Abstract. We continue our study of the class \( \mathcal{C}(D) \), where \( D \) is a uniform ultrafilter on a cardinal \( \kappa \) and \( \mathcal{C}(D) \) is the class of all pairs \( (\theta_1, \theta_2) \), where \( (\theta_1, \theta_2) \) is the cofinality of a cut in \( J^\kappa/D \) and \( J \) is some \( (\theta_1 + \theta_2)^+ \)-saturated dense linear order. We give a combinatorial characterization of the class \( \mathcal{C}(D) \). We also show that if \( (\theta_1, \theta_2) \in \mathcal{C}(D) \) and \( D \) is \( \aleph_1 \)-complete or \( \theta_1 + \theta_2 > 2^\kappa \), then \( \theta_1 = \theta_2 \).

1. Introduction

Assume \( \kappa \) is an infinite cardinal and \( D \) is an ultrafilter on \( \kappa \). Recall that \( \mathcal{C}(D) \) is defined to be the class of all pairs \( (\theta_1, \theta_2) \), where \( (\theta_1, \theta_2) \) is the cofinality of a cut in \( J^\kappa/D \) and \( J \) is some (equivalently any) \( (\theta_1 + \theta_2)^+ \)-saturated dense linear order. Also \( \mathcal{C}_{>\lambda}(D) \) is defined to be the class of all pairs \( (\theta_1, \theta_2) \in \mathcal{C}(D) \), such that \( \theta_1 + \theta_2 > \lambda \). The classes \( \mathcal{C}_{>\lambda}(D), \mathcal{C}_{<\lambda}(D) \) and \( \mathcal{C}_{\leq\lambda}(D) \) are defined similarly.

The works \cite{2}, \cite{3} and \cite{4} of Malliaris and Shelah have started the study of this class for the case \( \theta_1 + \theta_2 \leq 2^\kappa \) and \cite{1} started the study of the case \( \theta_1 + \theta_2 > 2^\kappa \). As it was observed in \cite{1}, the study of the class \( \mathcal{C}_{>2^\kappa}(D) \) is very different from the case \( \mathcal{C}_{\leq2^\kappa}(D) \), and to prove results about it, usually some extra set theoretic assumptions are needed. In this paper we continue \cite{1} and prove more results related to the class \( \mathcal{C}(D) \).

In the first part of the paper (Sections 2 and 3) we give a combinatorial characterization of \( \mathcal{C}(D) \). Using notions defined in section 2, we can state our first main theorem as follows.

**Theorem 1.1.** Assume \( D \) is an ultrafilter on \( \kappa \) and \( \lambda_1, \lambda_2 > \kappa \) are regular cardinals. The following are equivalent:

(a) There is \( \vec{a} \in \mathcal{S}_c \) which is not \( c \)-solvable, where \( c = \langle \kappa, D, \lambda_1, \lambda_2 \rangle \).
(b) \((\lambda_1, \lambda_2) \in C(D)\).

In the second part of the paper (Sections 4 and 5) we study the existence of non-symmetric pairs (i.e., pairs \((\lambda_1, \lambda_2)\) with \(\lambda_1 \neq \lambda_2\)) in \(C(D)\). By [5], we can find a regular ultrafilter \(D\) on \(\kappa\) such that

\[C(D) \supseteq \{ (\lambda_1, \lambda_2) : \aleph_0 < \lambda_1 < \lambda_2 \leq 2^\kappa, \lambda_1, \lambda_2 \text{ regular} \}.
\]

In particular, \(C(D)\) contains non-symmetric pairs. On the other hand, results of [1] show that if \((\lambda_1, \lambda_2) \in C_{>2^\kappa}(D)\), then we must have \(\lambda_1^\kappa = \lambda_2^\kappa\), in particular if SCH, the singular cardinals hypothesis, holds, then \(\lambda_1 = \lambda_2\), and so \(C_{>2^\kappa}(D)\) just contains symmetric pairs.

We then prove the following theorem (in ZFC):

**Theorem 1.2.** (a) Assume \(D\) is a uniform \(\aleph_1\)-complete ultrafilter on \(\kappa\) and \((\lambda_1, \lambda_2) \in C(D)\). Then \(\lambda_1 = \lambda_2\).

(b) Assume \(D\) is a uniform ultrafilter on \(\kappa\) and \((\lambda_1, \lambda_2) \in C_{>2^\kappa}(D)\). Then \(\lambda_1 = \lambda_2\).

The theorem shows some restrictions on the pairs \((\lambda_1, \lambda_2)\) that \(C(D)\) can have, in particular, it shows that in the result of [5] stated above, we can never take the ultrafilter \(D\) to be \(\aleph_1\)-complete and that \(C_{>2^\kappa}(D)\) can not have non-symmetric pairs.

The paper is organized as follows. In section 2 we give the required definitions, which lead us to the notion of \(c\)-solvability and in section 3 we complete the proof of Theorem 1.1. In section 4 we prove part (a) of Theorem 1.2 and in section 5 we complete the proof of part (b) of Theorem 1.2. We may note that parts one (Sections 2 and 3) and two (Sections 4 and 5) can be read independently of each other.

2. **On the notion of \(c\)-solvability**

In this section we give the required definitions which are used in Theorem 1.1.

**Definition 2.1.** (a) Let \(C\) be the class of tuples \(c = (\kappa_c, D_c, \lambda_{c,1}, \lambda_{c,2})\) where

(a-1) \(\lambda_{c,1}, \lambda_{c,2}\) are regular cardinals > \(\kappa_c\),

(a-2) \(D_c\) is a uniform ultrafilter on \(\kappa_c\).

Also let \(\lambda_c = 2^{<\lambda_{c,1}} + 2^{<\lambda_{c,2}}\) and \(\lambda_{c,0} = \min\{\lambda_{c,1}, \lambda_{c,2}\}\).
(b) For $c \in \mathcal{C}$ let $N_c = N_{c,1} + N_{c,2}$ be a linear order of size $\leq \lambda_c$ in such a way that $N_{c,1}$ has cofinality $\lambda_{c,1}$, $N_{c,2}$ has co-initiality $\lambda_{c,2}$ and both $N_{c,1}, N_{c,2}$ are $\lambda_{c,0}$-saturated dense linear orders \(^1\).

(c) For $c \in \mathcal{C}$ let $S_c$ be the set of all sequences $\bar{a} = \langle a_{s,t} : s, t \in N_c \rangle$ such that

(c-1) Each $a_{s,t}$ is a subset of $\kappa_c$,
(c-2) $a_{s,s} = \emptyset$,
(c-3) For $s \neq t, a_{s,t} = \kappa \setminus a_{t,s}$,
(c-4) $s <_{N_c} t \Rightarrow a_{s,t} \in D_c$,
(c-5) If $s_1 <_{N_c} s_2 <_{N_c} s_3$, then

\[(a_{s_1,s_3} \supseteq a_{s_1,s_2} \cap a_{s_2,s_3}) \& (a_{s_3,s_1} \supseteq a_{s_3,s_2} \cap a_{s_2,s_1}).\]

(d) For $c \in \mathcal{C}$ let $N_c^+ = N_{c,1} + N_0 + N_{c,2}$, where $N_0$ is a singleton, say $N_0 = \{s_0\}$.

We now define the notion of $c$-solvability.

**Definition 2.2.** Let $c \in \mathcal{C}$. We say $\bar{a} \in S_c$ is $c$-solvable, if there exists a sequence $\bar{b} = \langle b_s : s \in N_c \rangle$, such that the sequence $\bar{a}^1 = \bar{a} * \bar{b}$ satisfies clauses (c-1)-(c-5) above, where the sequence $\bar{a}^1 = \langle a_{s,t}^1 : s, t \in N_c^+ \rangle$ is defined as follows:

1. If $s, t \in N_c$, then $a_{s,t}^1 = a_{s,t}$,
2. For $s \in N_{c,1}, a_{s,s}^1 = b_s$ and $a_{s,s}^1 = \kappa \setminus b_s$,
3. For $s \in N_{c,2}, a_{s,s}^1 = b_s$ and $a_{s,s}^1 = \kappa \setminus b_s$,
4. $a_{s,s}^1 = \emptyset$.

Then $\bar{b}$ is called a $c$-solution for $\bar{a}$.

3. A combinatorial characterization of $\mathscr{C}(D)$

In this section we give a proof of Theorem 1.1.

**Lemma 3.1.** Assume $c \in \mathcal{C}$ and $\bar{a} \in S_c$. Then

(a) There are $M, \bar{f}$ such that

(a-1) $M$ is a $(\lambda_{c,1} + \lambda_{c,2})^+$-saturated dense linear order,

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\(^{1\text{N}_c\text{ is some fixed linear order which we choose in advance. We may assume global choice and let } N_c \text{ be the least such order.}}\)
(a-2) \( \bar{f} = \langle f_s : s \in N_c \rangle \),

(a-3) Each \( f_s \in {}^\kappa M \),

(a-4) If \( s <^N c t \), then \( a_{s,t} = \{ i < \kappa_c : f_s(i) <_M f_t(i) \} \),

(b) If \( M, \bar{f} \) are as in (a), then

(b-1) \( \langle f_s / D_c : s \in N_c \rangle \) is an increasing sequence in \( {}^\kappa M / D_c \),

(b-2) \( \bar{a} \) is \( c \)-solvable iff \( {}^\kappa M / D_c \) realizes the type

\[ q(x) = \{ f_s / D_c < x < f_t / D_c : s \in N_{c,1} and t \in N_{c,2} \}. \]

**Proof.** (a) Let \( A = \{(i,s) : i < \kappa_c, s \in N_c \} \), and define the order \( <_A \) on \( A \) by

\[ (i_1, s_1) <_A (i_2, s_2) \iff (i_1 < i_2) or (i_1 = i_2 \in a_{s_1, s_2}). \]

Also let \( \leq_A \) be defined on \( A \) in the natural way from \( <_A \), so

\[ (i_1, s_1) \leq_A (i_2, s_2) \iff (i_1, s_1) = (i_2, s_2) or (i_1, s_1) <_A (i_2, s_2). \]

It is easily seen that \( \leq_A \) is a linear order on \( A \). Now let \( M \) be a \((\lambda_{c,1} + \lambda_{c,2})^+\)-saturated dense linear order which contains \( (A,<_A) \) as a sub-order. Also let \( \bar{f} = \langle f_s : s \in N_c \rangle \), where for \( s \in N_c f_s \in {}^\kappa M \) is defined by \( f_s(i) = (i, s) \). It is clear that \( M \), and \( \bar{f} \) satisfy clauses (a-1)-(a-3). For (a-4), assume \( s <^N c t \) are given. Then

\[ a_{s,t} = \{ i < \kappa_c : i \in a_{s,t} \} = \{ i < \kappa_c : (i, s) <_A (i, t) \} = \{ i < \kappa_c : f_s(i) <_M f_t(i) \}. \]

Finally note that for \( s \neq t \) in \( N_c \),

\[ \text{range}(f_s) \cap \text{range}(f_t) = \{ (i, s) : i < \kappa_c \} \cap \{ (i, t) : i < \kappa_c \} = \emptyset. \]

So \( M \) and \( \bar{f} \) are as required.

(b) (b-1) follows from (a-4) and the fact that for \( s <^N c t, a_{s,t} \in D_c \). Let’s prove (b-2).

First assume that \( \bar{a} \) is \( c \)-solvable and let \( \bar{b} \) be a solution for \( \bar{a} \). For each \( i < \kappa_c \) let \( p_i(x) \) be the following type over \( M \):

\[ p_i(x) = \{ f_s(i) <_M x : s \in N_{c,1} and i \in b_s \} \cup \{ x <_M f_t(i) : t \in N_{c,2} and i \in \kappa_c \setminus b_t \}. \]

**Claim 3.2.** For each \( i < \kappa_c \), the type \( p_i(x) \) is finitely satisfiable in \( M \).
Proof. Let $s_0 <_{N_{c,1}} \cdots <_{N_{c,1}} s_{n-1}$ be in $N_{c,1}$ and $t_{m-1} <_{N_{c,2}} < \cdots <_{N_{c,2}} t_0$ be in $N_{c,2}$. Also suppose that $i \in \bigcap_{k<n} b_{s_k} \cap \bigcap_{l<m} (\kappa_c \setminus b_{t_l})$. Then for $k < n$ and $l < m$ we have

$$a_{s_k, t_l} \supseteq a^1_{s_k, s_\ast} \cap a^1_{s_\ast, t_l} = b_{s_k} \cap (\kappa_c \setminus b_{t_l}),$$

and so $i \in a_{s_k, t_l}$, which implies $f_{s_k}(i) < f_{t_l}(i)$. Take $x \in M$ so that

$$\forall k < n, \forall l < m, f_{s_k}(i) < x < f_{t_l}(i),$$

which exists as $M$ is dense. It follows that $p_i(x)$ is finitely satisfiable in $M$. \hfill \Box

It follows that there exists $f \in \kappa_c M$ such that for each $i < \kappa_c$, $f(i)$ realizes the type $p_i(x)$ over $M$. Then $f/D_c$ realizes $q(x)$ over $\kappa_c M/D_c$.

Conversely assume that $f \in \kappa_c M$ is such that $f/D_c$ realizes the type $q(x)$ over $\kappa_c M/D_c$.

Claim 3.3. We can assume that range($f$) is disjoint from $A$.

Proof. As $(\text{range}(f_s) : s \in N_c)$ is a sequence of pairwise disjoint sets and $\lambda_{c,1}, \lambda_{c,2} > \kappa_c$ are regular, there are $s_1 \in N_{c,1}$ and $s_2 \in N_{c,2}$ such that $s_1 <_{N_c} s <_{N_c} s_2$ implies range($f_s$) $\cap$ range($f$) = $\emptyset$. As $M$ is a $(\lambda_{c,1} + \lambda_{c,2})^+$-saturated dense linear order, there is $f'$ such that

- $f' \in \kappa_c M$,
- range($f'$) $\cap$ $A$ = $\emptyset$,
- If $s_1 <_{N_c} s <_{N_c} s_2$ and $i < \kappa_c$, then $f_s(i) <_{N_c} f'(i)$ \Rightarrow $f_s(i) <_{N_c} f(i)$ and $f'(i) <_{N_c} f_s(i) \Rightarrow f(i) <_{N_c} f_s(i)$.

So we can replace $f$ by $f'$ and $f'$ satisfies the requirements on $f$; i.e., $f'/D_c$ realizes $q(x)$ over $\kappa_c M/D_c$ and further range($f'$) $\cap$ $A$ = $\emptyset$. \hfill \Box

Now define $\bar{b} = (b_s : s \in N_c)$ by

$$b_s = \begin{cases} 
{i < \kappa_c : f_s(i) <_{N_c} f(i)} & \text{if } s \in N_{c,1}, \\
{i < \kappa_c : f(i) <_{N_c} f_s(i)} & \text{if } s \in N_{c,2}.
\end{cases}$$

Claim 3.4. $\bar{b}$ is a $c$-solution for $\bar{a}$. 

Definition 2.2). (c-1) and (c-2) are trivial and (c-3) follows from the fact that \( \forall i < \kappa_c, f_s(i) \neq f(i) \) (as \( \text{range}(f) \cap A = \emptyset \)).

For (c-4), suppose that \( s < N_c^+ t \). If both \( s, t \) are in \( N_c \), then we are done. So suppose otherwise. There are two cases to consider.

- If \( s = s_* \), then \( t \in N_{c,2} \) and as \( f/D_c \) realizes \( q(x) \), we have \( f/D_c < f_1/D_c \), which implies \( a_{s_*,s}^1 = b_s = \{i < \kappa_c : f(i) < N_c f_1(i)\} \in D_c \).
- If \( t = s_* \), then \( s \in N_{c,1} \) and as \( f/D_c \) realizes \( q(x) \), we have \( f_s < D_c f \), which implies \( a_{s,s_*}^1 = b_s = \{i < \kappa_c : f_s(i) < N_c f(i)\} \in D_c \).

For (c-5), assume \( s_1 < N_c^+ s_2 < N_c^+ s_3 \) are in \( N_c^+ \). If all \( s_1, s_2 \) and \( s_3 \) are in \( N_c \), then we are done. So assume otherwise. There are three cases to be considered:

- If \( s_1 = s_* \), then \( s_2, s_3 \in N_{c,2} \), and we have
  \[
  a_{s_*,s_2}^1 \cap a_{s_2,s_3}^1 = b_{s_2} \cap a_{s_2,s_3} = \{i < \kappa_c : (f(i) < N_c f_{s_2}(i)) \land (f_{s_2}(i) < N_c f_{s_3}(i))\} \\
  \subseteq \{i < \kappa_c : f(i) < N_c f_{s_3}(i)\} \\
  = b_{s_3} \\
  = a_{s_*,s_3}^1.
  \]
  Similarly,
  \[
  a_{s_3,s_2}^1 \cap a_{s_2,s_*}^1 = a_{s_3,s_2} \cap (\kappa_c \setminus b_{s_2}) = \{i < \kappa_c : (f_{s_2}(i) \geq N_c f_{s_3}(i)) \land (f(i) > N_c f_{s_2}(i))\} \\
  \subseteq \{i < \kappa_c : f(i) > N_c f_{s_3}(i)\} \\
  = \kappa_c \setminus b_{s_3} \\
  = a_{s_3,s_*}^1.
  \]

- If \( s_2 = s_* \), then \( s_1 \in N_{c,1}, s_3 \in N_{c,2} \) and we have
  \[
  a_{s_1,s_*}^1 \cap a_{s_*s_3}^1 = b_{s_1} \cap b_{s_3} = \{i < \kappa_c : (f_{s_1}(i) < N_c f(i)) \land (f(i) < N_c f_{s_3}(i))\} \\
  \subseteq \{i < \kappa_c : f_{s_1}(i) < N_c f_{s_3}(i)\} \\
  = a_{s_1,s_3} \\
  = a_{s_1,s_3}^1.
  \]

Also,
\[a_{s_3,s_*}^1 \cap a_{s_2,s_1}^1 = (\kappa_c \setminus b_{s_3}) \cap (\kappa_c \setminus b_{s_1})
\]
\[= \{i < \kappa_c : (f_{s_3}(i) <_{N_c} f(i)) \wedge (f(i) <_{N_c} f_{s_1}(i))\}
\]
\[\subseteq \{i < \kappa_c : f_{s_3}(i) \leq_{N_c} f_{s_1}(i)\}
\]
\[= a_{s_3,s_1}
\]
\[= a_{s_3,s_1}^1.
\]

- If \(s_3 = s_*\), then \(s_1, s_2 \in N_{c,1}\) and we have
  \[a_{s_1,s_2}^1 \cap a_{s_2,s_*}^1 = a_{s_1,s_2} \cap b_{s_2}
  \]
  \[= \{i < \kappa_c : (f_{s_1}(i) <_{N_c} f_{s_2}(i)) \wedge (f_{s_2}(i) <_{N_c} f(i))\}
  \]
  \[\subseteq \{i < \kappa_c : f_{s_1}(i) <_{N_c} f(i)\}
  \]
  \[= b_{s_1}
  \]
  \[= a_{s_1,s_*}^1.
  \]

Similarly, we have

\[a_{s_2,s_2}^1 \cap a_{s_2,s_1}^1 = (\kappa_c \setminus b_{s_2}) \cap a_{s_2,s_1}
\]
\[= \{i < \kappa_c : (f_{s_2}(i) >_{N_c} f(i)) \wedge (f_{s_1}(i) \geq_{N_c} f_{s_2}(i))\}
\]
\[\subseteq \{i < \kappa_c : f_{s_1}(i) >_{N_c} f(i)\}
\]
\[= \kappa_c \setminus b_{s_1}
\]
\[= a_{s_2,s_1}^1.
\]

Hence, \(\bar{b}\) is a \(c\)-solution for \(\bar{a}\), as required. \(\Box\)

The lemma follows. \(\Box\)

Given \(c \in C\), the next lemma gives a characterization, in terms of \(c\)-solubility, of when

\((\lambda_{c,1}, \lambda_{c,2})\) is in \(\mathcal{E}(D_c)\), which also completes the proof of Theorem 1.1.

**Lemma 3.5.** Assume \(c \in C\) and \(M\) is a \(\lambda_c^+\)-saturated dense linear order. The following are equivalent:

(a) There is \(\bar{a} \in S_c\) which is not \(c\)-solvable.

(b) \((\lambda_{c,1}, \lambda_{c,2}) \in \mathcal{E}(D_c)\).

**Proof.** First assume there exists \(\bar{a} \in S_c\) which is not \(c\)-solvable. By Lemma 3.1, there are

\(M, \bar{f}\) which satisfy clauses (1-a)-(1-e) of that lemma. But then as \(\bar{a}\) is not \(c\)-solvable, by
Lemma 2.3(b-2), the type
\[ q(x) = \{ f_s/D_c < x < f_t/D_c : s \in N_{c,1} \text{ and } t \in N_{c,2} \}. \]
is not realized. It follows that \((\lambda_{c,1}, \lambda_{c,2}) \in \mathcal{C}(D_c)\).

Conversely assume that \(M\) and \(\bar{f} = \langle f_s : s \in N_c \rangle\) witness \((\lambda_{c,1}, \lambda_{c,2}) \in \mathcal{C}(D_c)\). Let \(A = \bigcup \{ \text{range}(f_s) : s \in N_c \}\), and let \(\langle I_d : d \in A \rangle\) be a sequence of pairwise disjoint intervals of \(M\) such that \(d \in I_d\). For \(s \in N_c\), let \(f'_s \in \kappa \cdot M\) be such that \(f'_s(i) \in I_{f_s(i)}\) and \(\langle f'_s(i) : s \in N_c, i < \kappa \rangle\) is with no repetitions. Define the sequence \(\bar{a} = \langle a_{s,t} : s, t \in N_c \rangle\), such that for \(s < N_c, t, a_{s,t} = \{ i < \kappa : f_s(i) < f_t(i) \}\) and \(a_{t,s} = \kappa_c \setminus a_{s,t}\). Also set \(a_{s,s} = \emptyset\). It is evident that \(\bar{a} \in \mathcal{S}_c\).

**Claim 3.6.** \(\bar{a}\) is not \(c\)-solvable.

**Proof.** Assume not. Then by Lemma 3.1(b-2), the type
\[ q(x) = \{ f_s/D_c < x < f_t/D_c : s \in N_{c,1} \text{ and } t \in N_{c,2} \} \]
is realized in \(\kappa \cdot M/D_c\), which contradicts the choice of \(M, \bar{f}\).

The Lemma follows.

4. For \(\aleph_1\)-complete ultrafilter, \(\mathcal{C}(D)\) contains no non-symmetric pairs

In this section we prove part (a) of Theorem 1.2. In fact we will prove something stronger, that is of interest in its own sake.

**Definition 4.1.** Assume \(D\) is an ultrafilter on \(\kappa\), \(\langle I_i : i < \kappa \rangle\) is a sequence of linear orders and \(I = \prod_{i < \kappa} I_i/D\).

(a) a subset \(K\) of \(I\) is called internal if there are subsets \(K_i \subseteq I_i\) such that \(K = \prod_{i < \kappa} K_i/D\).

(b) The cut \((J^1, J^2)\) of \(I\) is called internal, if there are cuts \((J^1_i, J^2_i)\) of \(I_i, i < \kappa\), such that \(J^l = \prod_{i < \kappa} J^l_i/D \ (l = 1, 2)\).

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\(^2\)The existence of the sequence \(\langle I_d : d \in A \rangle\) follows from the fact that \(|A| \leq \kappa \cdot |N_c| \leq \lambda_c\) and \(M\) is \(\lambda^+\)-saturated.
Remark 4.2. Assume $J$ is an initial segment of $I$ which is internal and suppose that $(J, I \setminus J)$ is a cut of $I$. Then $(J, I \setminus J)$ is in fact an internal cut of $I$. Similarly, if $J$ is an end segment of $I$ which is internal and if $(I \setminus J, J)$ is a cut of $I$, then $(I \setminus J, J)$ is an internal cut of $I$.

Theorem 4.3. Assume $D$ is a uniform $\aleph_1$-complete ultrafilter on $\kappa$, $(I_i : i < \kappa)$ is a sequence of non-empty linear orders and $I = \prod_{i<\kappa} I_i / D$. Also assume $(J_1, J_2)$ is a cut of $I$ of cofinality $(\theta_1, \theta_2)$, where $\theta_1 \neq \theta_2$. Then the cut $(J_1, J_2)$ is internal.

Before giving the proof of Theorem 4.3, let us show that it implies Theorem 1.2(a).

Proof of Theorem 1.2(a) from Theorem 4.3. Suppose $D$ is an $\aleph_1$-complete ultrafilter on $\kappa$, $J$ is a $(\lambda_1 + \lambda_2)^+$-saturated dense linear order and $(J_1, J_2)$ is a cut of $I$ of cofinality $(\theta_1, \theta_2)$, where $\theta_1 \neq \theta_2$. Then the cut $(J_1, J_2)$ is internal.

We are now ready to complete the proof of Theorem 4.3.

Proof. We can assume that $\theta_1, \theta_2$ are infinite. Let $<^1_i = <^1_{I_i}$ ($i < \kappa$) and $<_1 = <_I$. Let $<_2$ be a well-ordering of $I_i$ with a last element and let $<_2$ be such that $(I, <_2) = \prod_{i<\kappa} (I_i, <^2_i)/D$. Then $<_2$ is a linear ordering of $I$ with a last element and since $D$ is $\aleph_1$-complete, it is well-founded, so $<_2$ is in fact a well-ordering of $I$ with a last element.

As $(J^1, <_1)$ has cofinality $\theta_1$, we can find $f_\alpha \in \prod_{i<\kappa} I_i$, for $\alpha < \theta_1$, such that

1. $\forall \alpha < \theta_1, f_\alpha / D \in J^1$,
2. $(f_\alpha / D : \alpha < \theta_1)$ is $<_1$-increasing,
3. $(f_\alpha / D : \alpha < \theta_1)$ is a $<_1$-cofinal subset of $J^1$. 

It follows that $\lambda_l = \prod_{i<\kappa} \lambda^l_i / D \geq (\lambda_1 + \lambda_2)^+$, which is a contradiction. \hfill \Box
Let $B = \{ t \in I : \{ s \in J^1 : s \prec_2 t \} \text{ is } \prec_1\text{-unbounded in } J^1 \}$. As $\theta_1$ is infinite, the $\prec_2$-last element of $I$ belongs to $B$, which implies $B \neq \emptyset$ and hence $B$ has a $\prec_2$-minimal element; call it $t_\ast$. Let $g_* \in \prod_{i < \kappa} I_i$ be such that $t_\ast = g_*/D$.

Note that for each $\alpha < \theta_1$ there are $s \in J^1$ and $\beta > \alpha$ such that $s \prec_2 g_*/D$ and $f_\alpha/D < s < f_\beta/D$, so we can assume that for all $\alpha < \theta_1$, $f_\alpha/D < 2 g_*/D$. This implies

$$
\bigwedge_{\alpha < \theta_1} \{ [i < \kappa : f_\alpha(i) \prec_1^2 g_*(i)] \in D \}.
$$

Also note that

$$
\{ i < \kappa : g_*(i) \text{ is } \prec_1^< \text{-minimal or } \prec_1^> \text{-maximal} \} \notin D,
$$

so, without loss of generality, it is empty. Hence, without loss of generality

$$
\bigwedge_{\alpha < \theta_1} \bigwedge_{i < \kappa} [f_\alpha(i) \prec_1^2 g_*(i) \text{ and } f_\alpha(i) \text{ is not } \prec_1^< \text{-minimal}].
$$

Let $f_{\theta_1} = g_*$ and for $\alpha \leq \theta_1$ set $K_\alpha = \{ s \in I : s \prec_2 f_\alpha/D \}$. Thus $K_\alpha$ is a $\prec_2$-initial segment of $I$.

**Claim 4.4.** $K_\alpha \cap J^1$ is $\prec_1$-bounded in $J^1$.

**Proof.** As $f_\alpha/D <_1 t_\ast$, it follows from our choice of $t_\ast$ that $K_\alpha$ is $\prec_1$ bounded in $J^1$. \qed

**Claim 4.5.** If $\alpha < \theta_1$, then $K_\alpha$ is an internal subset of $I$.

**Proof.** For each $i < \kappa$ set

$$
K_{\alpha,i} = \{ s \in I_i : s \prec_1^2 f_\alpha(i) \}.
$$

Then $K_\alpha = \prod K_{\alpha,i}/D$ and the result follows. \qed

Now consider the following statement:

(*) There is $\alpha < \theta_1$ such that $J^2 \cap K_\alpha$ is $\prec_1$-unbounded from below in $J^2$.

We split the proof into two cases.

**Case 1.** (*) holds: Fix $\alpha$ witnessing (*). It follows that $(J^1 \cap K_\alpha, J^2 \cap K_\alpha)$ is internal in $K_\alpha$, so there are end segments $L_i$ of $I_i \mid \{ s \in I_i : s \prec_1^2 f_\alpha(i) \}$, for $i < \kappa$, such that $J^2 \cap K_\alpha = \prod_{i < \kappa} L_i/D$, hence by the assumption, $J^2 = \prod_{i < \kappa} L_i'/D$, where $L_i' = \{ t \in I_i : \exists s \in L_i, s \leq_1^< t \}$, so $J^2$ is internal. It follows from Remark 4.2 that $(J^1, J^2)$ is an internal cut of $I$ and we are done.
**Case 2.** (⋆) fails: So for any $\alpha < \theta_1$, there is $s_\alpha \in J^2$ such that

$$\{s \in J^2 : s \preceq_1 s_\alpha\} \cap K_\alpha = \emptyset.$$ 

As $\theta_1 \neq \theta_2$ are regular cardinals, there is $s_* \in J^2$ such that

$$\sup\{\alpha < \theta_1 : s_* \preceq_1 s_\alpha\} = \theta_1,$$

hence

$$\{s \in J^2 : s \preceq_1 s_*\} \cap (\bigcup_{\alpha < \theta_1} K_\alpha) = \emptyset.$$

**Claim 4.6.** $\bigcup_{\alpha < \theta_1} K_\alpha = K_{\theta_1}$. 

**Proof.** It is clear that $\bigcup_{\alpha < \theta_1} K_\alpha \subseteq K_{\theta_1}$. Now suppose $s \in K_{\theta_1}$, so $s <_2 g_*/D$. If $s \notin \bigcup_{\alpha < \theta_1} K_\alpha$, then for any $\alpha < \theta_1$, $f_\alpha/D <_2 s$. So by the minimal choice of $t_*$ and the fact that $\langle f_\alpha/D : \alpha < \theta_1 \rangle$ is $<_1$-cofinal in $J^1$, we have $g_*/D \leq_2 s$ which is a contradiction. □

So we have $\{s \in J^2 : s \preceq_1 s_*\} \cap K_{\theta_1} = \emptyset$. Let $h_* \in \prod_{i<\kappa} I_i$ be such that $s_* = h_*/D$.

**Claim 4.7.**

(a) $K_{\theta_1}$ is internal.

(b) $J^1 \cap K_{\theta_1}$ is $<_1$-unbounded in $J^1$.

(c) $J^1 \cap K_{\theta_1}$ is internal.

**Proof.** (a) can be proved as in Claim 4.5 using $f_{\theta_1}$ instead of $f_\alpha$. (b) is also clear as $J^1 \cap K_{\theta_1} \supseteq \{f_\alpha/D : \alpha < \theta_1\}$ and $\langle f_\alpha/D : \alpha < \theta_1 \rangle$ is $<_1$-unbounded in $J^1$. Let’s prove (c). As $\{s \in J^2 : s \preceq_1 s_*\} \cap K_{\theta_1} = \emptyset$ and $I = J^1 \cup J^2$, we can easily see that

$$J^1 \cap K_{\theta_1} = \{s \in K_{\theta_1} : s \preceq_1 s_*\}.$$ 

For each $i < \kappa$ set

$$L_i = \{s \in I_i : s <_{i}^2 f_{\theta_1}(i) \text{ and } s <_{i}^1 h_*(i)\}.$$ 

It follows that $J^1 \cap K_{\theta_1} = \prod_{i<\kappa} L_i/D$, and so $J^1 \cap K_{\theta_1}$ is internal. □

It follows from the above claim that $J^1 = \prod_{i<\kappa} L'_i/D$, where for $i < \kappa$, $L'_i = \{t \in I_i : \exists s \in L_i, t \preceq_i^1 s\}$. Hence $J^1$ is internal and so by Remark 4.2, $(J^1, J^2)$ is an internal cut of $I$ which completes the proof of Case 2. The theorem follows. □
5. $\mathcal{C}_{2^\kappa}(D)$ Contains No Non-Symmetric Pairs

In this section we show that if $D$ is a uniform ultrafilter on $\kappa$, then $\mathcal{C}_{2^\kappa}(D)$ does not contain any non-symmetric pairs. Again, we prove a stronger result from which the above claim, and hence Theorem 1.2(b) follows.

**Theorem 5.1.** Assume $D$ is a uniform ultrafilter on $\kappa$, $\langle I_i : i < \kappa \rangle$ is a sequence of linear orders and $I = \prod_{i<\kappa} I_i/D$. Also assume $(J^1, J^2)$ is a cut of $I$ of cofinality $(\theta_1, \theta_2)$, where $\theta_1 \neq \theta_2$ are bigger than $2^\kappa$. Then the cut $(J^1, J^2)$ is internal.

**Proof.** Let $<^1_i = <_{I_i}$ ($i < \kappa$) and $<_1 = <_I$. Let $<^2_i$, for $i < \kappa$, be a well-ordering of $I_i$ with a last element and let $<_2$ be such that $(I, <_2) = \prod_{i<\kappa} (I_i, <^2_i)/D$; so $<_2$ is a linear ordering of $I$ with a last element.

We say a sequence $\bar{K} = \langle K_i : i < \kappa \rangle$ catches $(J^1, J^2)$ if each $K_i \subseteq I_i$ is non-empty and for every $s_1 \in J^1$ and $s_2 \in J^2$, there is $t \in \prod_{i<\kappa} K_i/D$ such that $s_1 \leq_1 t \leq_1 s_2$. Set

$$S = \{ K : \bar{K} \text{ catches } (J^1, J^2) \},$$

and

$$C = \{ \bar{\mu} = \langle \mu_i : i < \kappa \rangle : \text{There exists } K \in S \text{ such that } \bigwedge_{i<\kappa} |K_i| = \mu_i \}. $$

We can define an order on $C$ by

$$\bar{\mu}^1 = \langle \mu_i^1 : i < \kappa \rangle <_D \bar{\mu}^2 = \langle \mu_i^2 : i < \kappa \rangle \iff \{ i < \kappa : \mu_i^1 < \mu_i^2 \} \in D.$$ 

Now consider the following statement:

\((*)\) There is $\bar{\mu} \in C$ which is $<_D$-minimal.

We consider two cases.

**Case 1.** \((*)\) holds: Fix $\bar{\mu} = \langle \mu_i : i < \kappa \rangle$ witnessing (*), and let $\bar{K} \in S$ be such that for all $i < \kappa$, $|K_i| = \mu_i$. Let $<_3^1$ be a well-ordering of $K_i$ of order type $\mu_i$ and let $<_3$ be such that $(K, <_3) = \prod_{i<\kappa} (K_i, <_3^i)/D$, where $K = \prod_{i<\kappa} K_i/D \subseteq I$. Let $\theta_3 = \text{cf}(\prod_{i<\kappa} \mu_i/D)$ and let $g_\alpha \in \prod_{i<\kappa} K_i$, $\alpha < \theta_3$, be such that $\langle g_\alpha/D : \alpha < \theta_3 \rangle$ is $<_3$-increasing and cofinal in $(K, <_3)$.

As $\theta_1 \neq \theta_2$, for some $l \in \{1, 2\}$, $\theta_3 \neq \theta_l$. Assume without loss of generality that $\theta_3 \neq \theta_1$.

For $\alpha < \theta_3$ and $i < \kappa$ set

$$K_{\alpha, i} = \{ s \in K_i : s <_3^i g_\alpha(i) \}.$$
and

\[ K_\alpha = K \mid \{ s : s \prec_3 g_\alpha / D \} = \prod_{i<\kappa} K_{\alpha,i} / D. \]

Then the sequence \( \langle K_\alpha : \alpha < \theta_3 \rangle \) is \( \subseteq \)-increasing and \( K = \bigcup_{\alpha<\theta_3} K_\alpha \). The next claim is evident from our construction.

**Claim 5.2.** \( K \) is internal.

By our choice of \( \bar{\mu} \), the sequence \( \langle K_{\alpha,i} : i<\kappa \rangle \) does not catch \( (J^1, J^2) \), and hence we can find \( s_\alpha \in J^1 \) and \( t_\alpha \in J^2 \) such that

\[ K_\alpha \cap \{ s \in I : s_\alpha \prec_1 s < t_\alpha \} = \emptyset. \]

As \( \theta_3 \neq \theta_1 \), there is \( s_* \in J^1 \) such that

\[ \sup \{ \alpha < \theta_3 : s_\alpha \leq_1 s_* \} = \theta_3. \]

It follows that \( K \cap \{ s \in J^1 : s_* \leq_1 s \} = \emptyset \). As \( K \) catches \( (J^1, J^2) \), it follows that \( J^2 \cap K \) is \( <_1 \)-cofinal in \( J^2 \) from below, and since \( K \) is internal, the arguments of section 2 show that \( J^2 \) is also internal, and hence by Remark 4.2, \( (J^1, J^2) \) is an internal cut of \( I \), as required.

**Case 2.** \( \ast \) fails: Clearly \( \prec_D \) is a linear order on \( C \), so it has a co-initiality, call it \( \theta_3 \). As \( \ast \) fails, \( \prec_D \) is not well-founded and so \( \theta_3 \geq \aleph_0 \).

**Claim 5.3.** \( \theta_3 \leq 2^\kappa \).

**Proof.** Suppose not. Let \( \langle \bar{\mu}_\xi : \xi < (2^\kappa)^+ \rangle \) be a \( \prec_D \)-decreasing chain of elements of \( C \). Define a partition \( F : [(2^\kappa)^+]^2 \to \kappa \) by

\[ F(\xi, \zeta) = \min \{ i < \kappa : \mu_1^{\xi} < \mu_1^{\zeta} \}, \]

which is well-defined as \( \{ i < \kappa : \mu_1^{\xi} < \mu_1^{\zeta} \} \in D \), in particular it is non-empty. By the Erdős-Rado partition theorem, there are \( X \subseteq (2^\kappa)^+ \) of size \( \kappa^+ \) and some fixed \( i_* < \kappa \) such that for all \( \xi < \zeta \) in \( X \), \( F(\xi, \zeta) = i_* \). Thus

\[ \xi < \zeta \in X \Rightarrow \mu_1^{\xi} < \mu_1^{\zeta}, \]

which is impossible. \( \square \)
Let $\langle \bar{\mu}_\xi = (\mu_{\xi,i} : i < \kappa) : \xi < \theta_3 \rangle$ be $<_D$-decreasing which is unbounded from below in $(C,<_D)$. For $\xi < \theta_3$ choose $K_\xi = \langle K_{\xi,i} : i < \kappa \rangle \in S$ such that for all $i < \kappa$, $|K_{\xi,i}| = \mu_{\xi,i}$.

Let $K_\xi = \prod_{i<\kappa} K_{\xi,i} / D \subseteq I$. We consider two subcases.

**Subcase 2.1.** For some $\xi < \theta_3$, $K_\xi \cap J^1$ is bounded in $(J^1,<_1)$: Fix such a $\xi < \theta_3$ and let $s_\kappa \in J^1$ be a bound. Then as $K_\xi$ catches $(J^1, J^2)$, it follows that $K_\xi \cap J^2$ is unbounded in $J^2$ from below. Since $K_\xi$ is internal and $K_\xi \cap J^2$ is unbounded in $J^2$ from below, so $J^2$ is internal. In follows that the cut $(J^1, J^2)$ is internal and we are done.

**Subcase 2.2.** For all $\xi < \theta_3$, $K_\xi \cap J^1$ is unbounded in $(J^1,<_1)$: Since $\text{cf}(J^1,<_1) = \theta_1$, there are functions $f_\alpha \in \prod_{i<\kappa} I_i$, for $\alpha < \theta_1$, such that

1. $\forall \alpha < \theta_1, f_\alpha / D \in J^1$,
2. $\langle f_\alpha / D : \alpha < \theta_1 \rangle$ is $<_1$-increasing,
3. $\langle f_\alpha / D : \alpha < \theta_1 \rangle$ is a $<_1$-cofinal subset of $J^1$.

For every $\alpha < \theta_1$ and $\xi < \theta_3$ there are $\beta$ and $g$ such that

4. $\alpha < \beta < \theta_1$,
5. $g \in \prod_{i<\kappa} K_{\xi,i}$,
6. $f_\alpha / D <_1 g / D <_1 f_\beta / D$.

For $\alpha < \beta < \theta_1$ set

$$\Lambda_{\alpha,\beta} = \{ (\xi,i) : f_\alpha(i) \leq^1 f_\beta(i) \text{ and there is } s \in K_{\xi,i} \text{ such that } f_\alpha(i) \leq^1 s \leq^1 f_\beta(i) \}.$$ 

For $i < \kappa$ set $\Lambda_{\alpha,\beta,i} = \{ \xi < \theta_3 : (\xi,i) \in \Lambda_{\alpha,\beta} \}$ and $\Xi_{\alpha,\beta} = \{ i < \kappa : \Lambda_{\alpha,\beta,i} \neq \emptyset \}$. Also let $F^1_{\alpha,\beta}, F^2_{\alpha,\beta}$ be functions with domain $\kappa$ such that

- If $i \in \Xi_{\alpha,\beta}$, then
  $$F^1_{\alpha,\beta}(i) = \min \{ \mu_{\xi,i} : (\xi,i) \in \Lambda_{\alpha,\beta} \},$$

  and

  $$F^2_{\alpha,\beta}(i) = \min \{ \xi : \mu_{\xi,i} = F^1_{\alpha,\beta}(i) \}.$$

- If $i \in \kappa \setminus \Xi_{\alpha,\beta}$, then $F^1_{\alpha,\beta}(i) = F^2_{\alpha,\beta}(i) = 0$.

**Claim 5.4.** For each $\alpha < \theta_1$ there exist $A_\alpha \subseteq (\alpha, \theta_1)$, functions $F^1_{\alpha}, F^2_{\alpha}$ and a set $\Xi_{\alpha}$ such that
(1) $A_\alpha = \{ \beta \in (\alpha, \theta_1) : F^1_{\alpha,\beta} = F^1_\alpha, F^2_{\alpha,\beta} = F^2_\alpha \text{ and } \Xi_{\alpha,\beta} = \Xi_\alpha \}.$

(2) $\sup(A_\alpha) = \theta_1.$

Proof. As $\theta_3 \leq 2^\kappa$, we have

$$|\{(F^1_{\alpha,\beta}, F^2_{\alpha,\beta}, \Xi_{\alpha,\beta}) : \alpha < \beta < \theta_1\}| \leq \theta_3^\kappa = 2^\kappa < \theta_1.$$ 

So there is an unbounded subset $A_\alpha$ of $\theta_1$ such that all tuples $(F^1_{\alpha,\beta}, F^2_{\alpha,\beta}, \Xi_{\alpha,\beta})$, $\beta \in A_\alpha$, are the same. The result follows immediately. □

The next claim can be proved in a similar way.

Claim 5.5. There are $A \subseteq \theta_1$, functions $F_1, F_2$ and a set $\Xi$ such that

(1) $A = \{ \alpha < \theta_1 : F^1_\alpha = F_1, F^2_\alpha = F_2 \text{ and } \Xi_\alpha = \Xi \}.$

(2) $\sup(A) = \theta_1.$

Let $\bar{K}^* = \langle K^*_i : i < \kappa \rangle$ where $K^*_i = K_{F_2(i),i}$ and let $\bar{\mu}^* = \langle \mu^*_i : i < \kappa \rangle$ be defined by $\mu^*_i = |K^*_i|$. Note that

$$\mu^*_i = |K^*_i| = |K_{F_2(i),i}| = \mu_{F_2(i),i} = F_1(i).$$

Claim 5.6. For every $\xi < \theta_3$, $\bar{\mu}^* \leq_D \bar{\mu}_\xi$.

Proof. Choose $\alpha \in A$ and $\beta \in A_\alpha$. So we have $F_1 = F^1_{\alpha,\beta}, F_2 = F^2_{\alpha,\beta}$ and $\Xi = \Xi_{\alpha,\beta}$. By the construction, there is $t \in K_\xi$ such that $f_\alpha/D <_1 t <_1 f_\beta/D$. Let $t = g/D$, where $g \in \prod_{i<\kappa} K_{\xi,i}$. Then

$$f_\alpha(i) <_1 g(i) <_1 f_\beta(i) \Rightarrow (\xi, i) \in A_{\alpha,\beta} \Rightarrow \mu^*_i = F_1(i) = F^1_{\alpha,\beta}(i) \leq \mu_{\xi,i}.$$

So

$$\{i < \kappa : \mu^*_i \leq \mu_{\xi,i}\} \supseteq \{i < \kappa : f_\alpha(i) <_1 g(i) <_1 f_\beta(i)\} \in D,$$

and the result follows. □

Claim 5.7. $\bar{\mu}^* \in C.$
Proof. We show that $\bar{K}^*$ catches $(J^1, J^2)$, so that $\bar{K}^* \in S$ witnesses $\bar{\mu}^* \in C$. So let $s_1 \in J^1$ and $s_2 \in J^2$. Pick $\alpha \in A$ such that $s_1 <_1 f_\alpha / D$. Let $\beta \in A_\alpha$. By our construction there is $g \in \prod_{\iota < \kappa} K^*_\iota$ such that $f_\alpha / D <_1 g / D <_1 f_\beta / D$ and hence

$$s_1 <_1 f_\alpha / D <_1 g / D <_1 f_\beta / D <_1 s_2.$$

The claim follows. □

But Claims 5.6 and 5.7 give us a contradiction to the choice of the sequence $\langle \bar{\mu}_\xi : \xi < \theta_3 \rangle$. This contradiction finishes the proof. □

References


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