb{P}_{\bar{E}}-$generic over V. Then:

Theorem 2.1. *([6]) (a) V and $V[G]$ have the same cardinals,*

(b) κ remains an inaccessible cardinal in $V[G]$,

(c) In $V[G]$, there exists a club $\bar{C}$ of κ, such that if γ is a limit point of $\bar{C}$, then $2^\gamma = \gamma^{+\omega_1+1}$

By (b), V_κ of $V[G]$ is a model of ZFC, and the following lemma shows that in it, C is a proper class, which completes the proof of Theorem 1.1.

Lemma 2.2. *In $V[G], \{\alpha < \kappa : \alpha$ is a cardinal, $cf(\alpha)=\omega_1, \alpha^\omega > \alpha\} \supseteq \{\gamma^{+\omega_1} : \gamma$ is a limit point of $\bar{C}, cf(\gamma)=\omega\}$.*

Proof. Suppose γ is a limit point of $\bar{C}$ of cofinality ω. Then clearly $cf(\gamma^{+\omega_1})=\omega_1$. We also have $(\gamma^{+\omega_1})^\omega \geq \gamma^\omega = 2^\gamma = \gamma^{+\omega_1+1} > \gamma^{+\omega_1}$. □

2.2. **Second proof.** We now give our second proof of the main Theorem 1.1., assuming the existence of a proper class of $\kappa^{+\omega_1+1}-$strong cardinals κ. Thus assume GCH holds and suppose that there exists a proper class A of $\kappa^{+\omega_1+1}-$strong cardinals κ. We may assume that no element of A is a limit point of A.

Step 1) Let $\mathbb{P}$ be the reverse Easton iteration of $Sacks(\alpha, \alpha^{+\omega_1+1})$ for each inaccessible cardinal α, and let G be $\mathbb{P}-$generic over V. Then:

Theorem 2.3. *([2]) (a) V and $V[G]$ have the same cardinals and cofinalities,*

(b) $V[G] \models$ "for each inaccessible cardinal $\alpha, 2^\alpha = \alpha^{+\omega_1+1}$",

(c) Each $\alpha \in A$ is measurable in $V[G]$.

Step 2) Working in $V[G]$, let $\mathbb{Q}$ be the forcing defined in [1, §3.1], for changing the cofinality of each $\alpha \in A$ to ω, and let H be $\mathbb{Q}-$generic over $V[G]$.

Theorem 2.4. *([1]) (a) $V[G]$ and $V[G][H]$ have the same cardinals,*

(b) For each $\alpha \in A, V[G][H] \models$ "α is a strong limit cardinal, $cf(\alpha)=\omega$ and $2^\alpha = \alpha^{+\omega_1+1}$".

The following lemma completes the proof of the theorem:

Lemma 2.5. *In $V[G][H], C \supseteq \{\alpha^{+\omega_1} : \alpha \in A\}$.*

Proof. Work in $V[G][H]$ and let $\alpha \in A$. Clearly $cf(\alpha^{+\omega_1}) = \omega_1$. We also have $(\alpha^{+\omega_1})^\omega \geq \alpha^\omega = 2^\alpha = \alpha^{+\omega_1+1} > \alpha^{+\omega_1}$. □

2.3. **Third proof.** In this subsection we give our third proof of the main Theorem 1.1., assuming the existence of a proper class of $\kappa^{+\omega_1+1}-$strong cardinals κ. Again assume GCH holds and let A be a proper class of $\kappa^{+\omega_1+1}-$strong cardinals κ, such that no element of A is a limit point of A.

For each $\kappa \in A$, fix a $(\kappa, \kappa^{+\omega_1+1})-$extender $E(\kappa)$ and let $(\mathbb{P}_{E(\kappa)}, \leq_{\mathbb{P}_{E(\kappa)}}, \leq^*_{\mathbb{P}_{E(\kappa)}})$ (where $\leq^*_{\mathbb{P}_{E(\kappa)}}$ is the Prikry extension relation) be the corresponding extender based Prikry forcing for changing the cofinality of κ into ω, and making $2^\kappa = \kappa^{+\omega_1+1}$ [3].

Let $\mathbb{P}$ be the following version of iterated extender based Prikry forcing. Conditions in $\mathbb{P}$ are of the form $p = (X^p, F^p)$, where

(1) X^p is a subset of A,
(2) F^p is a function on X^p,
(3) For all $\kappa \in X^p, F^p(\kappa)$ is a condition in $\mathbb{P}_{E(\kappa)}$.

Given $p, q \in \mathbb{P}$, we define $p \leq q$ (p is stronger than q), if

(1) $X^p \supseteq X^q$,
(2) For all $\kappa \in X^q$, $F^p(\kappa) \leq_{\mathbb{P}_{E(\kappa)}} F^q(\kappa)$.

We also define the Prikry relation by $p \leq^* q$ iff

(1) $p \leq q$,
(2) For all $\kappa \in X^q$, $F^p(\kappa) \leq^*_{\mathbb{P}_{E(\kappa)}} F^q(\kappa)$.

Let G be $\mathbb{P}-$generic over V. Then using the methods of [1] and [3] we can prove the following:

Theorem 2.6. *(a) $\mathbb{P}$ is a tame class forcing notion; in particular $V[G] \models ZFC$,*

(b) $(\mathbb{P}, \leq, \leq^*)$ satisfies the Prikry property,

(c) V and $V[G]$ have the same cardinals,

(d) For each $\alpha \in A, V[G] \models$ "α is a strong limit cardinal, $cf(\alpha) = \omega$ and $2^\alpha = \alpha^{+\omega_1+1}$".

The rest of the argument is as in the second proof.

3. NECESSARY USE OF LARGE CARDINALS

In this section we show that some large cardinal assumptions are needed for the proof of Theorem 1.1.

Theorem 3.1. *Assume there is a model V of ZFC in which C is a proper class. Then there is an inner model of ZFC which contains a proper class of measurable cardinals.*

Proof. We may assume that there is no inner model with a strong cardinal, as otherwise we are done. Let $\mathcal{K}$ denote the core model of V below a strong cardinal. Assume on the contrary that the measurable cardinals of $\mathcal{K}$ are bounded, say by $\lambda > 2^{\omega_1}$. Then for all $\alpha > 2^\lambda$ with $cf(\alpha) = \omega_1$, we have

$$[\alpha]^\omega = \bigcup_{\delta<\alpha} [\delta]^\omega.$$

On the other hand, by the covering lemma,

$$[\delta]^\omega \subseteq [\delta]^{\leq\lambda} \subseteq \bigcup_{x \in \mathcal{K} \cap [\delta]^\lambda} P(x),$$

and hence

$$\delta^\omega \leq \sum_{x \in \mathcal{K} \cap [\delta]^\lambda} |P(x)| \leq |\mathcal{K} \cap [\delta]^\lambda| . 2^\lambda \leq \delta^+ . 2^\lambda < \alpha,$$

which implies

$$\alpha^\omega = \alpha$$

Thus $C \subseteq (2^\lambda)^+$ is bounded, and we get a contradiction. □

In fact we can prove more:

Theorem 3.2. *Assume there is a model V of ZFC in which C is a proper class. Then $\{\delta : \delta$ is a cardinal, $cf(\delta) = \omega$ and $\delta^\omega \geq \delta^{+\omega_1+1}\}$ is a proper class.*

Proof. Given $2^\omega < \alpha \in C$, we have $cf(\alpha) = \omega_1$ and $\alpha^\omega \geq \alpha^+$, hence there is $\gamma < \alpha$ such that $\gamma^\omega \geq \alpha^+$. Let δ be a singular cardinal of cofinality ω in the interval (γ, α). Then $\delta^\omega \geq \alpha^+ \geq \delta^{+\omega_1+1}$. □

It follows from the above theorem and the results of [4] that the large cardinal assumption made in our second and third proofs is almost optimal.

4. A GENERALIZATION

In general, for an infinite cardinal λ, set $C_\lambda = \{\alpha : \alpha$ is a cardinal, $cf(\alpha) = \lambda^+$ and $\alpha^\lambda > \alpha^{<\lambda} = \alpha\}$. Then by a simple modification of the above proofs we have the following:

Theorem 4.1. *Suppose GCH holds, κ is a strong cardinal and λ is an infinite cardinal less than κ. Then there is a cardinal preserving generic extension $V[G]$ of the universe in which κ remains inaccessible, no new subsets of λ^+ are added (in particular it remains regular in the extension), and $C_\lambda \cap \kappa$ is unbounded in κ.*

Theorem 4.2. *Suppose GCH holds, λ is an infinite cardinal, and there exists a proper class of $\kappa^{+\lambda^++1}-$strong cardinals κ. Then there is a generic extension $V[G]$ of V in which C_λ is a proper class.*

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