ON A QUESTION OF HAMKINS AND LÖWE ON THE MODAL LOGIC OF COLLAPSE FORCING

MOHAMMAD GOLSHANI AND WILLIAM MITCHELL

Abstract. Hamkins and Löwe asked whether there can be a model N of set theory with the property that $N \equiv N[H]$ whenever H is a generic collapse of a cardinal of N onto ω. We give a positive answer, assuming the existence of a cardinal κ with a measure sequence of length κ^+, a property which lies between $o(\kappa) = \kappa^{++}$ and $o(\kappa) = \kappa^{++} + 1$. We also show that our large cardinal assumption is necessary for the consistency proof.

1. introduction

In [4], Hamkins and Löwe introduced the modal logic of forcing. Suppose Γ is a definable class of forcing notions. A Γ-translation from propositional modal logic to set theory is a map $H : \mathrm{Sent}(\mathcal{L}_m) \to \mathrm{Sent}(\mathcal{L}_\in)$ from the set of sentences of the modal logic to the set of sentences of set theory which assigns to each propositional variable some sentence in the language of set theory, leaves the propositional connectives unchanged and in it the modal operations $\Diamond$ and $\Box$ are interpreted by

$$H(\Diamond\phi) = \Diamond_\Gamma \phi = \exists \mathbb{P} \in \Gamma, \exists p \in \mathbb{P}, p \Vdash \phi,$$

and

$$H(\Box\phi) = \Box_\Gamma \phi = \forall \mathbb{P} \in \Gamma, \forall p \in \mathbb{P}, p \Vdash \phi.$$

If $\mathcal{H}(\Gamma)$ denotes the set of Γ-translations, then we set

$$\mathrm{Force}(V, \Gamma) = \{\phi \in \mathrm{Sent}(\mathcal{L}_m) \mid \forall H \in \mathcal{H}(\Gamma), V \models H(\phi)\},$$

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where V is a model of ZFC. We call $\text{Force}(V, \Gamma)$ the modal logic of the class Γ over V. Also let

$$\text{Force}(\Gamma) = \bigcap \{\text{Force}(V, \Gamma) \mid V \models ZFC\}.$$

$\text{Force}(\Gamma)$ is called the modal logic of the class Γ. The main result of [4] says that the modal logic of the class of all forcing notions (over L, the Gödel's constructible universe) is exactly $S4.2$, i.e., using the above notation, $\text{Force}(\Gamma) = \text{Force}(L, \Gamma) = S4.2$, where Γ is the class of all forcing notions.

In [3], among many other things, it is proved that the modal logic of collapse forcing is $S4.3$, i.e., $\text{Force}(\text{Col}) = S4.3$, where Col denotes the class of all collapsing forcing notions.

Hamkins and Löwe have asked, in connection with results in [4], whether there can be a model N of ZFC such that $N \equiv N[H]$ whenever H is the generic collapse of any cardinal onto ω. This question appeared later in Hamkins's paper [2], as question 10.

Let us explain the impact of the existence of such model N for the class Col of collapse forcing notions. Recall that a sentence ϕ of the language of set theory is called a Γ-*button* over V if $V \models \Box_\Gamma \Diamond_\Gamma \Box_\Gamma \phi$ and it is called a Γ-*switch* over V if $V \models \Box_\Gamma \Diamond_\Gamma \phi \wedge \Box_\Gamma \Diamond_\Gamma \neg \phi$. By [3], if V contains arbitrary large finite independent families of buttons and switches, then $\text{Force}(V, \text{Col}) \subseteq S5$. So a model of set theory as N above would be an extreme counterexample in having no switches at all for the class of collapse forcing, and would have valid principles of collapse forcing that are beyond $S5$, a hard upper bound for the other natural classes of forcing.

Modal logics not contained in $S5$ have recently been studied by Inamdar and Löwe [5], where they determined that the modal logic of inner models is S4.2*Top*, a modal logic which is not contained in S5. *Top* is the axiom

$$\Diamond(\ (\Box\phi \leftrightarrow \phi) \wedge (\Box\neg\phi \leftrightarrow \neg\phi)\ ).$$

It is clear that if N is a model as above, then we have

$$N \models \Diamond_{\text{Col}} \phi \leftrightarrow \Box_{\text{Col}} \phi \leftrightarrow \phi$$

and so clearly $\textit{Top} \in \text{Force}(N, \text{Col})$.

Theorem 1.1 gives a positive answer to the question of Hamkins and Löwe.

Theorem 1.1. *Suppose that there is a cardinal κ with a measure sequence of length κ^+. Then there is a model N with the property that $N[H] \equiv N$ whenever H is a generic collapse of some cardinal λ of N onto ω.*

The definition of a "measure sequence of length κ^+" is given in Definition 2.2. Its strength is properly between those of $o(\kappa) = \kappa^{++}$ and $o(\kappa) = \kappa^{++} + 1$.

Our proof of Theorem 1.1 is based on an unpublished note [9] by the second author which used a cardinal κ with $o(\kappa) = \kappa^+$ to give a partial answer but with an incorrect proof. The second author gave an alternative proof using measure sequences, and it was then realized that this proof gives the full result.

Theorem 1.2 shows that the hypothesis in Theorem 1.1 is necessary:

Theorem 1.2 (Mitchell)**.** *Suppose κ is an inaccessible cardinal such that $V \equiv V[H]$ for any cardinal $\lambda < \kappa$ and any $\mathrm{Col}(\omega, \lambda)$-generic filter H over V. Then there is an inner model with a cardinal having a measure sequence of length κ^+.*

The proof of Theorem 1.1 is given in Section 2. and that of Theorem 1.2 is given in Section 3. A final Section 4 discusses some finer distinctions arising from the fact that the conclusion of Theorem 1.1 is not expressible in ZF set theory, and from the fact that the property Hamkins and Löwe asked for, a model V such that $V^{\mathrm{Col}(\omega,\lambda)} \equiv V$ for *all cardinals* λ, is substantially weaker than the conclusion of Theorem 1.1.

2. The consistency of a positive answer to Hamkins and Löwe

In this section we give a proof of Theorem 1.1. The strategy of the proof, given a measure sequence u on κ of length κ^+, is to use Radin forcing to obtain a closed unbounded set $C \subset \kappa$ of indiscernibles, with interleaved collapses so that in the resulting model the uncountable cardinals below κ are the limit points of C together with $\{\lambda^+ \mid \lambda \in C\}$. The result is that if $\eta < \kappa$ is any uncountable cardinal in the resulting model $V[G]$, then there is some $\lambda \in C$ such that $V[G]^{\mathrm{Col}(\omega,\eta)} = V[G]^{\mathrm{Col}(\omega,\lambda)}$ — if $\eta \in \mathrm{Lim}(C)$ then $\lambda = \eta$ and if $\eta = \omega$ or η is the successor of a member of C then $\lambda = \min(C \setminus \eta)$. Hence the indiscernibility of C implies that all of the models $V[G]^{\mathrm{Col}(\omega,\eta)}$ satisfy the same sentences.

Following this strategy directly would require collapsing, for each successive $\eta < \lambda$ in C, λ onto η^+; however the technique of interleaving collapses with Radin forcing only allows collapsing λ onto η^{++}. Thus we follow the first forcing with an Easton support product collapsing λ^{++} onto λ^+. Because of the homogeneity of the collapse forcing, this does not affect the indiscernibility of C for formulas which do not use the collapse functions as parameters. Thus this two stage forcing will have the desired property.

Most of the complexity of the proof comes in dealing with the Radin forcing. The following lemma gives the properties of this forcing, and we will follow it with the proof of Theorem 1.1 from Lemma 2.1. Following this we will give the proof of Lemma 2.1, beginning with the definition of a measure sequence.

Lemma 2.1. *Assume that w is a measure sequence on κ of length κ^+. Then there is a forcing $\mathbb{R}$ such that if $G \subset \mathbb{R}$ is generic then $V[G]$ has the following properties:*

(1) There is a closed unbounded subset C of κ such that the set of uncountable cardinals of $V[G]$ below κ is exactly $\mathrm{Lim}(C) \cup \{\lambda^+ \mid \lambda \in C\} \cup \{\lambda^{++} \mid \lambda \in C\}$.

(2) $V[G]$ satisfies the GCH, and cardinalities and cofinalities at and above κ are the same in $V[G]$ as in V. In particular κ is still inaccessible.

(3) For any sentence σ of set theory with no parameters,

$$\Vdash_{\mathbb{R}} (\sigma \iff (\forall \lambda \in \dot{C}) \Vdash_{\mathrm{Col}(\omega,\lambda)} \sigma).$$

(4) Suppose $\lambda \in C$ and $H \subseteq \mathrm{Col}(\omega, \lambda^+)$ is $V[G]$-generic. Then there is a $V[G]$-generic set $H' \subset \mathrm{Col}(\omega, \min(C \setminus \lambda^+))$ such that $V[G][H] = V[G][H']$.

For the purpose of some minor simplifications, we will assume that V is an extender model and is the least model having such a measure sequence.

Proof of Theorem 1.1 from Lemma 2.1. In $V[G]$, let $\mathbb{Q}$ be the Easton support product $\Pi\{\mathrm{Col}(\lambda^+, \lambda^{++}) \mid \lambda \in C\}$, and let $V' = V[G][H]$ where $H \subset \mathbb{Q}$ is $V[G]$ generic. By standard arguments, this does not change cofinalities or collapse any cardinals other than those specified; thus κ is still inaccessible in V' and the uncountable cardinals below κ are exactly $\mathrm{Lim}(C) \cup \{\lambda^+ \mid \lambda \in C\}$.

Now let σ be an arbitrary sentence of set theory, and suppose that $V' \models \sigma$. Then by the homogeneity of $\mathbb{Q}$, $V[G] \models \Vdash_{\mathbb{Q}} \sigma$, so by Clause 2.1(3), $V[G] \models \forall\lambda \in C \Vdash_{\mathrm{Col}(\omega,\lambda)} \Vdash_{\mathbb{Q}} (\omega, \sigma)$. Now $\mathrm{Col}(\omega,\lambda) \times \mathbb{Q}$ is equivalent to $\mathbb{Q} \times \mathrm{Col}(\omega,\lambda)$, so

$$V' \models \sigma \iff \forall\lambda \in C \Vdash_{\mathrm{Col}(\omega,\lambda)} \sigma. \tag{1}$$

Now suppose that η is any uncountable cardinal below κ in V'. If η is a limit cardinal then $\eta \in C$, so by Formula (1) $V' \models \Vdash_{\mathrm{Col}(\omega,\lambda)} \sigma$. Otherwise $\eta = \lambda^+$ for some $\eta \in C$. Then $V'] \models \Vdash_{\mathrm{Col}(\omega,\min(C\setminus\lambda+1))} \sigma$ by formula (1), and it follows by Lemma 2.1(4) that $V' \models \Vdash_{\mathrm{Col}(\omega,\eta)}$ σ. □

2.1. **Measure sequences.**

Definition 2.2. Suppose that $j : V \to M$ is an elementary embedding with critical point κ. The *measure sequence generated by* j is the sequence $u = \langle\, u(\tau) \mid \tau < \mathrm{lh}(\tau) \,\rangle$ defined by recursion on τ by setting $u(0) = \kappa$, and, for $\tau > 0$, setting $u(\tau)$ to be the ultrafilter on V_κ defined by

$$X \in u(\tau) \iff u \restriction \tau \in j(X).$$

The recursion stops, with $\mathrm{lh}(u) = \tau$, at the least τ such that $u{\restriction}\tau \notin M$.

Measure sequences were introduced by Radin in his original presentation of Radin forcing [10]. Note that $u(1)$ is the normal measure U induced by the embedding j. A measure sequence of length 3 is what was called in [7] a μ-measurable cardinals, which is stronger than $o(\kappa) = \kappa^{++}$. On the other hand, an extender of length κ^{++} generates an extender sequence of length κ^{++}, so any shorter measure sequence is weaker than $o(\kappa) = \kappa^{++} + 1$.

We regard a sequence $\langle\, \lambda \,\rangle$ of length 1 as the measure sequence on λ generated by the identity embedding. If u is a measure sequence generated by j then we write $\kappa(u)$ for $\mathrm{crit}(j) = u(0)$.

If u is a measure sequence and $\tau < \mathrm{lh}(u)$ then we write $j_{u(\tau)} \colon V \to M_{u(\tau)} = \mathrm{Ult}(V, u(\tau))$. If $\tau < \tau' < \mathrm{lh}(u)$ then there is an elementary embedding

$$k^u_{\tau',\tau} = k_{u(\tau),u(\tau')} \colon M_{u(\tau)} \to M_{u(\tau')} \quad \text{defined by} \quad k_{u(\tau),u(\tau')}([f]_{u(\tau)}) = [f]_{u(\tau')}.$$

These embeddings commute, that is, if $\tau'' < \tau' < \tau < \mathrm{lh}(u)$ then $k_{u(\tau''),u(\tau)} = k_{u(\tau'),u(\tau)} \circ k_{u(\tau''),u(\tau')}$. This enables us to define, for $\tau \leq \mathrm{lh}(u)$,

$$j_{u\restriction\tau}\colon V \to M_{u\restriction\tau} = \lim\operatorname{dir}\langle\langle M_{\tau'} \mid \tau' < \tau\rangle, \langle k_{u(\tau''),u(\tau')} \mid \tau'' \leq \tau' < \tau\rangle\rangle$$

In particular, if u is any measure sequence then u is generated by j_u.

Note that if $\tau + 1 < \mathrm{lh}(u)$ then $j_{u\restriction\tau+1}$ is the same as $j_{u(\tau)}$. Our notation will frequently use this observation. Note that the critical point of all embeddings j_τ is κ, and the critical point of all embeddings $k_{\tau',\tau}$ is $(\kappa^{++})^{M_{\tau'}}$.

Each of these models contains $V_{\kappa+1}$ and hence κ^+ is the same ordinal in all of them. In Definition 2.4 we use the fact that if $\tau < \mathrm{lh}(u)$ then $(\kappa^{+++})^{M_{u(\tau)}} < j_{u(\tau)}(\kappa) < \kappa^{++}$, and if $\tau + 1 < \mathrm{lh}(u)$ then $j_{u(\tau)}(\kappa) < (\kappa^{++})^{M_{u(\tau+1)}}$.

2.2. **Finding guiding generic filters.** We now define the forcing notions for which we will need "guiding generics".

Definition 2.3. Suppose that u is a measure sequence on a cardinal λ and $\tau < \mathrm{lh}(u)$. Then $\mathbb{Q}^u_\tau = \mathrm{Col}(\lambda^{++}, j_{u(\tau)}(\lambda))^{M_{u(\tau)}}$.

Note that $Q^u_\tau = [\lambda w\ \mathrm{Col}(\kappa(w)^{++}, \lambda)]_{u(\tau)}$ and that if $\tau' < \tau$ then $k^u_{\tau',\tau}(\mathbb{Q}^u_{\tau'}) = \mathbb{Q}^u_\tau$.

In the next definition, the assumption that V is an extender model $L[\mathcal{E}]$, and hence has a definable well order, is used to give an explicit definition of I_u. The word "first" means "first in the order of construction of $L[\mathcal{E}]$".

Definition 2.4. If u is a measure sequence on λ then we define a $M_{u(\tau)}$-generic subset $I^u_\tau = I_{u\restriction\tau}$ of Q^u_τ by recursion on $\tau < \mathrm{lh}(u)$. The definition can be carried out inside $M_{u(\tau+1)}$ whenever $\tau + 1 < \mathrm{lh}(u)$.

Suppose that $I^u_{\tau'}$ has been defined for $1 < \tau' < \tau$. Set $I' = \bigcup_{\tau'<\tau} k^u_{\tau',\tau}[I^u_{\tau'}]$. Then $I' \in M_{u(\tau)}$. Furthermore, I' has size κ^+ in $M_{u(\tau)}$ and is a filter on Q^u_τ, so $p_0 = \bigcup I' \in \mathbb{Q}^u_\tau$. Now the set of dense subsets of $\mathbb{Q}^u_\tau$ has cardinality at most $(2^{j_{u(\tau)}(\lambda)})^{M_{u(\tau}} < (\lambda^{++})^{M_{u(\tau+1)}}$, so the dense subsets of $\mathbb{Q}^u_\tau$ in $M_{u(\tau+1)}$ can be enumerated as $\langle\, D_\nu \mid \nu < \kappa^+\,\rangle$. Since $M_{u(\tau)}$ is closed under κ sequences, $\mathbb{Q}^u_\tau$ is κ-closed in V, so we can define p_ν by recursion on ν, taking $p_{\nu+1} < p_\nu$ to be the first condition such that $p_{\nu+1} \in D_\nu$ and, for limit ν, taking $p_\nu = \bigcup_{\nu'<\nu} p_{\nu'}$. Finally, I^u_τ is the filter generated by $\{p_\nu \mid \nu < \kappa^+\}$.

Proposition 2.5. *(1) If $1 \leq \tau' < \tau \leq \mathrm{lh}(u)$ then there are embeddings $\hat{k}^u_{\tau',\tau} \colon M[I_{u\restriction\tau'}] \to M[I_{u\restriction\tau}]$ extending $k^u_{\tau',\tau}$ such that if $\tau'' < \tau' < \tau$ then $\hat{k}^u_{\tau'',\tau} = \hat{k}^u_{\tau',\tau} \circ \hat{k}^u_{\tau'',\tau'}$.*

(2) If u is a measure sequence and $\tau' < \tau < \mathrm{lh}(u)$ then $I^u_{\tau'} = [\lambda w\ I^w_{g(w)}]_{u(\tau)}$ where $[g]_{u(\tau)} = \tau'$.

□

2.3. **Definition of the forcing notion $\mathbb{R}_w$.** For a measure sequence w, we write $S \in w$ if $S \in \bigcap_{1<\tau<\mathrm{lh}(w)} w(\tau)$.

Definition 2.6. If w is any measure sequence, then $\mathbb{P}_w$ is the set of tuples $s = (w, \lambda, f, S, g)$ such that

- $\lambda < \kappa(w)$.
- $f \in \mathrm{Col}(\lambda^{++}, \kappa(w))$.
- $S \in w$, and $(\forall u \in S)\ \mathrm{range}(f) \subset \kappa(u)$.
- g is a function with domain S such that $(\forall \tau < \mathrm{lh}(w))\ [g]_{w(\tau)} \in I^w_\tau$.

Definition 2.7. If w is a measure sequence, then $\mathbb{R}_w$ is the set of finite sequences $p = \langle\, p_i \mid i \leq k\,\rangle$ satisfying the following conditions:

- $p_i = (w_i, \lambda_i, f_i, S_i, g_i) \in \mathbb{P}_{w_i}$ for each $i \leq k$.
- $w_k = w$.
- If $i < k$ then $\lambda_{i+1} = \kappa(w_i)$.

Definition 2.8. If $s = (w, \lambda, f, S, g)$ and $s' = (w', \lambda', f', S', g')$ are in $\mathbb{P}_w$ then $s \leq^* s'$ if

- $w = w'$ and $\lambda = \lambda'$,
- $f \leq f'$ in $\mathrm{Col}(\lambda^+, \kappa(w))$ (i.e., $f \supseteq f'$),
- $S \subseteq S'$, and
- $(\forall u \in S)\, g(u) \supseteq g'(u)$.

If $p = \langle\, p_i \mid i \leq k\,\rangle$ and $p' = \langle\, p'_i \mid i \leq k'\,\rangle$ are in $\mathbb{R}_w$ then $p \leq^* p'$ if $k = k'$ and $p_i \leq^* p'_i$ for all $i \leq k$.

Proposition 2.9. *Suppose that* $s = (u, \lambda, f, S, g) \in \mathbb{P}_u$*.*

1. *The order* $(\mathbb{P}_u/s, \leq^*)$ *is closed under* $\leq^*$*-decreasing sequences of length* λ^+*.*
2. *The order* $(\mathbb{P}_u/s, \leq^*)$ *is closed under* $\leq^*$*-decreasing sequences of length less than* $\kappa(u)^+$ *provided that* f^s *is constant for* s *in the sequence.*
3. *If* $s', s'' \leq^* s$ *and* $f^{s'} = f^{s''}$ *then* s' *and* s'' *are compatible.* □

Definition 2.10. If $s = (w, \lambda, f, S, g) \in \mathbb{P}_w$ and $w' \in S$ then $\mathrm{Add}(s, w')$ is the condition $\langle s_0, s_1 \rangle \in \mathbb{R}_w$ defined by

- $s_0 = (w', \lambda, f, S \cap V_{\kappa(w')}, g \restriction V_{\kappa(w')})$, and
- $s_1 = (w, \kappa(w'), g(w'), S', g \restriction S')$

where $S' = \{v \in S \mid \kappa(v) > \sup(\mathrm{range}(g(w')))\}$. In the case that this does not yield a member of $\mathbb{R}_w$, then $\mathrm{Add}(s, w')$ is undefined.

If $p = \langle p_0, \dots, p_n \rangle \in \mathbb{R}_w$ and $u \in S^{p_i}$ for some $i \leq n$ then $\mathrm{Add}(p, u) \in \mathbb{R}_w$ is obtained by replacing p_i with the two members of $\mathrm{Add}(p_i, u)$. That is,

$$\mathrm{Add}(p, u) \restriction i = p \restriction i$$

$$\mathrm{Add}(p, u)_i = \mathrm{Add}(p_i, u)_0$$

$$\mathrm{Add}(p, u)_{i+1} = \mathrm{Add}(p_i, u)_1$$

$$\mathrm{Add}(p, u) \restriction [i + 2, n + 1] = p \restriction [i + 1, n].$$

The forcing order $\leq$ on $\mathbb{R}_w$ is the smallest transitive relation containing the direct order $\leq^*$ and all pairs of the form $(\mathrm{Add}(p, u), p)$.

2.4. **Basic properties of $\mathbb{R}_w$.** We now state the main properties of the forcing notion $\mathbb{R}_w$.

Lemma 2.11. *(The factorization lemma) Assume* $p = \langle p_0, \dots, p_n \rangle \in \mathbb{R}_w$ *with* $p_i = (w_i, \lambda_i, f_i, S_i, g_i)$ *and* $m < n$*. Set* $p^{\leq m} = \langle p_0, \dots, p_m \rangle$ *and* $p^{>m} = \langle p_{m+1}, \dots, p_n \rangle$*.*

1. $p^{\leq m} \in \mathbb{R}_{w_m}$, $p^{>m} \in \mathbb{R}_w$ *and there exists* $i : \mathbb{R}_w/p \to \mathbb{R}_{w_m}/p^{\leq m} \times \mathbb{R}_w/p^{>m}$ *which is an isomorphism with respect to both* $\leq^*$ *and* $\leq$*.*
2. *If* $m+1 < n$, $\mathrm{lh}(u^{p_{m+1}}) = 1$ *and* $S^{p_{m+1}} = \emptyset$ *then there exists* $i : \mathbb{R}_w/p \to \mathbb{R}_{w_m}/p^{\leq m} \times \mathrm{Col}(\kappa(w_m)^{++}, \kappa(w_{m+1})) \times \mathbb{R}_w/p^{>m+1}$ *which is an isomorphism with respect to both* $\leq^*$ *and* $\leq$*.* □

The next lemma can be proved as in [6]. The proof uses the critical observation of Proposition 2.9(3), which is the reason for invoking the generic filters $I^u_\tau \subset \mathbb{Q}^u_\tau$.

Lemma 2.12. $(\mathbb{R}_w, \leq, \leq^*)$ *satisfies the Prikry property.* □

Now suppose that $G \subseteq \mathbb{R}_w$ is generic over V and set

$$C = \{\kappa(u) \mid \exists p \in G, \exists i < \mathrm{lh}(p), p_i = (u, \lambda, f, S, g)\}.$$

By standard arguments, C is a club of κ. Let $\langle \kappa_\nu \mid \nu < \kappa \rangle$ be the increasing enumeration of the club C and for $\nu < \kappa$ let u_ν be the unique measure sequence u such that $\kappa(u) = \kappa_\nu$ and for some $p \in G$ and $i < \mathrm{lh}(p)$, $u = u^{p_i}$. Also let $\vec{F} = \langle F_\nu \mid \nu < \kappa \rangle$ be such that each F_ν is the $\mathrm{Col}(\kappa_\nu^{++}, \kappa_{\nu+1})$-generic function over V produced by G.

Lemma 2.13. *(a)* $V[G] = V[\vec{u}, \vec{F}]$.

(b) For every limit ordinal $\nu < \kappa$, $\langle \vec{u} \restriction \nu + 1, \vec{F} \restriction \nu + 1 \rangle$ *is* $\mathbb{R}_{u \restriction \nu}$*-generic over* V*, and* $\langle \vec{u} \restriction (\nu, \kappa), \vec{F} \restriction [\nu, \kappa) \rangle$ *is* $\mathbb{R}_w$*-generic over* $V[\vec{u} \restriction \nu, \vec{F} \restriction \nu]$.

(c) For every $\gamma < \kappa$ *and every* $A \subseteq \gamma$ *with* $A \in V[\vec{u}, \vec{F}]$*, we have* $A \in V[\vec{u} \restriction \nu, \vec{F} \restriction \nu + 1]$*, where* ν *is the least ordinal such that* $\gamma < \kappa_\nu$.

Proof. (a) It suffices to show that G is definable from $\vec{u}$ and $\vec{F}$. Let G' be the set of all conditions $p \in \mathbb{R}_w$ such that

- For all measure sequences $u \in V_\kappa$, if u appears in p, then $u = u_\xi$, for some $\xi < \kappa$,
- For all $\xi < \kappa$, there exists $q \leq p$ such that u_ξ appears in q,
- If $f \in V_\kappa$ appears in p, then $f \subset F_\xi$, for some $\xi < \kappa$,
- For all $\xi < \kappa$ and all $f \in \mathcal{P}(F_\xi) \cap \mathrm{Col}(\kappa_\xi^{++}, \kappa_{\xi+1})$, there exists $q \leq p$ such that f appears in q.

It is clear that $G' \in V[\vec{u}, \vec{F}]$. It is also easily seen that G' is a filter which includes G. It follows from the genericity of G that $G = G'$. So $G \in V[\vec{u}, \vec{F}]$, as required.

(b) Follows from (a) and the factorization lemma 2.11.

(c) First note that ν is not a limit ordinal, so assume $\nu = \xi + 1$ is a successor ordinal (if $\nu = 0$, then the proof is similar). Let $p \in G$ be such that p mentions both u_ξ and $u_{\xi+1}$, say $u_\xi = u^{p_m}$ and $u_{\xi+1} = u^{p_{m+1}}$. By the Factorization Lemma 2.11,

$$\mathbb{R}_w/p \simeq \mathbb{R}_{u_\xi}/p^{\leq m} \times \mathrm{Col}(\kappa_i^+, \kappa_{\xi+1}) \times \mathbb{R}_w/p^{>m+1}.$$

Let $\dot{A}$ be an $\mathbb{R}_w$-name for A such that $\Vdash_{\mathbb{R}_w} \dot{A} \subseteq \gamma$. Let $\dot{B}$ be an $\mathbb{R}_w/p^{>m+1}$-name for a subset of $\mathbb{R}_{u_\xi}/p^{\leq m} \times \mathrm{Col}(\kappa_\xi^+, \kappa_{\xi+1}) \times \gamma$ such that

$$\Vdash_{\mathbb{R}_w/p^{>m+1}} \forall \eta < \gamma, ((r, f, \eta) \in \dot{B} \iff (r, f) \Vdash_{\mathbb{R}_{u_\xi}/p^{\leq m} \times \mathrm{Col}(\kappa_i^+, \kappa_{\xi+1})} \eta \in \dot{A}).$$

Let $\langle y_\alpha : \alpha < \kappa_{\xi+1} \rangle$ be an enumeration of $\mathbb{R}_{u_\xi}/p^{\leq m} \times \mathrm{Col}(\kappa_\xi^+, \kappa_{\xi+1}) \times \gamma$. Define a $\leq^*$-decreasing sequence $\langle q_\alpha \mid \alpha < \kappa_{\xi+1} \rangle$ of conditions in $\mathbb{R}_w/p^{>m+1}$ such that for all α, q_α decides "$y_\alpha \in \dot{B}$". This is possible as $(\mathbb{R}_w/p^{>m+1}, \leq^*)$ is $\kappa_{\xi+1}^+$-closed and satisfies the Prikry property, Theorem 2.12. Let $q \leq^* q_\alpha$ for all $\alpha < \kappa_{\xi+1}$. Then q decides each "$y_\alpha \in \dot{B}$". It follows that $A \in V[\vec{u} \restriction \nu, \vec{F} \restriction \nu + 1]$ □

As $\mathrm{lh}(w) = \kappa^+$, it follows from Mitchell [8] (see also [1]) that

Lemma 2.14. *κ remains strongly inaccessible in $V[G]$.* □

Lemma 2.15. *(a) If $X \in \mathcal{P}(V_\kappa) \cap V$ then*

$$X \in w \iff \exists \eta < \kappa, \forall \nu \geq \eta, u_\nu \in X.$$

(b) If $X \in \mathcal{P}(\kappa) \cap V$ then one of $C \cap X$ or $C \setminus X$ is bounded in κ.

Proof. For clause (a), first assume $X \in w$ and let $p = p_d {}^\frown \langle w, \lambda, f, S, g \rangle \in \mathbb{R}_w$, where $p_d \in V_\kappa$ is the lower part of p. Then $q = p_d {}^\frown \langle w, \lambda, f, S \cap X, g \rangle \in \mathbb{R}_w$, q extends p and $q \Vdash_{\mathbb{R}_w}$ "some final segment of $\vec{u}$ is included in X". So the set of such q's is dense, and hence $\Vdash_{\mathbb{R}_w}$ "some final segment of $\vec{u}$ is included in X".

For the converse, assume that $X \notin w$ and let $0 < \tau < \kappa^+ = \mathrm{lh}(w)$ be such that $X \notin w(\beta)$. Then for any $\gamma < \kappa$, it is easily seen that the set

$$D_\gamma = \{p \in \mathbb{R}_w \mid p \text{ mentions some } u \notin X \text{ with } \kappa(u) > \gamma\}$$

is dense open, so X does not contain a final segment of w.

For clause (b), note that either $\kappa \in j(X)$ or $\kappa \in j(\kappa \setminus X)$. If $\kappa \in j(X)$, then $\{u \mid \kappa(u) \in X\} \in w$, so by (a), a final segment of C is included in X, hence $C \setminus X$ is bounded. Similarly, if $\kappa \in j(\kappa \setminus X)$, then $\{u \mid \kappa(u) \in \kappa \setminus X\} \in w$, so again by (a), a final segment of C is included in $\kappa \setminus X$, hence $C \cap X$ is bounded. □

2.5. **Completing the proof of Lemma 2.1.** In this subsection we complete the proof of Lemma 2.1 and hence of the Main Theorem 1.1. The forcing $\mathbb{R}$ of Lemma 2.1 will be $\mathbb{R}/p$ where $p \leq^* \mathbf{1}^{\mathbb{R}_w}$ will be determined in the proof of Lemma 2.1(3).

Proof of Lemma 2.1(1). It is clear that if $\gamma^{++} < \lambda \leq \gamma^*$, where $\gamma < \gamma^*$ are two successive points in C, then λ is collapsed in $V[G]$. We show that no other cardinals are collapsed. First, if w is any measure sequence then any two conditions $p = (w, \lambda, f, S, g)$ and $p' = (w, \lambda, f, S', g')$ in $\mathbb{P}_w$ are $\leq^*$-compatible, so $\mathbb{R}_w$ has the $\kappa(w)^+$-chain condition below any condition; thus $\mathbb{R}_w$ preserves all cardinals greater than $\kappa(w)$. Now assume $\lambda = \kappa_j \in C$, and assume by contradiction that λ^+ or λ^{++} is collapsed. Let $A \subseteq \lambda^+$ in $V[G]$ witness this. By 2.13(c) and the fact that $\mathrm{Col}(\kappa_j^{++}, \kappa_{j+1})$ is κ_j^+-closed, $A \in V[\vec{u} \restriction j, \vec{F} \restriction j]$, and so λ^+ or λ^{++} is collapsed after forcing with $\mathbb{R}_{u_j}$.n But $\mathbb{R}_{u_j}$ satisfies the λ^+-c.c. and we get a contradiction. So if $\lambda \in C$, then λ^+ and λ^{++} are preserved. It follows that the limit points of C are preserved as well. □

Proof of Lemma 2.1(2). This follows from 2.14 and the fact that $\mathbb{R}_w$ satisfies the κ^+-c.c. □

Proof of Lemma 2.1(3). Let $p \leq^* \mathbf{1}^{\mathbb{R}_w}$ be a condition in $\mathbb{R}_w$ such that $p \parallel_{\mathbb{R}_w} \sigma$ for every formula σ without parameters. Now suppose that $p \in G$ where $G \subset \mathbb{R}_w$ is generic, $\lambda \in C^G$, and $H \subseteq \mathrm{Col}(\sigma, \lambda)$ is $V[G]$-generic. Then $V[G][H] = V[G']$ where $G' \subseteq \mathbb{R}$ is generic and $p \in G'$. Hence

$$V[G][H] \models \sigma \iff p \Vdash_{\mathbb{R}} \sigma \iff V[G] \models \sigma.$$

□

Proof of Lemma 2.1(4). Let $\lambda = \kappa_j$ and $p = \langle p_0, p_1 \rangle \in G$ be such that u_j appears in p (note that this implies $p_0 = (u_j, \lambda, f, S, g)$ for some λ, f, S, g). By 2.11(b),

$$\mathbb{R}_w/p \simeq \mathbb{R}_{u_j}/p_0 \times \mathrm{Col}(\kappa_j^+, \kappa_{j+1}) \times \mathbb{R}_w/p_1. \quad (2)$$

So we have

$$
\begin{aligned}
(\alpha) \qquad & \mathbb{R}_w/p \times \mathrm{Col}(\omega, \kappa_j^+) \simeq \mathbb{R}_{u_j}/p_0 \times \mathrm{Col}(\omega, \kappa_j^+) \times \mathrm{Col}(\kappa_j^+, \kappa_{j+1}) \times \mathbb{R}_w/p_1 \\
(\beta) \qquad & \simeq \mathbb{R}_{u_j}/p_0 \times \mathrm{Col}(\omega, \kappa_{j+1}) \times \mathbb{R}_w/p_1 \\
(\gamma) \qquad & \simeq \mathbb{R}_{u_j}/p_0 \times \mathrm{Col}(\omega, \kappa_{j+1}) \times \mathbb{R}_w/p_1 \times \mathrm{Col}(\omega, \kappa_{j+1} \\
(\delta) \qquad & \simeq \mathbb{R}_w/p \times \mathrm{Col}(\omega, \kappa_{j+1}).
\end{aligned}
$$

Here (α) holds by (4.1), (β) holds as $\mathrm{Col}(\omega, \kappa_j^+) \times \mathrm{Col}(\kappa_j^+, \kappa_{j+1}) \simeq \mathrm{Col}(\omega, \kappa_{j+1})$, (γ) holds as $\mathrm{Col}(\omega, \kappa_{j+1}) \simeq \mathrm{Col}(\kappa_j^+, \kappa_{j+1}) \times \mathrm{Col}(\omega, \kappa_{j+1})$ and (δ) holds by 2. □

3. The lower bound

In this section we prove Theorem 1.2, showing that the hypothesis of Theorem 1.1 cannot be weakened. Our assumption is that for all sentences σ of set theory and all uncountable cardinals $\lambda < \kappa$,

$$
V \models \sigma \iff (\forall H \subseteq \mathrm{Col}(\omega, \lambda))\ (H \text{ is generic } \implies V[H] \models \sigma). \tag{3}
$$

By the homogeneity of the collapse forcing $\mathrm{Col}(\omega, \lambda)$, the truth of such sentences σ in $V[H]$ depends only on the cardinal λ and not on the choice of H. Hence we will write $V^{\mathrm{Col}(\omega,\lambda)}$ for a model $V[H]$, with H any generic subset of $\mathrm{Col}(\omega, \lambda)$.

We will frequently use the observation that equivalence (3) also holds for any sentence σ with a parameter a which is uniformly definable in the models $V^{\mathrm{Col}(\omega,\lambda)}$, since the use of a could be eliminated.

We can assume that every uncountable limit cardinal below κ is singular, as otherwise we could replace κ with the first regular limit cardinal. Furthermore, by Theorem 1.1 we can assume that there are no inner models with $\exists\kappa\, o(\kappa) = \kappa^{++} + 1$, and hence that the core model K exists.

3.1. **Indiscernibles for K.**

Proposition 3.1. *Every successor cardinal below κ is a successor cardinal in K.*[1]

[1]We thank Philip Welch for this observation and the fact (implied by our Lemma 3.5) that the predecessor is regular.

Proof. If λ is any singular limit cardinal below κ then the weak covering lemma implies that $\lambda^+ = (\lambda^+)^K$. Thus $V^{\mathrm{Col}(\omega,\lambda)} \models \exists\nu\ \omega_1 = (\nu^+)^K$. But by (3) this is then true in $V^{\mathrm{Col}(\omega,\eta)}$ for any $\eta < \kappa$, so η^+ is a successor in K for all $\eta < \kappa$. □

If $\omega \le \lambda < \kappa$ then we write λ^* for the K-predecessor of λ^+, and C for $\{\lambda^* \mid \omega \le \lambda < \kappa\}$. Note that if λ is a limit cardinal then, by our assumption that κ is the first inaccessible cardinal, λ is singular and therfore $\lambda^* = \lambda$. Thus C is closed and unbounded; in fact if V is the model constructed in Section 3 then this set C is the same as the set C of indiscernibles given by the Radin forcing.

In order to explore the indiscerniblity properties of C we consider formulas in the language of set theory with a unary predicate $\tilde{C}$ and a designated variable, usually λ. We write Γ for the set of such formulas (possibly with extra parameters). We will be concerned with the truth of these formulas in the structure $(K, \in, C \setminus \lambda + 1)$ for $\lambda \in C$, with the predicate $\tilde{C}$ interpreted as $C \setminus \lambda + 1$ and the designated variable as λ.

If $\lambda = \eta^*$ for a cardinal η of V then both $C \setminus \lambda + 1$ and λ are definable in $V^{\mathrm{Col}(\omega,\eta)}$, the former as the set of uncountable cardinals below the first inaccessible cardinal κ, and the latter as ω^*. Hence formula (3) immediately implies the following proposition:

Proposition 3.2. *If $\phi(\lambda)$ is any formula in Γ without parameters then either $(K, \in, C \setminus \lambda) \models \phi(\lambda)$ for all $\lambda \in C$ or $(K, \in, C \setminus \lambda) \models \neg\phi(\lambda)$ for all $\lambda \in C$.* □

The next two lemmas give a partial generalization of Proposition 3.2 to formulas with parameters from K.

Lemma 3.3. *Suppose that ϕ is a formula in Γ having a parameter $x \in K$. Then C is the union of a finite set of intervals I of C such that*

$$(4)\qquad (\forall \lambda, \lambda' \in I)\ \big((K, \in, C \setminus \lambda + 1) \models \phi(\lambda, x) \iff (K, \in, C \setminus \lambda' + 1) \models \phi(\lambda', x)\big).$$

Proof. Since K has a definable well order, we may assume that the parameter x is an ordinal α. Let $\tilde{\alpha}(\eta)$ be a term denoting, for $\eta \in C$, the least ordinal α for which the lemma is false of $\phi(\lambda, \alpha)$ in $(K, \in, C \setminus \eta + 1)$.

Now let $\phi'(\lambda)$ be the formula $\phi(\lambda, \tilde\alpha(\lambda))$. Then $\phi'(\lambda) \in \Gamma$ and $\phi'(\lambda)$ has no parameters, so by Proposition 3.2

$$\text{either} \quad (\forall\lambda \in C)\ (K, \in, C \setminus \lambda) \models \phi'(\lambda) \quad \text{or} \quad (\forall\lambda \in C)\ (K, \in, C \setminus \lambda) \models \neg\phi'(\lambda). \tag{5}$$

Now set $\alpha_0 = \tilde\alpha(\min(C))$. Since the lemma fails for $\phi(\lambda, \alpha_0)$ there is some $\eta \in \mathrm{Lim}(C)$ such that

$$\sup\{\lambda \in C \cap \eta \mid (K, \in, C \setminus \lambda) \models \phi(\lambda, \alpha_0)\} \\ = \sup\{\lambda \in C \cap \eta \mid (K, \in, C \setminus \lambda) \models \neg\phi(\lambda, \alpha_0)\} = \eta. \tag{6}$$

If η satisfies the formula (6) then $\tilde\alpha(\lambda) = \alpha_0$ for all $\lambda < \eta$, but this makes the formulas (5) and (6) contradictory. □

Lemma 3.4. *Suppose that $\phi(\lambda, \alpha)$ is a formula of Γ. Then κ is a finite union of intervals I such that for any $\lambda, \lambda' \in C \cap I$ such that $\lambda < \lambda'$,*

$$(\forall\alpha < \lambda)\ \big((K, \in, C \setminus \lambda) \models \phi(\lambda, \alpha) \iff (K, \in, C \setminus \lambda') \models \phi(\lambda', \alpha)\big). \tag{7}$$

Furthermore, if ϕ has no parameters other than α and λ then Formula (7) *holds with the single interval $I = \kappa$.*

Proof. First assume that ϕ has no additional parameters. By Lemma 3.3, for each $\alpha < \kappa$ there is a $\gamma(\alpha) \geq \alpha$ such that whenever $\lambda, \lambda' \in C \setminus \gamma(\alpha)$ we have $(K, \in, C \setminus \lambda) \models \phi(\alpha, \lambda)$ if and only if $(K, \in, C \setminus \lambda') \models \phi(\lambda', \alpha)$. Since κ is inaccessible in V there is closed and unbounded subset $C' \subseteq C$ of fixed points of the function $\gamma(\alpha)$. Then any $\lambda \in C'$ satisfies the following formula $\phi'(\lambda)$:

$$(\forall\alpha < \lambda)(\forall\lambda' \in C \setminus \lambda)\big((K, \in, C \setminus \lambda) \models \phi(\lambda, \alpha) \iff (K, \in, C \setminus \lambda') \models \phi(\lambda', \alpha)\big). \tag{8}$$

But this formula has no parameters other than λ, so it must be true of all members of C.

The case when ϕ does have other parameters, say ϕ is $\phi(\lambda, \alpha, \eta)$, is proved similarly to Lemma 3.3. Let $\tilde\eta(\lambda)$ be a name for the least ordinal η above λ for which the lemma fails and let $\phi'(\alpha, \lambda)$ be the formula $\phi(\lambda, \alpha, \tilde\eta(\lambda))$. Then ϕ' has no parameters other than λ and

α, hence for any $\alpha < \lambda < \lambda'$ with $\lambda, \lambda' \in C$ we have $(K, \in, C \setminus \lambda) \models \phi'(\lambda, \alpha)$ if and only if $(K, \in, C \setminus \lambda') \models \phi'(\lambda', \alpha)$.

On the other hand, if we set $\eta_0 = \tilde{\eta}(\min(C))$ then there is an increasing sequence $\langle \lambda_n \mid n < \omega \rangle$ of members of C such that for each $n < \omega$ there is $\alpha_n < \lambda_n$ such that $(K, \in, C \setminus \lambda_n) \models \phi(\lambda_n, \alpha_n, \eta_0)$ if and only if $(K, \in, C \setminus \lambda_{n+1}) \not\models \phi(\lambda_{n+1}, \alpha_n, \eta_0)$. Then $\bar{\eta}(\lambda_n) = \eta_0$ for all $n < \omega$, and this is a contradiction. □

If $\lambda \in \mathrm{Lim}(C)$ then we will write U_λ for $\{x \in \mathcal{P}^K(\lambda) \mid (C \cap \lambda) \setminus x \text{ is bounded in } \lambda\}$. Note that U_λ is not definable in $(K, \in, C \setminus \lambda)$, but U_η is definable there for any $\eta \in \mathrm{Lim}(C \setminus \lambda + 1) = \mathrm{Lim}(C) \setminus \lambda + 1$.

Lemma 3.5. *For each $\eta \in \mathrm{Lim}(C)$, the filter U_η is a normal measure in K.*

Proof. Lemma 3.3 implies that U_η is an ultrafilter, since for any set $x \in K$ all but boundedly many members of $C \cap \eta$ must be in the same interval for the formula $\lambda \in x$.

Lemma 3.4 implies that U_η is normal, since it implies that for any function $f\colon \eta \to \eta$ in K, the set of $\alpha < \lambda$ satisfying the formula $\alpha = f(\lambda)$ is constant for sufficiently large $\lambda < \eta$.

It remains to prove that $\mathrm{Ult}(K, U_\gamma)$ is well founded. This is immediate when $\mathrm{cf}(\gamma) > \omega$, since U_γ is countably complete. For the case of $\mathrm{cf}(\gamma) = \omega$ we use an argument due to Dodd and Jensen. Let $\bar{\gamma}(\lambda)$ be the term naming the least $\gamma \in \mathrm{Lim}(C) \setminus \lambda + 1$ such that $\mathrm{Ult}(K, U_\gamma)$ is not well founded, and for $n < \omega$ let $\bar{f}_n(\lambda)$ be a name for the function $f_n\colon \bar{\eta}(\lambda) \to \bar{\eta}(\lambda)$ such that, setting $\gamma_0 = \bar{\gamma}(\lambda)$ and $U = U_{\gamma_0}$,

- the ordinals of $\mathrm{Ult}(K, U)$ below $[f_n]_U$ are not well founded, and
- if $n > 0$ then $[f_n]U < [f_{n-1}]_U$.

Now if $\lambda \in C$ then $\bar{\gamma}(\lambda')$ is constant for $\lambda' \in [\lambda, \bar{\eta}(\lambda))$. Thus the functions $f_n = \bar{f}_n(\lambda')$ are identical for λ' in this interval, so for each $n \in \omega$ we have $f_{n+1}(\lambda') < f_n(\lambda')$ for all sufficiently large $\lambda' \in C \cap \bar{\gamma}(\lambda)$, but by Lemma 3.3 it follows that this holds for all $\lambda' \in [\lambda, \bar{\gamma}(\lambda))$. This is impossible, completing the proof that $\mathrm{Ult}(K, U_\gamma)$ is well founded for all $\gamma \in \mathrm{Lim}(C)$. □

3.2. **The measure sequence.** In order to extend this construction to obtain a measure sequence, we use canonical functions in K to define a sequence of closed unbounded subsets of C.

Definition 3.6. In K, define for $\lambda \leq \kappa$ and $\alpha < (\kappa^+)^K$:

- f^λ_α is the first function in K mapping λ onto α.
- $g^\lambda_\alpha(\lambda') = \operatorname{otp}(f^\lambda_\alpha[\lambda'])$.

Now define by recursion in V:

- $C^\lambda_0 = C$, and if $\alpha > 0$ then $C^\lambda_\alpha = \{\lambda' \in C \cap \lambda \mid (\forall \alpha' \in f^\lambda_\alpha[\lambda'])\ \lambda' \in \operatorname{Lim}(C^{\lambda'}_{\alpha'})\}$.
- If $\lambda \in \operatorname{Lim}(C^\lambda_\alpha)$ then $u^\lambda = \langle u^\lambda(\alpha) \mid \alpha < \operatorname{lh}(u^\lambda) \rangle$, where
  - $\operatorname{lh}(u^\lambda)$ is the least ordinal $1+\alpha$ where $\alpha \leq (\lambda^+)^K$ and C^λ_α is bounded in λ,
  - if $\lambda \in C \cup \{\kappa\}$ then $u^\lambda(0) = \lambda$, and
  - if $\lambda \in \operatorname{Lim}(C^\lambda_\alpha) \cup \{\kappa\}$ then $u^\lambda(1+\alpha)$ is the set of $x \in \mathcal{P}(V_\lambda) \cap K$ such that $u^{\lambda'} \restriction f^\lambda_{1+\alpha}(\lambda') \in x$ for all sufficiently large $\lambda' \in C^\lambda_\alpha$.

Proposition 3.7. *For all $\lambda < \kappa$,*

$$(\forall \lambda' \in (C \setminus \lambda \cup \{\kappa\})(\forall \alpha < \operatorname{lh}(u^{\lambda'}))\ \left(C^{\lambda'}_\alpha \setminus (\lambda+1) = (C^{\lambda'}_\alpha \setminus (\lambda+1))^{V^{\operatorname{Col}(\omega,\lambda)}}\right). \quad \square$$

Lemma 3.8. *For all $\lambda \in C \cup \{\kappa\}$ and all $\lambda < \operatorname{lh}(u^\lambda)$ the following holds for all but a boundedly many $\lambda' < \lambda$:*

$$(\forall \beta \in f^\lambda_\alpha[\lambda'])\ g^{\lambda'}_{g^\lambda_\beta(\lambda')} \restriction C = g^\lambda_\beta \restriction (C \cap \lambda'). \tag{9}$$

Proof. By standard arguments, carried out inside K, equation (9) holds for λ' in a subset of λ which is closed and unbounded in λ, and hence is in U_λ. $\square$

Corollary 3.9. *For all $\lambda \in C \cup \{\kappa\}$ and all $\lambda < \operatorname{lh}(u^\lambda)$, there is $\gamma < \lambda$ such that the following holds for all $\lambda' < \lambda$:*

$$(\forall \beta \in f^\lambda_\alpha[\lambda'])\ g^\lambda_\beta(\lambda') < \operatorname{lh}(u^{\lambda'}) \implies C^{\lambda'}_{g^\lambda_\beta(\lambda')} \setminus \gamma = C^\lambda_\beta \cap \lambda' \setminus \gamma \tag{10}$$

The following lemma will complete the proof of Theorem 1.2:

Lemma 3.10. *For all $\lambda \in C \cup \{\kappa\}$, u^λ is a measure sequence in K. In particular, u^κ is a measure sequence of length κ^+.*

Proof. The proof is similar to that of Lemma 3.3. We prove the lemma by induction on λ. Suppose that it is true for all $\lambda' < \lambda$.

First, we claim that $u^\lambda(\alpha)$ is an ultrafilter for $0 < \alpha < \mathrm{lh}(u^\lambda)$. If $x \in \mathcal{P}(V_\kappa) \cap K$ then Lemma 3.3, applied to the formula $\sigma(\lambda, x)$ asserting that " $u^{\lambda'}\restriction\alpha' \in x$" where $\lambda' = \min(C^\lambda_\alpha \setminus (\lambda+1)$ and $\alpha' = f^\lambda_\alpha(\lambda')$.

The well foundedness of $\mathrm{Ult}(K, u^\lambda)$ (and more generally its iteratability) can be proved as in the proof of Lemma 3.5. The coherence of the functions g^λ_α and sets C^λ_α ensure that u^λ is a measure sequence.

Finally, to see that $u^\lambda \in K$, consider the extender corresponding to the map j^{u^λ}. Core model theory implies that this extender is in K, and the measure sequence u^λ is constructible from it. □

4. Conclusion

Theorems 1.1 and 1.2 give an essential equiconsistency between the property which Hamkins and Löwe asked about and the existence of a κ^+-measure sequence on a cardinal κ; however there are two issues which lead to finer distinctions. The first of these issues is the fact that, by Tarski's theorem on the undefinability of truth, the conclusion of Theorem 1.1 and the hypothesis of Theorem 1.2 are not expressible in ZFC; the second issue is that our statement is stronger than that which Hampkins and Löwe actually asked, which omitted the inaccessible cardinal κ and required that $V^{\mathrm{Col}(\omega,\lambda)} \equiv V$ for *all cardinals* λ.

We consider the lower bound, Theorem 1.2, first as it is more straightforward. The undefinability of truth is, in this case, not an issue since the proof of the theorem only uses the assumption that $V^{\mathrm{Col}(\omega,\lambda)} \models \sigma \iff V \models \sigma$ for finitely many sentences σ. For the second issue, we have $\kappa = \mathrm{On}$ so κ^+ is not available in ZFC. With Gödel-Bernays set theory we can state and prove that the Hamkins-Löwe property implies that for every well ordering R of On there is a measure sequence of length $\mathrm{otp}(R)$. The proof takes "well ordering of R" to mean that R satisfies a recursion axiom, and represents the the measures by the closed unbounded sets C_α which generate them. Obtaining a measure sequence of length On^+ requires that On^+ exists and is well ordered, that is, that the superclass of well orderings of On is well founded under embedding as an initial segment. This is second order and thus requires a fragment of Morse-Kelly set theory.

The proof of the upper bound, Theorem 1.1, requires the existence of a set in $u(1)$ which is homogeneous for each of the countably many formulas $\sigma(\lambda)$. For this, it would be sufficient to have Σ^1_1 comprehension for subsets of κ over a model of Gödel-Bernays set theory.

The consistency of the Hamkins-Löwe property follows from Theorem 1.1, since that property only requires that $V_\kappa[G] \models V \equiv V^{\mathrm{Col}(\omega,\lambda)}$. However, the hypothesis is stronger than necessary for this conclusion. The proof could be carried out in a model in which On has a measure sequence of length On^+, together with enough strength above On to make the assumption of a measure sequence meaningful and to carry out the proof. We have not attempted to the minimum strength needed for this, but it appears to be more than can be obtained as described above by adapting Theorem 1.2. This suggests that a small gap remains.

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School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran-Iran.

E-mail address: golshani.m@gmail.com

Department of Mathematics, University of Florida, 358b Little Hall, PO Box 118105, Gainesville, FL 32611-8105, USA.

E-mail address: wjm@ufl.edu