SHELAH'S STRONG COVERING PROPERTY AND CH IN $V[r]$

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Abstract. In this paper we review Shelah's strong covering property and its applications. We also extend some of the results of Shelah and Woodin on the failure of CH by adding a real.

1. Introduction

In this paper we review Shelah's strong covering property and its applications, in particular, to pairs (W, V) of models of set theory with $V = W[r]$, for some real r. We also consider the consistency results of Shelah and Woodin on the failure of CH by adding a real and prove some related results. Some other results are obtained too.

The structure of the paper is as follows: In §2 we present an interesting result of Vanliere [6] on blowing up the continuum with a real. In §3 we give some applications of Shelah's strong covering property. In §4 we consider the work of Shelah and Woodin stated above and prove some new results. Finally in §5 we state some problems.

2. On a theorem of Vanliere

In this section we prove the following result of Vanliere [6]:

Theorem 2.1. *Assume $V = \mathbf{L}[X, a]$ where $X \subseteq \omega_n$ for some $n < \omega$, and $a \subseteq \omega$. If $\mathbf{L}[X] \models ZFC + GCH$ and the cardinals of $\mathbf{L}[X]$ are the true cardinals, then GCH holds in V.*

Proof. Let κ be an infinite cardinal. We prove the following:

$(*)_\kappa$: For any $Y \subseteq \kappa$ there is an ordinal $\alpha < \kappa^+$ and a set $Z \in \mathbf{L}[X], Z \subseteq \kappa$ such that $Y \in \mathbf{L}_\alpha[Z, a]$.

Then it will follow that $\mathcal{P}(\kappa) \subseteq \bigcup_{\alpha<\kappa^+} \bigcup_{Z \in \mathcal{P}^{\mathbf{L}[X]}(\kappa)} \mathbf{L}_\alpha[Z, a]$, and hence

The authors would like to thank Prof. Sy Friedman for some useful comments and remarks concerning §4. In particular Theorem 4.2 is suggested by him.

$$2^\kappa \leq \sum_{\alpha<\kappa^+} \sum_{Z \in \mathcal{P}^{\mathbf{L}[X]}(\kappa)} |\mathbf{L}_\alpha[Z,a]| \leq \kappa^+.(2^\kappa)^{\mathbf{L}[X]}.\kappa = \kappa^+$$

which gives the result. Now we return to the proof of $(*)_\kappa$.

Case 1. $\kappa \geq \aleph_n$.

Let $Y \subseteq \kappa$. Let θ be large enough regular such that $Y \in \mathbf{L}_\theta[X,a]$. Let $N \prec \mathbf{L}_\theta[X,a]$ be such that $|N| = \kappa, N \cap \kappa^+ \in \kappa^+$ and $\kappa \cup \{Y, X, a\} \subseteq N$. By the condensation lemma there are $\alpha < \kappa^+$ and π such that $\pi : N \cong \mathbf{L}_\alpha[X,a]$. then $Y = \pi(Y) \in \mathbf{L}_\alpha[X,a]$. Thus $(*)_\kappa$ follows.

Case 2. $\kappa < \aleph_n$.

We note that the above argument does not work in this case. Thus another approach is needed. To continue the work, we state a general result (again due to Vanliere) which is of interest in its own sake.

Lemma 2.2. *Suppose $\mu \leq \kappa < \lambda \leq \nu$ are infinite cardinals, λ regular. Suppose that $a \subseteq \mu, Y \subseteq \kappa, Z \subseteq \lambda$, and $X \subseteq \nu$ are such that $V = \mathbf{L}[X,a], Z \in \mathbf{L}[X], Y \in \mathbf{L}[Z,a]$ and $\lambda^+_{\mathbf{L}[X]} = \lambda^+$. Then there exists a proper initial segment $Z^{'}$ of Z such that $Z^{'} \in \mathbf{L}[X]$ and $Y \in \mathbf{L}[Z^{'},a]$.*

Proof. Let $\theta \geq \nu$ be regular such that $Y \in \mathbf{L}_\theta[Z,a]$. Let $N \prec \mathbf{L}_\theta[Z,a]$ be such that $|N| = \lambda, N \cap \lambda^+ \in \lambda^+$ and $\lambda \cup \{Y, Z, a\} \subseteq N$. By the condensation lemma we can find $\delta < \lambda^+$ and π such that $\pi : N \cong \mathbf{L}_\delta[Z,a]$.

In V, let $\langle M_i : i < \lambda \rangle$ be a continuous chain of elementary submodels of $\mathbf{L}_\delta[Z,a]$ with union $\mathbf{L}_\delta[Z,a]$ such that for each $i < \lambda, M_i \supseteq \kappa$, $|M_i| < \lambda$ and $M_i \cap \lambda \in \lambda$.

In $\mathbf{L}[Z]$ let $\langle W_i : i < \lambda \rangle$ be a continuous chain of elementary submodels of $\mathbf{L}_\delta[Z]$ with union $\mathbf{L}_\delta[Z]$ such that for each $i < \lambda, W_i \supseteq \kappa$, $|W_i| < \lambda$ and $W_i \cap \lambda \in \lambda$

Now we work in V. Let $E = \{i < \lambda : M_i \cap \mathbf{L}_\delta[Z] = W_i\}$. Then E is a club of λ. Pick $i \in E$ such that $Y \in M_i$, and let $M = M_i$, and $W = W_i$. By the condensation lemma let $\eta < \lambda$ and $\bar{\pi}$ be such that $\bar{\pi} : M \cong \mathbf{L}_\eta[Z^{'},a]$ where $Z^{'} = \bar{\pi}[M \cap Z] = \bar{\pi}[(M \cap \lambda) \cap Z] = (M \cap \lambda) \cap Z$, a proper initial segment of Z. Then $Y = \bar{\pi}(Y) \in \mathbf{L}_\eta[Z^{'},a]$ and $Z^{'} \subseteq \eta < \lambda$. It remains to observe that $Z^{'} \in \mathbf{L}[X]$ as $Z^{'}$ is an initial segment of Z. The lemma follows. □

We are now ready to complete the proof of Case 2. By Lemma 2.2 we can find a bounded subset X_n of ω_n such that $X_n \in \mathbf{L}[X]$ and $Y \in \mathbf{L}[X_n,a]$. Now trivially we can find a subset Z_{n-1} of ω_{n-1} such that $\mathbf{L}[X_n] = \mathbf{L}[Z_{n-1}]$, and hence $Z_{n-1} \in \mathbf{L}[X]$ and $Y \in \mathbf{L}[Z_{n-1},a]$.

Again by Lemma 2.2 we can find a bounded subset X_{n-1} of ω_{n-1} such that $X_{n-1} \in \mathbf{L}[X]$ and $Y \in \mathbf{L}[X_{n-1}, a]$, and then we find a subset Z_{n-2} of ω_{n-2} such that $\mathbf{L}[X_{n-1}] = \mathbf{L}[Z_{n-2}]$. In this way we can finally find a subset Z of κ such that $Z \in \mathbf{L}[X]$ and $Y \in \mathbf{L}[Z, a]$. Then as in case 1, for some $\alpha < \kappa^+, Y \in \mathbf{L}_\alpha[Z, a]$ and $(*)_\kappa$ follows. □

3. Shelah's strong covering property and its applications

In this section we study Shelah's strong covering property and give some of its applications. By a pair (W, V) we always mean a pair (W, V) of models of ZFC with the same ordinals such that $W \subseteq V$. Let us give the main definition.

Definition 3.1. (1) (W, V) *satisfies the strong* $(\lambda, \alpha)-$*covering property, where* λ *is a regular cardinal of* V *and* α *is an ordinal, if for every model* $M \in V$ *with universe* α *(in a countable language) and* $a \subseteq \alpha, |a| < \lambda$ *(in* V*), there is* $b \in W$ *such that* $a \subseteq b \subseteq \alpha, b \prec M$*, and* $|b| < \lambda$ *(in* V*).* (W, V) *satisfies the strong* $\lambda-$*covering property if it satisfies the strong* $(\lambda, \alpha)-$*covering property for every* α*.*

(2) (W, V) *satisfies the strong* $(\lambda^*, \lambda, \kappa, \mu)-$*covering property, where* $\lambda^* \geq \lambda \geq \kappa$ *are regular cardinals of* V *and* μ *is an ordinal, if player one has a winning strategy in the following game, called the* $(\lambda^*, \lambda, \kappa, \mu)-$*covering game, of length* λ*:*

In the $i-$*th move player I chooses* $a_i \in V$ *such that* $a_i \subseteq \mu, |a_i| < \lambda^*$ *(in* V*) and* $\bigcup_{j<i} b_j \subseteq a_i$*, and player II chooses* $b_i \in V$ *such that* $b_i \subseteq \mu, |b_i| < \lambda^*$ *(in* V*) and* $\bigcup_{j \leq i} a_j \subseteq b_i$*.*

Player I wins if there is a club $C \subseteq \lambda$ *such that for every* $\delta \in C \cup \{\lambda\}, cf(\delta) = \kappa \Rightarrow \bigcup_{i<\delta} a_i \in W$*.* (W, V) *satisfies the strong* $(\lambda^*, \lambda, \kappa, \infty)-$*covering property, if it satisfies the strong* $(\lambda^*, \lambda, \kappa, \mu)-$*covering property for every* μ*.*

The following theorem shows the importance of the first part of this definition and plays an important role in this section.

Theorem 3.2. *Suppose* $V = W[r], r$ *a real and* (W, V) *satisfies the strong* $(\lambda, \alpha)-$*covering property for* $\alpha < ([(2^{<\lambda})^W]^+)^V$*. Then* $(2^{<\lambda})^V = |(2^{<\lambda})^W|^V$*.*

Proof. Cf. [3, Theorem VII.4.5.]. □

It follows from Theorem 3.2 that if $V = W[r], r$ a real and (W, V) satisfies the strong $(\lambda^+, ([(2^\lambda)^W]^+)^V)-$covering property, then $(2^\lambda)^V = |(2^\lambda)^W|^V$.

We are now ready to give the applications of the strong covering property. For a pair (W,V) of models of ZFC consider the following conditions:

$(1)_\kappa$: • $V=W[r], r$ a real,

• V and W have the same cardinals $\leq \kappa^+$,

• $W \models \forall\lambda \leq \kappa, 2^\lambda = \lambda^+$

• $V \models 2^\kappa > \kappa^+$.

$(2)_\kappa$: $W \models GCH$.

$(3)_\kappa$: V and W have the same cardinals.

Theorem 3.3. (1) *Suppose there is a pair (W,V) satisfying $(1)_{\aleph_0}$ and $(2)_{\aleph_0}$. Then $\aleph_2^V$ in inaccessible in* $\mathbf{L}$.

(2) *Suppose there is a pair (W,V) as in (1) with $V \models 2^{\aleph_0} > \aleph_2$. Then $0^\sharp \in V$.*

(3) *Suppose there is a pair (W,V) as in (1) with $CARD^W \cap (\aleph_1^V, \aleph_2^V) = \emptyset$. Then $0^\sharp \in V$.*

(4) *Suppose $\kappa > \aleph_0$ and there is a pair (W,V) satisfying $(1)_\kappa$. Then $0^\sharp \in V$.*

Before we give the proof of Theorem 3.3 we state some conditions which imply Shelah's strong covering property. Suppose that in $V, 0^\sharp$ does not exist. Then:

(α) If $\lambda^* \geq \aleph_2^V$ is regular in V, then (W,V) satisfies the strong λ^*-covering property.

(β) If $CARD^W \cap (\aleph_1^V, \aleph_2^V) = \emptyset$ then (W,V) satisfies the strong $\aleph_1^V-$covering property.

Remark 3.4. *For $\lambda^* \geq \aleph_3^V, (\alpha)$ follows from [3, Theorem VII.2.6], and (β) follows from [3, Theorem VII.2.8]. In order to obtain (α) for $\lambda^* = \aleph_2^V$ we can proceed as follows: As in the proof of [3, Theorem VII.2.6] proceed by induction on μ to show that $(\mathbf{L}, V)$ satisfies the strong $(\aleph_2^V, \aleph_1^V, \aleph_0^V, \mu)-$covering property. For successor μ (in $\mathbf{L}$) use [3, Lemma VII.2.2] and for limit μ use [3, Remark VII.2.4] (instead of [3, Lemma VII.2.3]). It then follows that $(\mathbf{L}, V)$ and hence (W,V) satisfies the strong $\aleph_2^V-$covering property.*

Proof. (1) We may suppose that $0^\sharp \notin V$. Then by $(\alpha), (W,V)$ satisfies the strong $\aleph_2^V-$covering property. On the other hand by Jensen's covering lemma and [3, Claim VII.1.11], W has squares. By [3, Theorem VII.4.10], $\aleph_2^V$ is inaccessible in W, and hence in $\mathbf{L}$.

(2) Suppose not. Then by $(\alpha), (W, V)$ satisfies the strong $\aleph_2^V-$covering property. By Theorem 3.2, $(2^{\aleph_0})^V \leq (2^{\aleph_1})^V = |(2^{\aleph_1})^W|^V = |\aleph_2^W| = \aleph_2^V$, which is a contradiction.

(3) Suppose not. Then by $(\beta), (W, V)$ satisfies the strong $\aleph_1^V-$covering property, hence by Theorem 3.2, $(2^{\aleph_0})^V = |(2^{\aleph_0})^W|^V = \aleph_1^V$, which is a contradiction.

(4) Suppose not. Then by $(\alpha), (W, V)$ satisfies the strong κ^+-covering property. By Theorem 3.2, $(2^\kappa)^V = |(2^\kappa)^W|^V = \kappa^+$, and we get a contradiction. □

Theorem 3.5. (1) *Suppose there is a pair* (W, V) *satisfying* $(1)_\kappa, (2)_\kappa$ *and* $(3)_\kappa$. *Then there is in* V *an inner model with a measurable cardinal.*

(2) *Suppose there is a pair* (W, V) *satisfying* $(1)_\kappa$, *where* $\kappa \geq \aleph_\omega$. *Further suppose that* $\kappa_W^{++} = \kappa_V^{++}$ *and* (W, V) *satisfies the* κ^+-*covering property. Then there is in* V *an inner model with a measurable cardinal.*

Proof. (1) Suppose not. Then by [3, conclusion VII.4.3(2)], (W, V) satisfies the strong κ^+-covering property, hence by Theorem 3.2, $(2^\kappa)^V = |(2^\kappa)^W|^V = \kappa^+$, which is a contradiction.

(2) Suppose not. Let $\kappa = \mu^{+n}$, where μ is a limit cardinal, and $n < \omega$. By [3, Theorem VII.2.6, Theorem VII.4.2(2) and Conclusion VII.4.3(3)], we can show that (W, V) satisfies the strong $(\kappa^+, \kappa, \aleph_1, \mu)-$covering property. On the other hand since (W, V) satisfies the κ^+-covering property and V and W have the same cardinals $\leq \kappa^+, (W, V)$ satisfies the $\mu^{+i}-$covering property for each $i \leq n + 1$. By repeatedly use of [3, Lemma VII.2.2], (W, V) satisfies the strong $(\kappa^+, \kappa, \aleph_1, \kappa^{++})-$covering property, and hence the strong $(\kappa^+, \kappa^{++})-$covering property. By Theorem 3.2, $(2^\kappa)^V = |(2^\kappa)^W|^V = \kappa^+$, which is a contradiction. □

Remark 3.6. *In [3] (see also [4]), Theorem 3.4(1), for* $\kappa = \aleph_0$, *is stated under the additional assumption* $2^{\aleph_0} > \aleph_\omega$ *in* V.

Let us close this section by noting that the hypotheses in [3, Conclusion VII.4.6] are inconsistent. In other words we are going to show that the following hypotheses are not consistent:

(a) V has no inner model with a measurable cardinal.

(b) $V = W[r], r$ a real, V and W have the same cardinals $\leq \lambda$, where $(2^{\aleph_0})^V \geq \lambda \geq \aleph_\omega^V, \lambda$ is a limit cardinal.

(c) $W \models 2^{\aleph_0} < \lambda$.

To see this, note that by (b) and [3, Theorem VII.4.2], $K_\lambda(W) = K_\lambda(V)$. Then by [3, conclusion VII.4.3(3)], (W, V) satisfies the strong $(\aleph_1, \lambda)-$covering property. On the other hand $\lambda > ([(2^{\aleph_0})^W]^+)^V$, and hence by Theorem 3.2, $\lambda \leq (2^{\aleph_0})^V = |(2^{\aleph_0})^W|^V < \lambda$. Contradiction.

4. Some consistency results

In this section we consider the work of Shelah and Woodin in [4] and prove some related results.

Theorem 4.1. *There is a generic extension W of L and two reals a and b such that:*

(a) *Both of $W[a]$ and $W[b]$ satisfy CH.*

(b) *CH fails in $W[a, b]$.*

Furthermore $2^{\aleph_0}$ can be arbitrary large in $W[a, b]$.

Proof. Let $\lambda \geq \aleph_5^{\mathbf{L}}$ be regular in $\mathbf{L}$. By [4, Theorem 1] there is a pair (W, V) of generic extensions of $\mathbf{L}$ such that:

- (W, V) satisfies $(1)_{\aleph_0}$.
- $V \models 2^{\aleph_0} = \lambda$.

Let $V = W[r]$ where r is a real. Working in V, let $\mathrm{P} = Col(\aleph_0, \aleph_1)$ and let G be $\mathrm{P}-$generic over V. In $V[G]$ the set $\{D \in V : D$ is open dense in $Add(\omega, 1)\}$ is countable, hence we can easily find two reals a and b in $V[G]$ such that both of a and b are $Add(\omega, 1)-$generic over V, and $r \in L[a, b]$. Then the model W and the reals a and b are as required. $\square$

Note that for $\kappa > \aleph_0$, by Theorem 3.3(4) we can not expect to obtain [4, Theorems 1 and 2] without assuming the existence of $0^\sharp$. However it is natural to ask whether it is possible to extend them under the assumption "$0^\sharp$ exists". The following result (Cf. [1, Lemma 1.6]) shows that for $\kappa > \aleph_0$, there is no pair (W, V) satisfying $(1)_\kappa$ such that $0^\sharp \notin W \subseteq \mathbf{L}[0^\sharp]$ and W and $\mathbf{L}[0^\sharp]$ have the same cardinals $\leq \kappa^+$.

Theorem 4.2. *Let $\kappa = \aleph_1^V$. If $0^\sharp$ exists and M is an inner model in which $\kappa_{\mathbf{L}}^+$ is collapsed, then $0^\sharp \in M$.*

Proof. Let I be the class of Silver indiscernibles. There are constructible clubs $C_n, n < \omega$, such that $I \cap \kappa = \bigcap_{n<\omega} C_n$. if $\kappa_{\mathbf{L}}^+$ is collapsed in M, then in M there is a club C of κ, almost contained in every constructible club. Hence C is almost contained in the $\bigcap_{n<\omega} C_n$ and hence in $I \cap \kappa$. It follows that $0^\sharp \in M$. □

We now prove a strengthening of Theorem 4.1 under stronger hypotheses.

Theorem 4.3. *Suppose $cf(\lambda) > \aleph_0$, there are $\lambda-$many measurable cardinals and GCH holds. Then there is a cardinal preserving generic extension W of the universe and two reals a and b such that:*

- (a) *The models $W, W[a], W[b]$ and $W[a,b]$ have the same cardinals.*
- (b) *$W[a]$ and $W[b]$ satisfy GCH.*
- (c) *$W[a,b] \models 2^{\aleph_0} = \lambda$.*

Proof. By [4, Theorem 4] there is a pair (W, V) of cardinal preserving generic extensions of the universe such that:

- (W, V) satisfies $(1)_{\aleph_0}, (2)_{\aleph_0}$ and $(3)_{\aleph_0}$.
- $V \models 2^{\aleph_0} = \lambda$.

Working in V, let $\mathrm{P} = Col(\aleph_0, \aleph_1)$ and let G be $\mathrm{P}-$generic over V. As in the proof of Theorem 4.1 we can find two reals a^* and b^* such that a^* is $Add(\omega, 1)-$generic over V and b^* is $Add(\omega, 1)-$generic over $V[a^*]$, where $Add(\omega, 1)$ is the Cohen forcing for adding a new real. Note that $V[a^*]$ and $V[a^*, b^*]$ are cardinal preserving generic extensions of V. Working in $V[a^*, b^*]$ let $\langle k_N : N < \omega \rangle$ be an increasing enumeration of $\{N : a^*(N) = 0\}$ and let $a = a^*$ and $b = \{N : b^*(N) = a^*(N) = 1\} \cup \{k_N : r(N) = 1\}$ where $V = W[r]$. Then clearly $r \in \mathbf{L}[\langle k_N : N < \omega \rangle, b] \subseteq \mathbf{L}[a, b]$ as $r = \{N : k_N \in b\}$.

We show that b is $Add(\omega, 1)-$generic over V. It suffices to prove the following:

$(*)$ For any $(p, q) \in Add(\omega, 1) * \underset{\sim}{A}dd(\omega, 1)$ and any open dense subset $D \in V$ of $Add(\omega, 1)$ there is $(\bar{p}, \bar{q}) \leq (p, q)$ such that $(\bar{p}, \bar{q}) \Vdash$ "$\dot{b}$ extends some element of D".

Let (p, q) and D be as above. By extending one of p or q if necessary, we can assume that $lh(p) = lh(q)$. Let $\langle k_N : N < M \rangle$ be an increasing enumeration of $\{N < lh(p) : p(N) = 0\}$. Let $s : lh(p) \to 2$ be such that considered as a subset of ω,

$$s = \{N < lh(p) : p(N) = q(N) = 1\} \cup \{k_N : N < M, r(N) = 1\}.$$

Let $t \in D$ be such that $t \leq s$.

Claim 4.4. *There is an extension $(\bar{p}, \bar{q})$ of (p, q) such that $lh(\bar{p}) = lh(\bar{q}) = lh(t)$ and*

$$t = \{N < lh(t) : \bar{p}(N) = \bar{q}(N) = 1\} \cup \{k_N : N < \bar{M}, r(N) = 1\},$$

where $\langle k_N : N < \bar{M} \rangle$ is an increasing enumeration of $\{N < lh(\bar{p}) : \bar{p}(N) = 0\}$.

Proof. Extend p, q to $\bar{p}, \bar{q}$ of length $lh(t)$ so that for i in the interval $[lh(s), lh(t))$

- $\bar{p}(i) = 1$,
- $\bar{q}(i) = 1$ iff $i \in t$.

Then

$$t = \{N < lh(t) : \bar{p}(N) = \bar{q}(N) = 1\} \cup \{k_N : N < M, r(N) = 1\}.$$

But (using our definitions) $M = \bar{M}$ so

$$t = \{N < lh(t) : \bar{p}(N) = \bar{q}(N) = 1\} \cup \{k_N : N < \bar{M}, r(N) = 1\}.$$

as desired. □

It follows that

$$(\bar{p}, \bar{q}) \Vdash \dot{b} \text{ extends t}$$

and $(*)$ follows.

It follows that a and b are $Add(\omega, 1)-$generic over W and $r \in \mathbf{L}[a, b]$. Hence the model W and the reals a and b are as required and the theorem follows. □

Remark 4.5. *The above kind of argument is widely used in [2] to prove the genericity of a $\lambda-$sequence of reals over $Add(\omega, \lambda)$, the Cohen forcing for adding $\lambda-$many new reals.*

5. Open problems

We close this paper by some remarks and open problems. Our first problem is related to Vanliere's Theorem.

Problem 5.1. *Find the least κ such that there are $X \subseteq \kappa$ and $a \subseteq \omega$ such that $\mathbf{L}[X] \models ZFC + GCH, \mathbf{L}[X]$ and $\mathbf{L}[X,a]$ have the same cardinals and $\mathbf{L}[X,a] \models 2^{\aleph_0} > \aleph_1$*

Now consider the following property:

(*): If P is a non-trivial forcing notion and G is P$-$generic over V, then for any cardinal $\chi \geq \aleph_2^V$ and $x \in H^{V[G]}(\chi)$, there is $N \prec \langle H^{V[G]}(\chi), \in, <^*_\chi, H^V(\chi)\rangle$, such that $x \in N$ and $N \cap H^V(\chi) \in V$, where $<^*_\chi$ is a well-ordering of $H^V(\chi)$.

Note that if $(*)$ holds, and $p \in G$ is such that p forces "$x \in N$ and $N \cap H^V(\chi) \in V$" and decides a value for $N \cap H^V(\chi)$, then p is $(N \cap H^V(\chi), \mathrm{P})-$generic . Using Shelah's work on strong covering property we can easily show that:

(a) If $0^\sharp \notin V$, then $(*)$ holds.

(b) If in V there is no inner model with a measurable cardinal, then $(*)$ holds for any cardinal preserving forcing notion P.

Now we state the following problem:

Problem 5.2. *Suppose $0^\sharp \in V$. Does $(*)$ fail for any non-trivial constructible forcing notion P with $o^{\mathbf{L}}(P) \geq \omega_1^V$, where $o^{\mathbf{L}}(P)$ is the least β such that forcing with P over $\mathbf{L}$ adds a new subset to β.*

This problem is motivated by the fact that if $0^\sharp \in V$, then for any non-trivial constructible forcing notion P, forcing with P over V collapses $o^{\mathbf{L}}(\mathrm{P})$ into ω (Cf. [5]).

Problem 5.3. *Assume $V \models GCH, \lambda$ is a cardinal in $V, A \subseteq \lambda$ and $V[A]$ is a model of set theory with the same cardinals as V. Can we have more than $\lambda-$many reals in $H^V(\lambda^+)[A]$.*

Let us note that if λ is regular in V, then the answer is no, and if λ has countable cofinality in V, then the answer is yes. Also if there is a stationary subset of $[\lambda]^{\leq\aleph_0}$ in $V[A]$ of size $\leq \lambda$, then the answer is no (Cf. [3, Theorem VII.0.5(1)]). Let us note that the Theorem as stated in [3] is wrong. The conclusion should be: There are $\leq \lambda$ many reals in $H^V(\lambda^+)[A]$).

Concerning Problem 5.3, the main question is for $\lambda = \aleph_{\omega_1}$. We restate it for this special case.

Problem 5.4. *(Mathias). Is Problem 5.3 true for* $\lambda = \aleph_{\omega_1}$.

References

[1] Sy D. Friedman, $0^\sharp$ and inner models, JSL 67(3), 924-932 (2002).

[2] M. Gitik, M. Golshani, Adding a lot of Cohen reals by adding a few I, to appear.

[3] S. Shelah, Cardinal Arithmetic. Oxford Logic Guides, 1994.

[4] S. Shelah, H. Woodin, Forcing the failure of CH by adding a real, JSL 49 (4), 1185-1189 (1984).

[5] M. Stanley, Forcing disabled, JSL 57(4), 1153-1175, (1992).

[6] M. B. Vanliere, Splitting the reals into two small pieces, PhD thesis, University of California, Berkeley, 1982.