Completeness of the provability logic GL with respect to the filter sequence of normal measures

Mohammad Golshani (IPM)

joint work with Reihane Zoghifard

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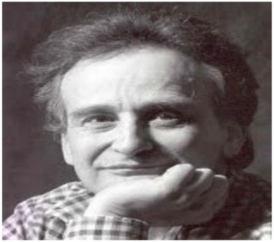
Gauss: Few, but ripe



Shelah: Few is beautiful

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- The Gödel-Löb provability logic **GL** deals with the study of modality $\Box$ as provability in a formal theory T such as Peano arithmetic. Then the dual-modal operator $\Diamond$ is interpreted as the consistency in T.

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- The system **GL** is defined by the following axioms schemata and rules:
  1. propositional tautologies,
  2. K. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,
  3. Löb. $\Box(\Box\varphi \to \varphi) \to \Box\varphi$,
  4. MP. $\vdash \varphi$, $\vdash \varphi \to \psi \Rightarrow \vdash \psi$,
  5. Nec. $\vdash \varphi \Rightarrow \Box\varphi$.

Provability logic GL

- Leo Esakia investigated the topological semantics for **GL** and perceived that the modal operator $\Diamond$ has the same behavior as the derivative operator in topological scattered spaces. Then he proved that **GL** is (strongly) complete with respect to the class of all scattered spaces.

Provability logic GL

- In 1990, Andreas Blass improved Esakia's result. He interpreted modal operators over filters associated with specific uncountable cardinals. He showed the soundness of **GL** concerning some natural classes of filters. Then he studied the completeness of **GL** for two classes of filters: end-segment filters and closed unbounded (club) filters.

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- He proved that (in *ZFC*) **GL** is complete with respect to end-segment filters.
- Then he proved the completeness of **GL** for club filters by assuming the Gödel's axiom of constructibility or more precisely, Jensen's square principle $\square_\kappa$ for all uncountable cardinals $\kappa < \aleph_\omega$. Building on some deep results of Harrington and Shelah, he also showed that the incompleteness of **GL** for club filters is equiconsistent with the existence of a Mahlo cardinal.

Provability logic GL

- By considering the topological interpretation of modal operators, the first Blass completeness result expresses the completeness of **GL** with respect to any ordinal $\alpha \geq \omega^\omega$ equipped with the interval (order) topology. This result was independently proved by Abashidze.

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- Using Blass's construction, Beklemishev showed the completeness of the bimodal provability logic, **GLB**, for any ordinal $\alpha \geq \aleph_\omega$ equipped with the interval and club topologies, under the assumption of the axiom of constructibility.

Provability logic GL

- Suppose that $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha : \alpha \in On \rangle$ is a family of filters where $\mathcal{F}_\alpha$ is a filter on α, for each $\alpha \in On$.

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- A valuation ν on this family is a function which assigns a class of ordinals to each propositional variable p. Then the valuation function ν is extended to all formulas by the standard rules for boolean connections and the following for $\Box$ operator:

$$\nu(\Box\varphi) = \{\alpha \mid \nu(\varphi) \in \mathcal{F}_\alpha\}.$$

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- Then we have

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- A formula φ is $\vec{\mathcal{F}}$-valid if for every valuation ν on $\vec{\mathcal{F}}$ we have $\nu(\varphi) = On$.

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- For each ordinal α, let $\mathcal{M}_\alpha$ be the intersection of all normal measures on α.
- (Blass) **GL** is sound with respect to the class of normal filters $\vec{\mathcal{M}}$.
- (Blass-1990) Is it consistent that **GL** is complete with respect to the class of normal filters $\vec{\mathcal{M}}$?
- (Joosten-Beklemishev-2012) Is **GL** complete w.r.t. the derivative operator of the topology corresponding to measurable filter, under suitable set-theoretic assumptions?

Trees $\mathbf{K}_n$

- For each fixed natural number n, the nodes of $\mathbf{K}_n$ consists of every finite sequences of pairs $\langle (i_1, j_1), \ldots, (i_k, j_k) \rangle$ where $n > i_1 > \cdots > i_k \geq 0$ and $j_1, \ldots, j_k \in \omega$ are arbitrary. The order of $\mathbf{K}_n$ is the end extension order, thus t extends s iff $s \lhd t$.

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- If $\mathbf{GL} \vdash \varphi$, then φ is valid in the root $\langle \rangle$ of $\mathbf{K}_n$ for every n.

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- If $\mathbf{GL} \vdash \varphi$, then φ is valid in the root $\langle \rangle$ of $\mathbf{K}_n$ for every n.
- (Blass) Let $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha : \alpha \in On \rangle$ be a family of filters $\mathcal{F}_\alpha$ on α. Suppose that for each $n < \omega$ there exists a function $\Gamma : \mathbf{K}_n \to \mathcal{P}(On)$ satisfying the following conditions:
  1. $\Gamma(\langle \rangle)$ in non-empty,
  2. if $s \neq t$ are in $\mathbf{K}_n$, then $\Gamma(s) \cap \Gamma(t)$ is empty,
  3. If $s \lhd t$ are in $\mathbf{K}_n$ and $\alpha \in \Gamma(s)$, then $\Gamma(t) \cap \alpha$ has positive measure with respect to $\mathcal{F}_\alpha$,
  4. If $s \in \mathbf{K}_n$ and $\alpha \in \Gamma(s)$, then $\bigcup_{s \lhd t} \Gamma(t) \cap \alpha \in \mathcal{F}_\alpha$.

  Then every $\vec{\mathcal{F}}$-valid modal formula is provable in $\mathbf{GL}$.

Mitchell order of normal measures

- (Mitchell) Suppose κ is a measurable cardinal and $\mathcal{U}, \mathcal{W}$ are normal measures on it. Then $\mathcal{W} \lhd \mathcal{U}$ if and only if $\mathcal{W} \in \mathrm{Ult}(V, \mathcal{U})$.

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- Mitchell proved that $\lhd$ is a well founded order now known as the Mitchell ordering. Thus given any normal measure $\mathcal{U}$ on κ, we can define its Mitchell order as

$$o(\mathcal{U}) = \sup\{o(\mathcal{W}) + 1 : \mathcal{W} \lhd \mathcal{U}\}.$$

The Mitchell order of κ is also defined as

$$o(\kappa) = \sup\{o(\mathcal{U}) + 1 : \mathcal{U} \text{ is a normal measure on } \kappa\}.$$

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$$o(\kappa) = \sup\{o(\mathcal{U}) + 1 : \mathcal{U} \text{ is a normal measure on } \kappa\}.$$

- Suppose κ is a measurable cardinal. Then

$$\lhd(\kappa) = (\{\mathcal{U} : \mathcal{U} \text{ is a normal measure on } \kappa\}, \lhd).$$

Mitchell order of normal measures

- (Omer Ben-Neria-2015) Let $V = L[E]$ be a core model. Suppose there is a strong cardinal κ and infinitely many measurable cardinals above it. Let $(\mathbf{S}, <)$ be a countable well-founded order of rank at most ω. Then there exists a generic extension V^* of V in which $\lhd(\kappa)^{V^*} \simeq (\mathbf{S}, <)$.

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- (Corollary to Ben-Neria's result) Let $V = L[E]$ be a core model. Suppose there is an ω-sequence $\langle \kappa_n : n < \omega \rangle$ of strong cardinals and suppose $\langle (\mathbf{S}_n, <_n) : n < \omega \rangle$ is a sequence of countable well-founded orders, each of rank at most ω. Then there exists a generic extension V^* of V in which for each $n < \omega$, $\lhd(\kappa_n)^{V^*} \simeq (\mathbf{S}_n, <_n)$.

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- The proof is by suitable Magidor iteration of Prikry type forcing notions, at stage n, making sure that $\lhd(\kappa_n) \simeq (\mathbf{S}_n, <_n)$.

Main result and its proof

- (G-Zoghifard) Assume there are infinitely many strong cardinals. Then there exists a generic extension of the canonical core model in which the provability logic **GL** is complete with respect to the filter sequence $\langle \mathcal{M}_\eta : \eta \in On \rangle$.

Main result and its proof

- (G-Zoghifard) Assume there are infinitely many strong cardinals. Then there exists a generic extension of the canonical core model in which the provability logic **GL** is complete with respect to the filter sequence $\langle \mathcal{M}_\eta : \eta \in On \rangle$.
- The rest of this talk is devoted to the main ideas of the proof of the above theorem.

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- The rest of this talk is devoted to the main ideas of the proof of the above theorem.
- As noticed by Blass, some large cardinals are needed to get the result. For example it implies the existence of measurable cardinals of any finite Mitchell order.

Main result and its proof

- Let $L[E]$ be the canonical extender model and suppose in it there is an ω sequence $\langle \kappa_n : 0 < n < \omega \rangle$ of strong cardinals.

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- We show that in V, the provability logic **GL** is complete with respect to the normal filter sequence.

Main result and its proof

- It suffices to show that for each $n < \omega$ there exists a function $\Gamma : \mathbf{K}_n \to \mathcal{P}(\kappa)$ satisfying the following conditions:

$(\dagger)_1$ $\Gamma(\langle\rangle)$ in non-empty,

$(\dagger)_2$ if $s \neq t$ are in $\mathbf{K}_n$, then $\Gamma(s) \cap \Gamma(t)$ is empty,

$(\dagger)_3$ If $s \lhd t$ are in $\mathbf{K}_n$ and $\eta \in \Gamma(s)$, then $\Gamma(t) \cap \eta$ has positive measure with respect to $\mathcal{M}_\eta$, i.e., $\Gamma(t) \cap \eta$ belongs to at least one normal measure on η,

$(\dagger)_4$ If $s \in \mathbf{K}_n$ is not maximal and $\eta \in \Gamma(s)$, then $\bigcup_{s \lhd t} \Gamma(t) \cap \eta \in \mathcal{M}_\eta$.

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- Pick sets A_ℓ so that:
  1. $(\beth)_1$ $A_\ell \in \mathcal{U}(\ell)$,
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- Then $\mathbf{S} = \{\langle (0, \ell) \rangle : \ell < \omega\}$, and in V, η has exactly ω-many normal measures $\mathcal{U}(\ell), \ell < \omega$, all of Mitchell order 0.
- Pick sets $A_{0,\ell}$ so that:
  1. $(\sqsupset)_1$ $A_\ell \in \mathcal{U}(\ell)$,
  2. $(\sqsupset)_2$ for all $\ell \neq \ell', A_\ell \cap A_{\ell'} = \emptyset$.
- Define $\Gamma : \mathbf{K}_1 \to \mathcal{P}(\kappa)$ by

$$\Gamma(s) = \begin{cases} \{\eta\} & \text{if } s = \langle\rangle, \\ A_\ell & \text{if } s = \langle (0, \ell) \rangle. \end{cases}$$

It is clear that Γ is as required.

Main result and its proof

- Now suppose that $n \geq 2$. Let $\mathbf{S} = \mathbf{S}_n$ and $\eta = \kappa_n$. Thus in V, $\lhd(\eta) \simeq \mathbf{S}$. Let

$$\lhd(\eta) = \{\mathcal{U}(s) : s \in \mathbf{S}\},$$

where for each $s, t \in \mathbf{S}$

$$t < s \iff \mathcal{U}(t) \lhd \mathcal{U}(s).$$

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- Pick the sets A_s for $s \in \mathbf{S}$ such that:
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- It is tempting to define the function Γ using the sets A_s, but this does not work.

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- It is tempting to define the function Γ using the sets A_s, but this does not work.
- For $t < s$ in $\mathbf{S}$, let $g^s_t : \eta \to V$ represents $\mathcal{U}(t)$ in the ultrapower by $\mathcal{U}(s)$, i.e., $\mathcal{U}(t) = [g^s_t]_{\mathcal{U}(s)}$.

Main result and its proof

- (Mitchell) Suppose $t < s$ are in $\mathbf{S}$ and $X \subseteq \eta$. Then

$$X \in \mathcal{U}(t) \iff \{\nu \in A_s : X \cap \nu \in g^s_t(\nu)\} \in \mathcal{U}(s).$$

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- Suppose $u < t < s$ are in $\mathbf{S}$. Then $A^1_{s,t,u} = \{\nu \in A_s : g^s_u(\nu) \lhd g^s_t(\nu)$ are normal measures on $\nu\} \in \mathcal{U}(s)$.

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- Suppose $u < t < s$ are in $\mathbf{S}$. Then $A^2_{s,t,u} = \{\nu \in A_s : g^s_u(\nu) = [g^t_u \restriction \nu]_{g^s_t(\nu)}\} \in \mathcal{U}(s)$.

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- Suppose $s \in \mathbf{S}$ is not minimal. Then

$$B_s = \{\nu \in A_s : \lhd(\nu) \simeq \mathbf{S}/(<s)\} \in \mathcal{U}(s).$$

Furthermore, for each $\nu \in B_s$,

$$\lhd(\nu) = \{g^s_t(\nu) : t < s\}.$$

Main result and its proof

- For each $s \in \mathbf{S}$, set

$$C_s = B_s \cap \bigcap_{u<t<s} A^1_{s,t,u} \cap \bigcap_{u<t<s} A^2_{s,t,u}.$$

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- By shrinking the sets $C_s, s \in \mathbf{S}$, we may assume that:
  - $(\sqsupset)_3$ for all $t < s$ in $\mathbf{S}$ and all $\nu \in C_s$, $C_t \cap \nu \in g^s_t(\nu)$.

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  1. $(\sqsupset)_3$ for all $t < s$ in $\mathbf{S}$ and all $\nu \in C_s$, $C_t \cap \nu \in g^s_t(\nu)$.
- Define $\Gamma : \mathbf{K}_n \to \mathcal{P}(\kappa)$ by

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- One can show that Γ satisfies the requirements $(\dagger)_1$-$(\dagger)_4$.

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$$C_s = B_s \cap \bigcap_{u<t<s} A^1_{s,t,u} \cap \bigcap_{u<t<s} A^2_{s,t,u}.$$

- By shrinking the sets C_s, $s \in \mathbf{S}$, we may assume that:
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- One can show that Γ satisfies the requirements $(\dagger)_1$-$(\dagger)_4$.
- Corollary. Assuming the existence of infinitely many strong cardinals $\langle \kappa_n : n < \omega \rangle$, it is consistent that **GL** is complete with respect to the ordinal space (α, τ_M), where $\alpha \geq \sup_{n<\omega} \kappa_n$ and τ_M is the derivative operator of the topology corresponding to the normal measure filter.

Thank You for your attention