ON CUTS IN ULTRAPRODUCTS OF LINEAR ORDERS II

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Abstract. We continue our study of the class $\mathscr{C}(D)$, where D is a uniform ultrafilter on a cardinal κ and $\mathscr{C}(D)$ is the class of all pairs (θ_1, θ_2), where (θ_1, θ_2) is the cofinality of a cut in J^κ/D and J is some $(\theta_1+\theta_2)^+$-saturated dense linear order. We give a combinatorial characterization of the class $\mathscr{C}(D)$. We also show that if $(\theta_1, \theta_2) \in \mathscr{C}(D)$ and D is $\aleph_1$-complete or $\theta_1 + \theta_2 > 2^\kappa$, then $\theta_1 = \theta_2$.

1. Introduction

Assume κ is an infinite cardinal and D is an ultrafilter on κ. Recall that $\mathscr{C}(D)$ is defined to be the class of all pairs (θ_1, θ_2), where (θ_1, θ_2) is the cofinality of a cut in J^κ/D and J is some (equivalently any) $(\theta_1+\theta_2)^+$-saturated dense linear order. Also $\mathscr{C}_{>\lambda}(D)$ is defined to be the class of all pairs $(\theta_1, \theta_2) \in \mathscr{C}(D)$, such that $\theta_1+\theta_2 > \lambda$. The classes $\mathscr{C}_{\geq\lambda}(D), \mathscr{C}_{<\lambda}(D)$ and $\mathscr{C}_{\leq\lambda}(D)$ are defined similarly.

The works [2], [3] and [4] of Malliaris and Shelah have started the study of this class for the case $\theta_1+\theta_2 \leq 2^\kappa$ and [1] started the study of the case $\theta_1+\theta_2 > 2^\kappa$. As it was observed in [1], the study of the class $\mathscr{C}_{>2^\kappa}(D)$ is very different from the case $\mathscr{C}_{\leq 2^\kappa}(D)$, and to prove results about it, usually some extra set theoretic assumptions are needed. In this paper we continue [1] and prove more results related to the class $\mathscr{C}(D)$.

In the first part of the paper (Sections 2 and 3) we give a combinatorial characterization of $\mathscr{C}(D)$. Using notions defined in section 2, we can state our first main theorem as follows.

Theorem 1.1. *Assume D is an ultrafilter on κ and $\lambda_1, \lambda_2 > \kappa$ are regular cardinals. The following are equivalent:*

(a) There is $\bar{a} \in \mathcal{S}_c$ which is not c-solvable, where $c = \langle \kappa, D, \lambda_1, \lambda_2 \rangle$.

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(*b*) $(\lambda_1, \lambda_2) \in \mathscr{C}(D)$.

In the second part of the paper (Sections 4 and 5) we study the existence of non-symmetric pairs (i.e., pairs (λ_1, λ_2) with $\lambda_1 \neq \lambda_2$) in $\mathscr{C}(D)$. By [5], we can find a regular ultrafilter D on κ such that

$$\mathscr{C}(D) \supseteq \{(\lambda_1, \lambda_2) : \aleph_0 < \lambda_1 < \lambda_2 \leq 2^\kappa, \lambda_1, \lambda_2 \text{ regular}\}.$$

In particular, $\mathscr{C}(D)$ contains non-symmetric pairs. On the other hand, results of [1] show that if $(\lambda_1, \lambda_2) \in \mathscr{C}_{>2^\kappa}(D)$, then we must have $\lambda_1^\kappa = \lambda_2^\kappa$, in particular if SCH, the singular cardinals hypothesis, holds, then $\lambda_1 = \lambda_2$, and so $\mathscr{C}_{>2^\kappa}(D)$ just contains symmetric pairs. We then prove the following theorem (in ZFC):

Theorem 1.2. (*a*) *Assume D is a uniform $\aleph_1$-complete ultrafilter on κ and $(\lambda_1, \lambda_2) \in \mathscr{C}(D)$. Then $\lambda_1 = \lambda_2$.*

(*b*) *Assume D is a uniform ultrafilter on κ and $(\lambda_1, \lambda_2) \in \mathscr{C}_{>2^\kappa}(D)$. Then $\lambda_1 = \lambda_2$.*

The theorem shows some restrictions on the pairs (λ_1, λ_2) that $\mathscr{C}(D)$ can have, in particular, it shows that in the result of [5] stated above, we can never take the ultrafilter D to be $\aleph_1$-complete and that $\mathscr{C}_{>2^\kappa}(D)$ can not have non-symmetric pairs.

The paper is organized as follows. In section 2 we give the required definitions, which lead us to the notion of c-solvability and in section 3 we complete the proof of Theorem 1.1. In section 4 we prove part (*a*) of Theorem 1.2 and in section 5 we complete the proof of part (*b*) of Theorem 1.2. We may note that parts one (Sections 2 and 3) and two (Sections 4 and 5) can be read independently of each other.

2. On the notion of c-solvability

In this section we give the required definitions which are used in Theorem 1.1.

Definition 2.1. (a) *Let $\mathcal{C}$ be the class of tuples $c = \langle \kappa_c, D_c, \lambda_{c,1}, \lambda_{c,2} \rangle$ where*

(a-1) *$\lambda_{c,1}, \lambda_{c,2}$ are regular cardinals $> \kappa_c$,*

(a-2) *D_c is a uniform ultrafilter on κ_c.*

Also let $\lambda_c = 2^{<\lambda_{c,1}} + 2^{<\lambda_{c,2}}$ and $\lambda_{c,0} = \min\{\lambda_{c,1}, \lambda_{c,2}\}$.

(b) *For* $c \in \mathcal{C}$ *let* $N_c = N_{c,1} + N_{c,2}$ *be a linear order of size* $\leq \lambda_c$ *in such a way that* $N_{c,1}$ *has cofinality* $\lambda_{c,1}$, $N_{c,2}$ *has co-initiality* $\lambda_{c,2}$ *and both* $N_{c,1}, N_{c,2}$ *are* $\lambda_{c,0}$*-saturated dense linear orders* [1].

(c) *For* $c \in \mathcal{C}$ *let* $\mathcal{S}_c$ *be the set of all sequences* $\bar{a} = \langle a_{s,t} : s, t \in N_c \rangle$ *such that*

(c-1) *Each* $a_{s,t}$ *is a subset of* κ_c,

(c-2) $a_{s,s} = \emptyset$,

(c-3) *For* $s \neq t, a_{s,t} = \kappa \setminus a_{t,s}$,

(c-4) $s <_{N_c} t \Rightarrow a_{s,t} \in D_c$,

(c-5) *If* $s_1 <_{N_c} s_2 <_{N_c} s_3$, *then*

$$(a_{s_1,s_3} \supseteq a_{s_1,s_2} \cap a_{s_2,s_3}) \ \& \ (a_{s_3,s_1} \supseteq a_{s_3,s_2} \cap a_{s_2,s_1}).$$

(d) *For* $c \in \mathcal{C}$ *let* $N_c^+ = N_{c,1} + N_0 + N_{c,2}$, *where* N_0 *is a singleton, say* $N_0 = \{s_*\}$.

We now define the notion of c-solvability.

Definition 2.2. *Let* $c \in \mathcal{C}$. *We say* $\bar{a} \in \mathcal{S}_c$ *is c-solvable, if there exists a sequence* $\bar{b} = \langle b_s : s \in N_c \rangle$, *such that the sequence* $\bar{a}^1 = \bar{a} * \bar{b}$ *satisfies clauses* (c-1)-(c-5) *above, where the sequence* $\bar{a}^1 = \langle a^1_{s,t} : s, t \in N_c^+ \rangle$ *is defined as follows:*

(1) *If* $s, t \in N_c$, *then* $a^1_{s,t} = a_{s,t}$,

(2) *For* $s \in N_{c,1}, a^1_{s,s_*} = b_s$ *and* $a^1_{s_*,s} = \kappa_c \setminus b_s$,

(3) *For* $s \in N_{c,2}, a^1_{s_*,s} = b_s$ *and* $a^1_{s,s_*} = \kappa_c \setminus b_s$,

(4) $a^1_{s_*,s_*} = \emptyset$.

Then $\bar{b}$ *is called a c-solution for* $\bar{a}$.

3. A combinatorial characterization of $\mathscr{C}(D)$

In this section we give a proof of Theorem 1.1.

Lemma 3.1. *Assume* $c \in \mathcal{C}$ *and* $\bar{a} \in \mathcal{S}_c$. *Then*

(a) *There are* $M, \bar{f}$ *such that*

(a-1) M *is a* $(\lambda_{c,1} + \lambda_{c,2})^+$*-saturated dense linear order,*

[1] N_c is some fixed linear order which we choose in advance. We may assume global choice and let N_c be the least such order.

(a-2) $\bar{f} = \langle f_s : s \in N_c \rangle$,

(a-3) *Each* $f_s \in {}^{\kappa_c}M$,

(a-4) *If* $s <_{N_c} t$, *then* $a_{s,t} = \{i < \kappa_c : f_s(i) <_M f_t(i)\}$,

(a-4) $\langle \mathrm{range}(f_s) : s \in N_c \rangle$ *is a sequence of pairwise disjoint sets.*

(b) *If* $M, \bar{f}$ *are as in (a), then*

(b-1) $\langle f_s/D_c : s \in N_c \rangle$ *is an increasing sequence in* ${}^{\kappa_c}M/D_c$,

(b-2) $\bar{a}$ *is* c*-solvable iff* ${}^{\kappa_c}M/D_c$ *realizes the type*

$$q(x) = \{f_s/D_c < x < f_t/D_c : s \in N_{c,1} \text{ and } t \in N_{c,2}\}.$$

Proof. (a) Let $A = \{(i, s) : i < \kappa_c, s \in N_c\}$, and define the order $<_A$ on A by

$$(i_1, s_1) <_A (i_2, s_2) \iff (i_1 < i_2) \text{ or } (i_1 = i_2 \in a_{s_1,s_2}).$$

Also let $\leq_A$ be defined on A in the natural way from $<_A$, so

$$(i_1, s_1) \leq_A (i_2, s_2) \iff (i_1, s_1) = (i_2, s_2) \text{ or } (i_1, s_1) <_A (i_2, s_2).$$

It is easily seen that $\leq_A$ is a linear order on A. Now let M be a $(\lambda_{c,1} + \lambda_{c,2})^+$-saturated dense linear order which contains $(A, <_A)$ as a sub-order. Also let $\bar{f} = \langle f_s : s \in N_c \rangle$, where for $s \in N_c$ $f_s \in {}^{\kappa_c}M$ is defined by $f_s(i) = (i, s)$. It is clear that M, and $\bar{f}$ satisfy clauses (a-1)-(a-3). For (a-4), assume $s <_{N_c} t$ are given. Then

$$a_{s,t} = \{i < \kappa_c : i \in a_{s,t}\} = \{i < \kappa_c : (i, s) <_A (i, t)\} = \{i < \kappa_c : f_s(i) <_M f_t(i)\}.$$

Finally note that for $s \neq t$ in N_c,

$$\mathrm{range}(f_s) \cap \mathrm{range}(f_t) = \{(i, s) : i < \kappa_c\} \cap \{(i, t) : i < \kappa_c\} = \emptyset.$$

So M and $\bar{f}$ are as required.

(b) (b-1) follows from (a-4) and the fact that for $s <_{N_c} t, a_{s,t} \in D_c$. Let's prove (b-2). First assume that $\bar{a}$ is c-solvable and let $\bar{b}$ be a solution for $\bar{a}$. For each $i < \kappa_c$ let $p_i(x)$ be the following type over M:

$$p_i(x) = \{f_s(i) <_M x : s \in N_{c,1} \text{ and } i \in b_s\} \cup \{x <_M f_t(i) : t \in N_{c,2} \text{ and } i \in \kappa_c \setminus b_t\}.$$

Claim 3.2. *For each* $i < \kappa_c$, *the type* $p_i(x)$ *is finitely satisfiable in* M.

Proof. Let $s_0 <_{N_{c,1}} \cdots <_{N_{c,1}} s_{n-1}$ be in $N_{c,1}$ and $t_{m-1} <_{N_{c,2}} < \cdots <_{N_{c,2}} t_0$ be in $N_{c,2}$. Also suppose that $i \in \bigcap_{k<n} b_{s_k} \cap \bigcap_{l<m} (\kappa_c \setminus b_{t_l})$. Then for $k < n$ and $l < m$ we have

$$a_{s_k,t_l} \supseteq a^1_{s_k,s_*} \cap a^1_{s_*,t_l} = b_{s_k} \cap (\kappa_c \setminus b_{t_l}),$$

and so $i \in a_{s_k,t_l}$, which implies $f_{s_k}(i) < f_{t_l}(i)$. Take $x \in M$ so that

$$\forall k < n, \forall l < m,\ f_{s_k}(i) < x < f_{t_l}(i),$$

which exists as M is dense. It follows that $p_i(x)$ is finitely satisfiable in M. □

It follows that there exists $f \in {}^{\kappa_c}M$ such that for each $i < \kappa_c, f(i)$ realizes the type $p_i(x)$ over M. Then f/D_c realizes $q(x)$ over ${}^{\kappa_c}M/D_c$.

Conversely assume that $f \in {}^{\kappa_c}M$ is such that f/D_c realizes the type $q(x)$ over ${}^{\kappa_c}M/D_c$.

Claim 3.3. *We can assume that* range(f) *is disjoint from* A.

Proof. As $\langle \text{range}(f_s) : s \in N_c \rangle$ is a sequence of pairwise disjoint sets and $\lambda_{c,1}, \lambda_{c,2} > \kappa_c$ are regular, there are $s_1 \in N_{c,1}$ and $s_2 \in N_{c,2}$ such that $s_1 <_{N_c} s <_{N_c} s_2$ implies $\text{range}(f_s) \cap \text{range}(f) = \emptyset$. As M is a $(\lambda_{c,1} + \lambda_{c,2})^+$-saturated dense linear order, there is f' such that

- $f' \in {}^{\kappa_c}M$,
- $\text{range}(f') \cap A = \emptyset$,
- If $s_1 <_{N_c} s <_{N_c} s_2$ and $i < \kappa_c$, then $f_s(i) <_{N_c} f'(i) \Rightarrow f_s(i) <_{N_c} f(i)$ and $f'(i) <_{N_c} f_s(i) \Rightarrow f(i) <_{N_c} f_s(i)$.

So we can replace f by f' and f' satisfies the requirements on f; i.e., f'/D_c realizes $q(x)$ over ${}^{\kappa_c}M/D_c$ and further $\text{range}(f') \cap A = \emptyset$. □

Now define $\bar{b} = \langle b_s : s \in N_c \rangle$ by

$$b_s = \begin{cases} \{i < \kappa_c : f_s(i) <_{N_c} f(i)\} & \text{if } s \in N_{c,1}, \\ \{i < \kappa_c : f(i) <_{N_c} f_s(i)\} & \text{if } s \in N_{c,2}. \end{cases}$$

Claim 3.4. $\bar{b}$ *is a c-solution for* $\bar{a}$.

Proof. We show that conditions (c-1)-(c-5) of Definition 2.1 are satisfied by $\bar{a}^1 = \bar{a} * \bar{b}$ (see Definition 2.2). (c-1) and (c-2) are trivial and (c-3) follows from the fact that $\forall i < \kappa_c, f_s(i) \neq f(i)$ (as $\mathrm{range}(f) \cap A = \emptyset$).

For (c-4), suppose that $s <_{N_c^+} t$. If both s, t are in N_c, then we are done. So suppose otherwise. There are two cases to consider.

- If $s = s_*$, then $t \in N_{c,2}$ and as f/D_c realizes $q(x)$, we have $f/D_c < f_t/D_c$, which implies $a^1_{s_*,t} = b_t = \{i < \kappa_c : f(i) <_{N_c} f_t(i)\} \in D_c$.
- If $t = s_*$, then $s \in N_{c,1}$ and as f/D_c realizes $q(x)$, we have $f_s <_{D_c} f$, which implies $a^1_{s,s_*} = b_s = \{i < \kappa_c : f_s(i) <_{N_c} f(i)\} \in D_c$.

For (c-5), assume $s_1 <_{N_c^+} s_2 <_{N_c^+} s_3$ are in N_c^+. If all s_1, s_2 and s_3 are in N_c, then we are done. So assume otherwise. There are three cases to be considered:

- If $s_1 = s_*$, then $s_2, s_3 \in N_{c,2}$, and we have

$$\begin{aligned} a^1_{s_*,s_2} \cap a^1_{s_2,s_3} &= b_{s_2} \cap a_{s_2,s_3} \\ &= \{i < \kappa_c : (\ f(i) <_{N_c} f_{s_2}(i)\ ) \wedge (\ f_{s_2}(i) <_{N_c} f_{s_3}(i)\ )\} \\ &\subseteq \{i < \kappa_c : f(i) <_{N_c} f_{s_3}(i)\} \\ &= b_{s_3} \\ &= a^1_{s_*,s_3}. \end{aligned}$$

  Similarly,

$$\begin{aligned} a^1_{s_3,s_2} \cap a^1_{s_2,s_*} &= a_{s_3,s_2} \cap (\kappa_c \setminus b_{s_2}) \\ &= \{i < \kappa_c : (\ f_{s_2}(i) \geq_{N_c} f_{s_3}(i)\ ) \wedge (\ f(i) >_{N_c} f_{s_2}(i)\ )\} \\ &\subseteq \{i < \kappa_c : f(i) >_{N_c} f_{s_3}(i)\} \\ &= \kappa_c \setminus b_{s_3} \\ &= a^1_{s_3,s_*}. \end{aligned}$$

- If $s_2 = s_*$, then $s_1 \in N_{c,1}$, $s_3 \in N_{c,2}$ and we have

$$\begin{aligned} a^1_{s_1,s_*} \cap a^1_{s_*,s_3} &= b_{s_1} \cap b_{s_3} \\ &= \{i < \kappa_c : (\ f_{s_1}(i) <_{N_c} f(i)\ ) \wedge (\ f(i) <_{N_c} f_{s_3}(i)\ )\} \\ &\subseteq \{i < \kappa_c : f_{s_1}(i) <_{N_c} f_{s_3}(i)\} \\ &= a_{s_1,s_3}. \\ &= a^1_{s_1,s_3}. \end{aligned}$$

  Also,

$$\begin{aligned} a^1_{s_3,s_*} \cap a^1_{s_*,s_1} &= (\kappa_c \setminus b_{s_3}) \cap (\kappa_c \setminus b_{s_1}) \\ &= \{i < \kappa_c : (\ f_{s_3}(i) <_{N_c} f(i)\ ) \wedge (\ f(i) <_{N_c} f_{s_1}(i)\ )\} \\ &\subseteq \{i < \kappa_c : f_{s_3}(i) \leq_{N_c} f_{s_1}(i)\} \\ &= a_{s_3,s_1} \\ &= a^1_{s_3,s_1}. \end{aligned}$$

- If $s_3 = s_*$, then $s_1, s_2 \in N_{c,1}$ and we have

$$\begin{aligned} a^1_{s_1,s_2} \cap a^1_{s_2,s_*} &= a_{s_1,s_2} \cap b_{s_2} \\ &= \{i < \kappa_c : (\ f_{s_1}(i) <_{N_c} f_{s_2}(i)\ ) \wedge (\ f_{s_2}(i) <_{N_c} f(i)\ )\} \\ &\subseteq \{i < \kappa_c : f_{s_1}(i) <_{N_c} f(i)\} \\ &= b_{s_1}. \\ &= a^1_{s_1,s_*}. \end{aligned}$$

  Similarly, we have

$$\begin{aligned} a^1_{s_*,s_2} \cap a^1_{s_2,s_1} &= (\kappa_c \setminus b_{s_2}) \cap a_{s_2,s_1} \\ &= \{i < \kappa_c : (\ f_{s_2}(i) >_{N_c} f(i)\ ) \wedge (\ f_{s_1}(i) \geq_{N_c} f_{s_2}(i)\ )\} \\ &\subseteq \{i < \kappa_c : f_{s_1}(i) >_{N_c} f(i)\} \\ &= \kappa_c \setminus b_{s_1} \\ &= a^1_{s_*,s_1}. \end{aligned}$$

Hence, $\bar{b}$ is a c-solution for $\bar{a}$, as required. □

The lemma follows. □

Given $c \in \mathcal{C}$, the next lemma gives a characterization, in terms of c-solvability, of when $(\lambda_{c,1}, \lambda_{c,2})$ is in $\mathscr{C}(D_c)$, which also completes the proof of Theorem 1.1.

Lemma 3.5. *Assume $c \in \mathcal{C}$ and M is a λ_c^+-saturated dense linear order. The following are equivalent:*

(*a*) *There is $\bar{a} \in \mathcal{S}_c$ which is not c-solvable.*

(*b*) *$(\lambda_{c,1}, \lambda_{c,2}) \in \mathscr{C}(D_c)$.*

Proof. First assume there exists $\bar{a} \in \mathcal{S}_c$ which is not c-solvable. By Lemma 3.1, there are $M, \bar{f}$ which satisfy clauses (1-a)-(1-e) of that lemma. But then as $\bar{a}$ is not c-solvable, by

Lemma 2.3(b-2), the type

$$q(x) = \{f_s/D_c < x < f_t/D_c : s \in N_{c,1} \text{ and } t \in N_{c,2}\}.$$

is not realized. It follows that $(\lambda_{c,1}, \lambda_{c,2}) \in \mathscr{C}(D_c)$.

Conversely assume that M and $\bar{f} = \langle f_s : s \in N_c\rangle$ witness $(\lambda_{c,1}, \lambda_{c,2}) \in \mathscr{C}(D_c)$. Let $A = \bigcup\{\text{range}(f_s) : s \in N_c\}$, and let $\langle I_d : d \in A\rangle$ be a sequence of pairwise disjoint intervals of M such that $d \in I_d$ [2]. For $s \in N_c$, let $f'_s \in {}^{\kappa_c}M$ be such that $f'_s(i) \in I_{f_s(i)}$ and $\langle f'_s(i) : s \in N_c, i < \kappa\rangle$ is with no repetitions. Define the sequence $\bar{a} = \langle a_{s,t} : s, t \in N_c\rangle$, such that for $s <_{N_c} t$, $a_{s,t} = \{i < \kappa : f_s(i) < f_t(i)\}$ and $a_{t,s} = \kappa_c \setminus a_{s,t}$. Also set $a_{s,s} = \emptyset$. It is evident that $\bar{a} \in \mathcal{S}_c$.

Claim 3.6. *$\bar{a}$ is not c-solvable.*

Proof. Assume not. Then by Lemma 3.1(b-2), the type

$$q(x) = \{f_s/D_c < x < f_t/D_c : s \in N_{c,1} \text{ and } t \in N_{c,2}\}$$

is realized in ${}^{\kappa_c}M/D_c$, which contradicts the choice of $M, \bar{f}$. □

The Lemma follows. □

4. For $\aleph_1$-complete ultrafilter, $\mathscr{C}(D)$ contains no non-symmetric pairs

In this section we prove part (a) of Theorem 1.2. In fact we will prove something stronger, that is of interest in its own sake.

Definition 4.1. *Assume D is an ultrafilter on κ, $\langle I_i : i < \kappa\rangle$ is a sequence of linear orders and $I = \prod_{i<\kappa} I_i/D$.*

- (a) *a subset K of I is called internal if there are subsets $K_i \subseteq I_i$ such that $K = \prod_{i<\kappa} K_i/D$.*
- (b) *The cut (J^1, J^2) of I is called internal, if there are cuts (J^1_i, J^2_i) of I_i, $i < \kappa$, such that $J^l = \prod_{i<\kappa} J^l_i/D$ $(l = 1, 2)$.*

[2] The existence of the sequence $\langle I_d : d \in A\rangle$ follows from the fact that $|A| \leq \kappa_c \cdot |N_c| \leq \lambda_c$ and M is λ_c^+-saturated.

Remark 4.2. *Assume J is an initial segment of I which is internal and suppose that $(J, I \setminus J)$ is a cut of I. Then $(J, I \setminus J)$ is in fact an internal cut of I. Similarly, if J is an end segment of I which is internal and if $(I \setminus J, J)$ is a cut of I, then $(I \setminus J, J)$ is an internal cut of I.*

Theorem 4.3. *Assume D is a uniform $\aleph_1$-complete ultrafilter on κ, $\langle I_i : i < \kappa \rangle$ is a sequence of non-empty linear orders and $I = \prod_{i<\kappa} I_i/D$. Also assume (J^1, J^2) is a cut of I of cofinality (θ_1, θ_2), where $\theta_1 \neq \theta_2$. Then the cut (J^1, J^2) is internal.*

Before giving the proof of Theorem 4.3, let us show that it implies Theorem 1.2(a).

Proof of Theorem 1.2(a) from Theorem 4.3. Suppose D is an $\aleph_1$-complete ultrafilter on κ, J is a $(\lambda_1+\lambda_2)^+$-saturated dense linear order and (J^1, J^2) is a cut of J^κ/D of cofinality (λ_1, λ_2). Towards contradiction assume that $\lambda_1 \neq \lambda_2$. It follows from Theorem 4.3 that the cut (J^1, J^2) is internal, and so that there are cuts (J^1_i, J^2_i) of J, $i < \kappa$, such that $J^l = \prod_{i<\kappa} J^l_i/D$ (for $l = 1, 2$). Let $(\lambda^1_i, \lambda^2_i) = \mathrm{cf}(J^1_i, J^2_i)$. It follows that $\lambda_l = \prod_{i<\kappa} \lambda^l_i/D, l = 1, 2$.

By the choice of J, for every $i < \kappa$, either $\lambda^1_i \geq (\lambda_1 + \lambda_2)^+$ or $\lambda^2_i \geq (\lambda_1 + \lambda_2)^+$, hence for some $l \in \{1, 2\}$, we have

$$A = \{i < \kappa : \lambda^l_i \geq (\lambda_1 + \lambda_2)^+\} \in D.$$

It follows that $\lambda_l = \prod_{i<\kappa} \lambda^l_i/D \geq (\lambda_1 + \lambda_2)^+$, which is a contradiction. □

We are now ready to complete the proof of Theorem 4.3.

Proof. We can assume that θ_1, θ_2 are infinite. Let $<^1_i = <_{I_i}$ $(i < \kappa)$ and $<_1 = <_I$. Let $<^2_i$ be a well-ordering of I_i with a last element and let $<_2$ be such that $(I, <_2) = \prod_{i<\kappa}(I_i, <^2_i)/D$. Then $<_2$ is a linear ordering of I with a last element and since D is $\aleph_1$-complete, it is well-founded, so $<_2$ is in fact a well-ordering of I with a last element.

As $(J^1, <_1)$ has cofinality θ_1, we can find $f_\alpha \in \prod_{i<\kappa} I_i$, for $\alpha < \theta_1$, such that

(1) $\forall \alpha < \theta_1, f_\alpha/D \in J^1$,

(2) $\langle f_\alpha/D : \alpha < \theta_1 \rangle$ is $<_1$-increasing,

(3) $\langle f_\alpha/D : \alpha < \theta_1 \rangle$ is a $<_1$-cofinal subset of J^1.

Let $B = \{t \in I : \{s \in J^1 : s <_2 t\}$ is $<_1$-unbounded in $J^1\}$. As θ_1 is infinite, the $<_2$-last element of I belongs to B, which implies $B \neq \emptyset$ and hence B has a $<_2$-minimal element; call it t_*. Let $g_* \in \prod_{i<\kappa} I_i$ be such that $t_* = g_*/D$.

Note that for each $\alpha < \theta_1$ there are $s \in J^1$ and $\beta > \alpha$ such that $s <_2 g_*/D$ and $f_\alpha/D <_1 s <_1 f_\beta/D$, so we can assume that for all $\alpha < \theta_1$, $f_\alpha/D <_2 g_*/D$. This implies

$$\bigwedge_{\alpha<\theta_1} [\{i < \kappa : f_\alpha(i) <^2_i g_*(i)\} \in D].$$

Also note that

$$\{i < \kappa : g_*(i) \text{ is } <^1_i \text{-minimal or } <^1_i \text{-maximal}\} \notin D,$$

so, without loss of generality, it is empty. Hence, without loss of generality

$$\bigwedge_{\alpha<\theta_1} \bigwedge_{i<\kappa} [f_\alpha(i) <^2_i g_*(i) \text{ and } f_\alpha(i) \text{ is not } <^1_i \text{-minimal}].$$

Let $f_{\theta_1} = g_*$ and for $\alpha \leq \theta_1$ set $K_\alpha = \{s \in I : s <_2 f_\alpha/D\}$. Thus K_α is a $<_2$-initial segment of I.

Claim 4.4. *$K_\alpha \cap J^1$ is $<_1$-bounded in J^1.*

Proof. As $f_\alpha/D <_2 t_*$, it follows from our choice of t_* that K_α is $<_1$ bounded in J^1. □

Claim 4.5. *If $\alpha < \theta_1$, then K_α is an internal subset of I.*

Proof. For each $i < \kappa$ set

$$K_{\alpha,i} = \{s \in I_i : s <^2_i f_\alpha(i)\}.$$

Then $K_\alpha = \prod K_{\alpha,i}/D$ and the result follows. □

Now consider the following statement:

(∗) There is $\alpha < \theta_1$ such that $J^2 \cap K_\alpha$ is $<_1$-unbounded from below in J^2.

We split the proof into two cases.

Case 1. (∗) holds: Fix α witnessing (∗). It follows that $(J^1 \cap K_\alpha, J^2 \cap K_\alpha)$ is internal in K_α, so there are end segments L_i of $I_i \restriction \{s \in I_i : s <^2_i f_\alpha(i)\}$, for $i < \kappa$, such that $J^2 \cap K_\alpha = \prod_{i<\kappa} L_i/D$, hence by the assumption, $J^2 = \prod_{i<\kappa} L'_i/D$, where $L'_i = \{t \in I_i : \exists s \in L_i,\ s \leq^1_i t\}$, so J^2 is internal. It follows from Remark 4.2 that (J^1, J^2) is an internal cut of I and we are done.

Case 2. (∗) fails: So for any $\alpha < \theta_1$, there is $s_\alpha \in J^2$ such that

$$\{s \in J^2 : s <_1 s_\alpha\} \cap K_\alpha = \emptyset.$$

As $\theta_1 \neq \theta_2$ are regular cardinals, there is $s_* \in J^2$ such that

$$\sup\{\alpha < \theta_1 : s_* \leq_1 s_\alpha\} = \theta_1,$$

hence

$$\{s \in J^2 : s <_1 s_*\} \cap (\bigcup_{\alpha<\theta_1} K_\alpha) = \emptyset.$$

Claim 4.6. $\bigcup_{\alpha<\theta_1} K_\alpha = K_{\theta_1}$.

Proof. It is clear that $\bigcup_{\alpha<\theta_1} K_\alpha \subseteq K_{\theta_1}$. Now suppose $s \in K_{\theta_1}$, so $s <_2 g_*/D$. If $s \notin \bigcup_{\alpha<\theta_1} K_\alpha$, then for any $\alpha < \theta_1$, $f_\alpha/D <_2 s$. So by the minimal choice of t_* and the fact that $\langle f_\alpha/D : \alpha < \theta_1\rangle$ is $<_1$-cofinal in J^1, we have $g_*/D \leq_2 s$ which is a contradiction. □

So we have $\{s \in J^2 : s <_1 s_*\} \cap K_{\theta_1} = \emptyset$. Let $h_* \in \prod_{i<\kappa} I_i$ be such that $s_* = h_*/D$.

Claim 4.7. (a) *K_{θ_1} is internal.*

(b) *$J^1 \cap K_{\theta_1}$ is $<_1$-unbounded in J^1.*

(c) *$J^1 \cap K_{\theta_1}$ is internal.*

Proof. (a) can be proved as in Claim 4.5 using f_{θ_1} instead of f_α. (b) is also clear as $J^1 \cap K_{\theta_1} \supseteq \{f_\alpha/D : \alpha < \theta_1\}$ and $\langle f_\alpha/D : \alpha < \theta_1\rangle$ is $<_1$-unbounded in J^1. Let's prove (c). As $\{s \in J^2 : s <_1 s_*\} \cap K_{\theta_1} = \emptyset$ and $I = J^1 \cup J^2$, we can easily see that

$$J^1 \cap K_{\theta_1} = \{s \in K_{\theta_1} : s <_1 s_*\}.$$

For each $i < \kappa$ set

$$L_i = \{s \in I_i : s <^2_i f_{\theta_1}(i) \text{ and } s <^1_i h_*(i)\}.$$

It follows that $J^1 \cap K_{\theta_1} = \prod_{i<\kappa} L_i/D$, and so $J^1 \cap K_{\theta_1}$ is internal. □

It follows from the above claim that $J^1 = \prod_{i<\kappa} L'_i/D$, where for $i < \kappa$, $L'_i = \{t \in I_i : \exists s \in L_i,\ t \leq^1_i s\}$. Hence J^1 is internal and so by Remark 4.2, (J^1, J^2) is an internal cut of I which completes the proof of Case 2. The theorem follows. □

5. $\mathscr{C}_{>2^\kappa}(D)$ CONTAINS NO NON-SYMMETRIC PAIRS

In this section we show that if D is a uniform ultrafilter on κ, then $\mathscr{C}_{>2^\kappa}(D)$ does not contain any non-symmetric pairs. Again, we prove a stronger result from which the above claim, and hence Theorem 1.2(b) follows.

Theorem 5.1. *Assume D is a uniform ultrafilter on κ, $\langle I_i : i < \kappa\rangle$ is a sequence of linear orders and $I = \prod_{i<\kappa} I_i/D$. Also assume (J^1, J^2) is a cut of I of cofinality (θ_1, θ_2), where $\theta_1 \neq \theta_2$ are bigger than 2^κ. Then the cut (J^1, J^2) is internal.*

Proof. Let $<^1_i = <_{I_i}$ $(i < \kappa)$ and $<_1 = <_I$. Let $<^2_i$, for $i < \kappa$, be a well-ordering of I_i with a last element and let $<_2$ be such that $(I, <_2) = \prod_{i<\kappa}(I_i, <^2_i)/D$; so $<_2$ is a linear ordering of I with a last element.

We say a sequence $\bar{K} = \langle K_i : i < \kappa\rangle$ catches (J^1, J^2) if each $K_i \subseteq I_i$ is non-empty and for every $s_1 \in J^1$ and $s_2 \in J^2$ there is $t \in \prod_{i<\kappa} K_i/D$ such that $s_1 \leq_1 t \leq_1 s_2$. Set

$$S = \{\bar{K} : \bar{K} \text{ catches } (J^1, J^2)\},$$

and

$$C = \{\bar{\mu} = \langle \mu_i : i < \kappa\rangle : \text{ There exists } \bar{K} \in S \text{ such that } \bigwedge_{i<\kappa} |K_i| = \mu_i\}.$$

We can define an order on C by

$$\bar{\mu}^1 = \langle \mu^1_i : i < \kappa\rangle <_D \bar{\mu}^2 = \langle \mu^2_i : i < \kappa\rangle \iff \{i < \kappa : \mu^1_i < \mu^2_i\} \in D.$$

Now consider the following statement:

$(*)$ $\qquad$ There is $\bar{\mu} \in C$ which is $<_D$-minimal.

We consider two cases.

Case 1. $(*)$ holds: Fix $\bar{\mu} = \langle \mu_i : i < \kappa\rangle$ witnessing $(*)$, and let $\bar{K} \in S$ be such that for all $i < \kappa$, $|K_i| = \mu_i$. Let $<^3_i$ be a well-ordering of K_i of order type μ_i and let $<_3$ be such that $(K, <_3) = \prod_{i<\kappa}(K_i, <^3_i)/D$, where $K = \prod_{i<\kappa} K_i/D \subseteq I$. Let $\theta_3 = \mathrm{cf}(\prod_{i<\kappa} \mu_i/D)$ and let $g_\alpha \in \prod_{i<\kappa} K_i$, $\alpha < \theta_3$, be such that $\langle g_\alpha/D : \alpha < \theta_3\rangle$ is $<_3$-increasing and cofinal in $(K, <_3)$. As $\theta_1 \neq \theta_2$, for some $l \in \{1, 2\}$, $\theta_3 \neq \theta_l$. Assume without loss of generality that $\theta_3 \neq \theta_1$.

For $\alpha < \theta_3$ and $i < \kappa$ set

$$K_{\alpha,i} = \{s \in K_i : s <^3_i g_\alpha(i)\}.$$

and

$$K_\alpha = K \restriction \{s : s <_3 g_\alpha/D\} = \prod_{i<\kappa} K_{\alpha,i}/D.$$

Then the sequence $\langle K_\alpha : \alpha < \theta_3\rangle$ is $\subseteq$-increasing and $K = \bigcup_{\alpha<\theta_3} K_\alpha$. The next claim is evident from our construction.

Claim 5.2. *K is internal.*

By our choice of $\bar{\mu}$, the sequence $\langle K_{\alpha,i} : i < \kappa\rangle$ does not catch (J^1, J^2), and hence we can find $s_\alpha \in J^1$ and $t_\alpha \in J^2$ such that

$$K_\alpha \cap \{s \in I : s_\alpha <_1 s <_1 t_\alpha\} = \emptyset.$$

As $\theta_3 \neq \theta_1$, there is $s_* \in J^1$ such that

$$\sup\{\alpha < \theta_3 : s_\alpha \leq_1 s_*\} = \theta_3.$$

It follows that $K \cap \{s \in J^1 : s_* \leq_1 s\} = \emptyset$. As K catches (J^1, J^2), it follows that $J^2 \cap K$ is $<_1$-cofinal in J^2 from below, and since K is internal, the arguments of section 2 show that J^2 is also internal, and hence by Remark 4.2, (J^1, J^2) is an internal cut of I, as required.

Case 2. $(*)$ fails: Clearly $<_D$ is a linear order on C, so it has a co-initiality, call it θ_3. As $(*)$ fails, $<_D$ is not well-founded and so $\theta_3 \geq \aleph_0$.

Claim 5.3. $\theta_3 \leq 2^\kappa$.

Proof. Suppose not. Let $\langle \bar{\mu}_\xi : \xi < (2^\kappa)^+\rangle$ be a $<_D$-decreasing chain of elements of C. Define a partition $F : [(2^\kappa)^+]^2 \to \kappa$ by

$$F(\xi, \zeta) = \min\{i < \kappa : \mu_i^\zeta < \mu_i^\xi\},$$

which is well-defined as $\{i < \kappa : \mu_i^\zeta < \mu_i^\xi\} \in D$, in particular it is non-empty. By the Erdös-Rado partition theorem, there are $X \subseteq (2^\kappa)^+$ of size κ^+ and some fixed $i_* < \kappa$ such that for all $\xi < \zeta$ in X, $F(\xi, \zeta) = i_*$. Thus

$$\xi < \zeta \in X \implies \mu_{i_*}^\zeta < \mu_{i_*}^\xi,$$

which is impossible. □

Let $\langle \bar{\mu}_\xi = \langle \mu_{\xi,i} : i < \kappa \rangle : \xi < \theta_3 \rangle$ be $<_D$-decreasing which is unbounded from below in $(C, <_D)$. For $\xi < \theta_3$ choose $\bar{K}_\xi = \langle K_{\xi,i} : i < \kappa \rangle \in S$ such that for all $i < \kappa$, $|K_{\xi,i}| = \mu_{\xi,i}$. Let $K_\xi = \prod_{i<\kappa} K_{\xi,i}/D \subseteq I$. We consider two subcases.

Subcase 2.1. For some $\xi < \theta_3$, $K_\xi \cap J^1$ is bounded in $(J^1, <_1)$: Fix such a $\xi < \theta_3$ and let $s_* \in J^1$ be a bound. Then as $\bar{K}_\xi$ catches (J^1, J^2), it follows that $K_\xi \cap J^2$ is unbounded in J^2 from below. Since K_ξ is internal and $K_\xi \cap J^2$ is unbounded in J^2 from below, so J^2 is internal. In follows that the cut (J^1, J^2) is internal and we are done.

Subcase 2.2. For all $\xi < \theta_3$, $K_\xi \cap J^1$ is unbounded in $(J^1, <_1)$: Since $\mathrm{cf}(J^1, <_1) = \theta_1$, there are functions $f_\alpha \in \prod_{i<\kappa} I_i$, for $\alpha < \theta_1$, such that

(1) $\forall \alpha < \theta_1, f_\alpha/D \in J^1$,

(2) $\langle f_\alpha/D : \alpha < \theta_1 \rangle$ is $<_1$-increasing,

(3) $\langle f_\alpha/D : \alpha < \theta_1 \rangle$ is a $<_1$-cofinal subset of J^1.

For every $\alpha < \theta_1$ and $\xi < \theta_3$ there are β and g such that

(4) $\alpha < \beta < \theta_1$,

(5) $g \in \prod_{i<\kappa} K_{\xi,i}$,

(6) $f_\alpha/D <_1 g/D <_1 f_\beta/D$.

For $\alpha < \beta < \theta_1$ set

$$\Lambda_{\alpha,\beta} = \{(\xi, i) : f_\alpha(i) \leq^1_i f_\beta(i) \text{ and there is } s \in K_{\xi,i} \text{ such that } f_\alpha(i) \leq^1_i s \leq^1_i f_\beta(i)\}.$$

For $i < \kappa$ set $\Lambda_{\alpha,\beta,i} = \{\xi < \theta_3 : (\xi, i) \in \Lambda_{\alpha,\beta}\}$ and $\Xi_{\alpha,\beta} = \{i < \kappa : \Lambda_{\alpha,\beta,i} \neq \emptyset\}$. Also let $F^1_{\alpha,\beta}, F^2_{\alpha,\beta}$ be functions with domain κ such that

- If $i \in \Xi_{\alpha,\beta}$, then

$$F^1_{\alpha,\beta}(i) = \min\{\mu_{\xi,i} : (\xi, i) \in \Lambda_{\alpha,\beta}\},$$

  and

$$F^2_{\alpha,\beta}(i) = \min\{\xi : \mu_{\xi,i} = F^1_{\alpha,\beta}(i)\}.$$

- If $i \in \kappa \setminus \Xi_{\alpha,\beta}$, then $F^1_{\alpha,\beta}(i) = F^2_{\alpha,\beta}(i) = 0$.

Claim 5.4. *For each $\alpha < \theta_1$ there exist $A_\alpha \subseteq (\alpha, \theta_1)$, functions F^1_α, F^2_α and a set Ξ_α such that*

(1) $A_\alpha = \{\beta \in (\alpha, \theta_1) : F^1_{\alpha,\beta} = F^1_\alpha, F^2_{\alpha,\beta} = F^2_\alpha$ *and* $\Xi_{\alpha,\beta} = \Xi_\alpha\}$.

(2) $\sup(A_\alpha) = \theta_1$.

Proof. As $\theta_3 \leq 2^\kappa$, we have

$$|\{(F^1_{\alpha,\beta}, F^2_{\alpha,\beta}, \Xi_{\alpha,\beta}) : \alpha < \beta < \theta_1\}| \leq \theta_3^\kappa = 2^\kappa < \theta_1.$$

So there is an unbounded subset A_α of θ_1 such that all tuples $(F^1_{\alpha,\beta}, F^2_{\alpha,\beta}, \Xi_{\alpha,\beta}), \beta \in A_\alpha$, are the same. The result follows immediately. □

The next claim can be proved in a similar way.

Claim 5.5. *There are* $A \subseteq \theta_1$*, functions* F_1, F_2 *and a set* Ξ *such that*

(1) $A = \{\alpha < \theta_1 : F^1_\alpha = F_1, F^2_\alpha = F_2$ *and* $\Xi_\alpha = \Xi\}$.

(2) $\sup(A) = \theta_1$.

Let $\bar{K}^* = \langle K^*_i : i < \kappa\rangle$ where $K^*_i = K_{F_2(i),i}$ and let $\bar{\mu}^* = \langle \mu^*_i : i < \kappa\rangle$ be defined by $\mu^*_i = |K^*_i|$. Note that

$$\mu^*_i = |K^*_i| = |K_{F_2(i),i}| = \mu_{F_2(i),i} = F_1(i).$$

Claim 5.6. *For every* $\xi < \theta_3$, $\bar{\mu}^* \leq_D \bar{\mu}_\xi$.

Proof. Choose $\alpha \in A$ and $\beta \in A_\alpha$. So we have $F_1 = F^1_{\alpha,\beta}, F_2 = F^2_{\alpha,\beta}$ and $\Xi = \Xi_{\alpha,\beta}$. By the construction, there is $t \in K_\xi$ such that $f_\alpha/D <_1 t <_1 f_\beta/D$. Let $t = g/D$, where $g \in \prod_{i<\kappa} K_{\xi,i}$. Then

$$f_\alpha(i) <^1_i g(i) <^1_i f_\beta(i) \Rightarrow (\xi, i) \in \Lambda_{\alpha,\beta} \Rightarrow \mu^*_i = F_1(i) = F^1_{\alpha,\beta}(i) \leq \mu_{\xi,i}.$$

So

$$\{i < \kappa : \mu^*_i \leq \mu_{\xi,i}\} \supseteq \{i < \kappa : f_\alpha(i) <^1_i g(i) <^1_i f_\beta(i)\} \in D,$$

and the result follows. □

Claim 5.7. $\bar{\mu}^* \in C$.

Proof. We show that $\bar{K}^*$ catches (J^1, J^2), so that $\bar{K}^* \in S$ witnesses $\bar{\mu}^* \in C$. So let $s_1 \in J^1$ and $s_2 \in J^2$. Pick $\alpha \in A$ such that $s_1 <_1 f_\alpha/D$. Let $\beta \in A_\alpha$. By our construction there is $g \in \prod_{i<\kappa} K_i^*$ such that $f_\alpha/D <_1 g/D <_1 f_\beta/D$ and hence

$$s_1 <_1 f_\alpha/D <_1 g/D <_1 f_\beta/D <_1 s_2.$$

The claim follows. □

But Claims 5.6 and 5.7 give us a contradiction to the choice of the sequence $\langle \bar{\mu}_\xi : \xi < \theta_3 \rangle$. This contradiction finishes the proof. □

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