ON A QUESTION OF SILVER ABOUT GAP-TWO CARDINAL TRANSFER PRINCIPLES

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Abstract. Assuming the existence of a Mahlo cardinal, we produce a generic extension of Gödel's constructible universe L, in which the GCH holds and the transfer principles $(\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1)$ and $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$ fail simultaneously. The result answers a question of Silver from 1971. We also extend our result to higher gaps.

1. Introduction

In this paper we study cardinal transfer principles introduced by Vaught [6], [7], and prove some consistency results related to them.

Assume $\mathcal{L}$ is a first order language which contains a unary predicate U. By a (κ, λ)-model for $\mathcal{L}$, we mean a model $\mathcal{M} = (M, U^{\mathcal{M}}, \dots)$, where $|M| = \kappa$ and $|U^{\mathcal{M}}| = \lambda$, where $U^{\mathcal{M}}$ is the interpretation of U in $\mathcal{M}$. Following Devlin [2], we use the notation

$$(\kappa, \lambda) \to (\kappa', \lambda')$$

to mean the following transfer principle:

> For every countable first order language $\mathcal{L}$ as above, and every first order theory T of $\mathcal{L}$, if T has a (κ, λ)-model, then it has a (κ', λ')-model.

For any natural number $n \geq 1$, by the *gap-n-cardinal transfer principle* we mean the statement

$$\forall \kappa \ \forall \lambda \ (\kappa^{+n}, \kappa) \to (\lambda^{+n}, \lambda).$$

In [5], Silver proved the independence of gap-2-cardinal transfer principle. Starting from an inaccessible cardinal, he was able to produce a model in which the cardinal transfer $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$ fails. His proof is simply as follows: By a result of Vaught [7], there

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exists a sentence ϕ_{KH} in a suitable first order language, such that for any infinite cardinal β,

$$\phi_{KH} \text{ has a } (\beta^{++}, \beta)\text{-model} \iff \text{there exists a } \beta^{+}\text{-Kurepa tree.}$$

Now, starting from an inaccessible cardinal κ, Silver shows that in the generic extension by the Levy collapse $\mathrm{Col}(\aleph_1, < \kappa)$, there are no $\aleph_1$-Kurepa trees. If we start with $V = L$, then in the resulting extension, there are $\aleph_2$-Kurepa trees, and so the transfer principle $(\aleph_3, \aleph_1) \rightarrow (\aleph_2, \aleph_0)$ fails in it. Similarly if we force with $\mathrm{Col}(\aleph_2, < \kappa)$, then in the extension there are no $\aleph_2$-Kurepa trees, and we can use it to prove the independence of $(\aleph_2, \aleph_0) \rightarrow (\aleph_3, \aleph_1)$. The following question was asked by Silver [5].

Question 1.1. [1] *Is it consistent with GCH that both transfer principles* $(\aleph_3, \aleph_1) \rightarrow (\aleph_2, \aleph_0)$ *and* $(\aleph_2, \aleph_0) \rightarrow (\aleph_3, \aleph_1)$ *fail simultaneously?*

Remark 1.2. *If we drop the GCH assumption from the question, then one can easily answer the above question. Assume κ is an inaccessible cardinals and let $G * H$ be* $\mathrm{Col}(\aleph_1, < \kappa) * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$*-generic over L. In the generic extension $L[G * H]$ there are no $\aleph_1$-Kurepa trees (see Devlin* [3]*) but there exists an $\aleph_2$-Kurepa tree, and hence by the remarks above, the transfer principle* $(\aleph_3, \aleph_1) \rightarrow (\aleph_2, \aleph_0)$ *fails in* $L[G * H]$.

On the other hand $L[G * H]$ satisfies "$2^{\aleph_0} = 2^{\aleph_1} = \kappa = \aleph_2$". Let $\mathcal{L} = (U, F)$, where U is a unary predicate symbol and F is a binary predicate symbol. let T be an $\mathcal{L}$-theory which says the following:

(1) *$\forall x, y \ F(x, y) \rightarrow U(y)$. In particular, for each x, F determines a subset F_x of U, namely, $F_x = \{y : F(x, y)\}$.*
(2) *For all $x \neq x', F_x \neq F_y$.*

Then T has an $(\aleph_2, \aleph_0)$ *model but it does not have an* $(\aleph_3, \aleph_1)$*-model (as otherwise we should have $2^{\aleph_1} \geq \aleph_3$). Thus the transfer principle* $(\aleph_2, \aleph_0) \rightarrow (\aleph_3, \aleph_1)$ *fails in* $L[G * H]$.

We give an affirmative answer to this question by proving the following theorem:

[1] On page 388 of [5], Silver writes "One can also get a *GCH* model in which $(\aleph_7, \aleph_5) \rightarrow (\aleph_3, \aleph_1)$ fails and a *GCH* model which $(\aleph_3, \aleph_1) \rightarrow (\aleph_7, \aleph_5)$ fails (though I don't see how to get the $\rightarrow$ both ways to fail simultaneously)".

Theorem 1.3. *Assume κ is a Mahlo cardinal. Then there is a generic extension of L, the Gödel's constructible universe, in which the GCH holds and the cardinal transfer principles $(\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1)$ and $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$ fail.*

Then we prove a general model theoretic fact, and use it to extend the above result to higher gaps:

Theorem 1.4. *Assume κ is a Mahlo cardinal. Then there is a generic extension of L in which the GCH holds and for all $n \geq 2$, the cardinal transfer principles $(\aleph_n, \aleph_0) \to (\aleph_{n+1}, \aleph_1)$ and $(\aleph_{n+1}, \aleph_1) \to (\aleph_n, \aleph_0)$ fail.*

Remark 1.5. *Our proofs can be easily extended to get the following consistency result: assume $\alpha < \beta$ are regular cardinals and assume there exists a Mahlo cardinal above them. Then in a generic extension of L, the GCH holds and both transfer principles $(\alpha^{+n}, \alpha) \to (\beta^{+n}, \beta)$ and $(\beta^{+n}, \beta) \to (\alpha^{+n}, \alpha)$ fail.*

In Section 2 we prove Theorem 1.3 and in Section 3, we prove Theorem 1.4. In the last section, we discuss the same problem for the case of gap-1.

2. Proof of Theorem 1.3

In this section we prove Theorem 1.3.

2.1. **On a result of Jensen.** In this subsection we state a result of Jensen [4] and mention some of its basic properties which are needed. Let $\mathcal{L} = \{\in, A, \mathcal{C}\}$, where A is a unary predicate and $\mathcal{C}$ is a function symbol. Let T_J be the following theory in $\mathcal{L}$:

$$\text{“}ZFC^{-}+GCH + A^{+} \text{ is the largest cardinal}+\mathcal{C} \text{ is a } \Box_{A^{+}}\text{-sequence”}.$$

By a (κ, λ)-model of T_J we mean a model $\mathcal{M} = (M, \in^{\mathcal{M}}, A^{\mathcal{M}}, \mathcal{C}^{\mathcal{M}})$ of T_J, where $|M| = \kappa$ and $|A^{\mathcal{M}}| = \lambda$.

Theorem 2.1. *(Jensen [4]) Assume $GCH + \Diamond_{\beta^+}$ holds, where β is a regular cardinal, and suppose $\kappa > \beta$ is a Mahlo cardinal. Then there is a forcing notion $\mathbb{P}_{\beta,\kappa}$ such that if K is $\mathbb{P}_{\beta,\kappa}$-generic over V, then the following hold in $V[K]$:*

(a) $V[K] \models$ *“GCH”.*

(b) *The principle* $\diamondsuit^+_{\beta^+}$ *holds.*

(c) *The theory* T_J *does not have any* (β^{++}, β)*-model.*

Proof. As requested by the referees, we sketch the proof of the theorem, by providing the forcing construction $\mathbb{P}_{\beta,\kappa}$, and refer to [4] for details. Let G be $\mathrm{Col}(\beta^+, <\kappa)$-generic over V, where

$$\mathrm{Col}(\beta^+, <\kappa) = \{p : \beta^+ \times \kappa \to \kappa : |p| \leq \beta \text{ and for all } (\alpha, \lambda) \in \mathrm{dom}(p),\ p(\alpha, \lambda) < \lambda\}$$

is the Levy collapse. The next claim is standard.

Claim 2.2. (a) *The forcing* $\mathrm{Col}(\beta^+, <\kappa)$ *is* β^+*-closed and* κ*-c.c.*

(b) $V[G] \models$ "$GCH + \diamondsuit_{\beta^+}$".

(c) $V[G] \models$ "$\kappa = \beta^{++}$ *and* $\square_{\beta^{++}}$ *fails*".

In [4], the following strengthening of Claim 2.2(c) is proved.

Claim 2.3. *In* $V[G]$, *the theory* T_J *has no* (β^{++}, β)*-model*

From now on we work in $V[G]$. Let $\mathcal{S} = \langle S_\alpha : \alpha < \beta^+ \rangle$ witness $\diamondsuit_{\beta^+}$. For each $\alpha < \beta^+$ let $d_\alpha : \beta \to \alpha$ be an onto function and set $d = \langle d_\alpha : \alpha < \beta^+ \rangle$. For $\alpha < \beta^+$ set

$$M_\alpha = L_{\gamma_\alpha}[\mathcal{S} \restriction \alpha + 1, d \restriction \alpha + 1],$$

where γ_α is the least ordinal $\gamma > \alpha$ such that $\gamma > \sup_{\nu < \alpha} \gamma_\nu$ and

$$L_\gamma[\mathcal{S} \restriction \alpha + 1, d \restriction \alpha + 1] \models \text{ "}ZFC^-\text{"}.$$

Define

$$\mathcal{S}^* = \langle S^*_\alpha : \alpha < \beta^+ \rangle,$$

where $S^*_\alpha = P(\alpha) \cap M_\alpha$. We find a generic extension of $V[G]$ in which $\mathcal{S}^*$ is a $\diamondsuit^+_{\beta^+}$-sequence.

Let $A \subseteq \kappa$ be such that $L_\kappa[A] = H(\kappa)$ and define the sequence $\langle \rho_\nu : \nu < \kappa \rangle$ by recursion on ν as follows: ρ_ν is the least ordinal $\rho > \beta^+$ such that

- $\rho > \sup_{\xi < \nu} \rho_\xi$.
- $\langle M_\alpha : \alpha < \beta^+ \rangle \in L_\rho[A]$.
- $cf(\rho) = \beta^+$.

- $L_\rho[A] \models$"$ZFC^- + \forall x, |x| \leq \beta^+$.

Set $\tilde{\rho}_\nu = \beta^+ \cup \sup_{\xi<\nu} \rho_\xi$,

$$\mathcal{U}_\nu = \langle L_{\rho_\nu}[A], \in, A \cap \rho_\nu, \langle M_\alpha : \alpha < \beta^+ \rangle \rangle,$$

and for $\nu > 0$ set

$$\tilde{\mathcal{U}}_\nu = \bigcup_{\xi<\nu} \tilde{\mathcal{U}}_\xi = \langle L_{\tilde{\rho}_\nu}[A], \in, A \cap \tilde{\rho}_\nu, \langle M_\alpha : \alpha < \beta^+ \rangle \rangle.$$

Then set

$$f_\nu = \text{the } \mathcal{U}_\nu\text{-least bijection } f : \beta^+ \leftrightarrow \tilde{\rho}_\nu.$$

$$a_\xi = \text{the } \xi\text{-th } a \subseteq \beta^+ \text{ in } L_\kappa[A].$$

$$\tilde{a}_\nu = \{(\xi, \mu) : \xi \in a_{f_\nu(\mu)}\}.$$

We are now ready to define the desired forcing notion, that we denote by $Add(\diamondsuit^+_{\beta^+})$. First we define the forcing notions $Add(\diamondsuit^+_{\beta^+})_\nu, \nu < \kappa$, which are the building blocks of the main forcing construction [2].

A condition in $Add(\diamondsuit^+_{\beta^+})_\nu$ is a subset p of β^+ such that

(1) $p \subseteq \beta^+$ is closed and bounded.
(2) $\alpha \in p \implies \tilde{a}_\nu \cap \alpha \in M_\alpha$.

$Add(\diamondsuit^+_{\beta^+})_\nu$ is ordered by end extension:

$$p \leq q \iff q = p \cap (\max(p) + 1).$$

Let us now define $Add(\diamondsuit^+_{\beta^+})$. A condition in $Add(\diamondsuit^+_{\beta^+})$ is a function p such that

(1) $\mathrm{dom}(p) \subseteq \kappa$ and $|\mathrm{dom}(p)| \leq \beta$.
(2) $\forall \nu \in \mathrm{dom}(p), p(\nu) \in Add(\diamondsuit^+_{\beta^+})_\nu$.
(3) If $\nu \in \mathrm{dom}(p)$, then
    (a) $f''_\nu[\max(p(\nu))] \subseteq \mathrm{dom}(p)$.
    (b) For each $\xi \in f''_\nu[\max(p(\nu))]$, $\max(p(\xi)) \geq \max(p(\nu))$.
    (c) $\alpha \in p(\nu) \implies \tilde{C}_{p,\nu} \cap \alpha \in M_\alpha$, where

$$\tilde{C}_{p,\nu} = \{(\mu, \xi) \in \max(p(\nu)) \times \max(p(\nu)) : \mu \in p(f_\nu(\xi))\}.$$

[2]In [4], the forcing notion $Add(\diamondsuit^+_{\beta^+})_\nu$ is denoted by $\mathbb{P}^A_\nu$ and the forcing notion $Add(\diamondsuit^+_{\beta^+})$ is denoted by $\mathbb{P}^A$

The forcing $Add(\diamondsuit^+_{\beta^+})$ is ordered as follows: $p \leq q$ if and only if

$$\text{dom}(p) \supseteq \text{dom}(q) \text{ and for all } \nu \in \text{dom}(q), p(\nu) \leq_{Add(\diamondsuit^+_{\beta^+})_\nu} q(\nu).$$

Let H be $Add(\diamondsuit^+_{\beta^+})$-generic over $V[G]$. The next claim is proved in [4].

Claim 2.4. (a) *$Add(\diamondsuit^+_{\beta^+})$ is β^+-distributive and $\kappa = \beta^{++}$-c.c.".*

(b) *$V[G * H] \models$ "GCH".*

(c) *$\mathcal{S}^*$ witnesses that $\diamondsuit^+_{\beta^+}$ holds in $V[G * H]$.*

(d) *The theory T_J does not have a (β^{++}, β)-model in $V[G * H]$.*

Then $\mathbb{P}_{\beta,\kappa} = \text{Col}(\beta^+, < \kappa) * \underset{\sim}{Add}(\diamondsuit^+_{\beta^+})$ is as required. □

Suppose $K = G * H$ is $\mathbb{P}_{\beta,\kappa}$-generic over V. As $\diamondsuit^+_{\beta^+}$ implies the existence of a β^+-Kurepa tree [2], in $V[K]$, we have β^+-Kurepa trees.

2.2. **Completing the proof of Theorem 1.3.** In this subsection we complete the proof of Theorem 1.3. Thus assume $V = L$ and let κ be a Mahlo cardinal. Let λ be the least inaccessible cardinal. So $\lambda < \kappa$. Let G be $\text{Col}(\aleph_1, < \lambda)$-generic over L. Then:

Lemma 2.5. (a) *$L[G] \models$ "There are no $\aleph_1$-Kurepa trees".*

(b) *$L[G] \models$ " GCH holds".*

(c) *$L[G] \models$ " κ is a Mahlo cardinal".*

Proof. (a) and (b) hold by [5], and (c) is clear, as the forcing $\text{Col}(\aleph_1, < \lambda)$ has size $< \kappa$. □

Let K be $\mathbb{P}^{L[G]}_{\aleph_1,\kappa}$-generic over $L[G]$. We show that $L[G * K]$ is the required model. First note that by Theorem 2.1,

$$L[G * K] \models \text{" there exists an } \aleph_2\text{-Kurepa tree"}.$$

But by Lemma 2.5, $L[G] \models$ "There are no $\aleph_1$-Kurepa trees". On the other hand, $L[G] \models$"$\mathbb{P}_{\aleph_1,\kappa}$ is $\lambda = \aleph_2$-distributive", in particular

$$L[G * K] \models \text{"There are no } \aleph_1\text{-Kurepa trees"}.$$

It follows that

$$L[G * K] \models \text{" } (\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0) \text{ fails "}.$$

On the other hand, by Theorem 2.1(b), $L[G*K] \models$"T_J does not have an $(\aleph_3, \aleph_1)$-model". We show that T_J has an $(\aleph_2, \aleph_0)$-model in $L[G*K]$. First note that $\aleph_2^{L[G*K]} = \lambda$, which is inaccessible but not Mahlo in L, so it follows from results of Jensen and Solovay (see [2]) that $\square_{\aleph_1}$ holds in both $L[G]$ and $L[G*K]$. Let $\mathcal{C} = \langle C_\alpha : \alpha < \lambda, \lim(\alpha) \rangle \in L[G]$ witness this. Consider the model

$$\mathcal{M} = (H(\lambda)^{L[G]}, \in, \aleph_0, \mathcal{C}),$$

where $\aleph_0$ is considered as the interpretation of A. Then $\mathcal{M}$ is an $(\aleph_2, \aleph_0)$-model of T. So

$$L[G*K] \models \text{ `` } (\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1) \text{ fails ''}.$$

The theorem follows.

3. A general model theoretic fact and the proof of Theorem 1.4

In this section we prove a general model theoretic fact, and use it to prove Theorem 1.4.

3.1. **A general model theoretic fact.** In this subsection we prove the following lemma and consider some of its consequences.

Lemma 3.1. *Assume $n \geq 1$, $\mathcal{L}$ is a first order language which contains a unary predicate U, and T is a theory in $\mathcal{L}$. Then there are $\mathcal{L}^+ \supseteq \mathcal{L}$ and a theory T^+ in $\mathcal{L}^+$, such that for all infinite cardinals β:*

$$T \text{ has a } (\beta^{+n}, \beta)\text{-model} \iff T^+ \text{ has a } (\beta^{+n+1}, \beta)\text{-model.}$$

Proof. Let $\mathcal{L}^+ = \mathcal{L} \cup \{<, W_0, \dots, W_n, F_{-1}, F_0, \dots, F_n\}$ where $<$ is a binary predicate symbol, W_i's are unary predicate symbols, F_{-1} is a binary predicate symbol and F_i's, $0 \leq i \leq n$, are ternary predicate symbols. Let T^+ consists of the following axioms:

(1) ϕ^{W_n}, for each $\phi \in T$, where ϕ^{W_n} is the relativization of ϕ to W_n.
(2) $<$ is a linear ordering of the universe.
(3) Under $<$, each W_i is an initial segment of $W_{i+1}, i < n$, and W_n is an initial segment of the universe (in particular $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n$).
(4) $U \subseteq W_n$ (i.e., $\forall x (U(x) \to W_n(x))$).
(5) $F_{-1} \subseteq U \times W_0$ defines a bijection from U onto W_0.

(6) For each $0 \le i < n, F_i \subseteq (W_{i+1} \setminus W_i) \times W_i \times W_{i+1}$ is such that if $x \in W_{i+1} \setminus W_i$, then $\{(y,z) : F_i(x,y,z)\}$ is a bijection from W_i onto $\{z \in W_{i+1} : z < x\}$.

(7) F_n is such that if $x \notin W_n$, then $\{(y,z) : F_n(x,y,z)\}$ is a bijection from W_n onto $\{z : z < x\}$.

Now suppose that T has a (β^{+n}, β)-model $\mathcal{M} = (\beta^{+n}, U^{\mathcal{M}}, \dots)$. Consider the model

$$\mathcal{M}^+ = (\beta^{+n+1}, \mathcal{M}, <, \beta, \dots, \beta^{+n}, f_{-1}, f_0, \dots, f_n),$$

where $f_{-1} : U^{\mathcal{M}} \leftrightarrow \beta$, each $f_i, 0 \le i \le n$ is such that for each $\beta^{+i} \le \gamma < \beta^{+i+1}, \{(\zeta, \eta) : (\gamma, \zeta, \eta) \in f_i\}$ defines a bijection $\beta^{+i} \leftrightarrow \gamma$. It is easily seen that $\mathcal{M}^+$ is a (β^{+n+1}, β)-model for T^+.

Conversely assume that $\mathcal{M}^+$ is a (β^{+n+1}, β)-model for T^+. Consider the model $\mathcal{M}$ which is obtained from $\mathcal{M}^+ \restriction \mathcal{L}$, by replacing its universe with $W_n^{\mathcal{M}^+}$. It follows from (1) that $\mathcal{M}$ is a model of T. We show that it is a (β^{+n}, β)-model. We have $U^{\mathcal{M}} = U^{\mathcal{M}^+}$, which has size β. On the other hand, axioms (4)-(6) can be used to show that $|W_0^{\mathcal{M}^+}| = \beta$, $|W_{i+1}^{\mathcal{M}^+}| \le |W_i^{\mathcal{M}^+}|^+$ and $|W_m^{\mathcal{M}^+}| \ge \beta^{+n}$, so by induction on $i \le n$, we have $|W_i^{\mathcal{M}^+}| = \beta^{+i}$. In particular $|W_n^{\mathcal{M}^+}| = \beta^{+n}$, and the result follows. □

Corollary 3.2. *For each $n \ge 1$, the gap-$(n+1)$-cardinal transfer principle implies the gap-n-cardinal transfer principle.*

Remark 3.3. *In personal communication, Ali Enayat informed us that Corollary 3.2 is an immediate consequence of the downward Löwenheim-Skolem theorem, i.e., the fact that if $\mathcal{M} = (M, \dots)$ is an infinite structure in a countable language and X is any subset of M, then there is an elementary substructure $\mathcal{M}_0 = (M_0, \dots)$ of $\mathcal{M}$ that includes X and whose cardinality is $\max\{\aleph_0, |X|\}$. Using this theorem, it is easy to see that every model $\mathcal{M}$ that exhibits a gap-m model, say (κ^{+m}, κ), for some $m > 0$ has an elementary sub-model $\mathcal{M}_0$ that exhibits a gap-n model (κ^{+n}, κ) for all $n < m$.*

3.2. **Proof of Theorem 1.4.** In this subsection we complete the proof of Theorem 1.4. Let $L[G * H]$ be the model obtained in Subsection 2.2. So in $L[G * H]$ both transfer principles $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$ and $(\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1)$ fail. So, by induction, and using Lemma 3.1, for each $n \geq 2$, the transfer principles

$$(\aleph_n, \aleph_0) \rightarrow (\aleph_{n+1}, \aleph_1)$$

and

$$(\aleph_{n+1}, \aleph_1) \rightarrow (\aleph_n, \aleph_0)$$

fail in $L[G * H]$.

4. The case of gap-1 and some problems

In general, we can not hope to prove a result as above for gap-1-cardinal transfer principles. This is because of Vaught's theorem [7] that the transfer principle $(\beta^+, \beta) \rightarrow (\aleph_1, \aleph_0)$ is a theorem of ZFC. However we do not know the answer to the following question:

Question 4.1. *Is it consistent that both transfer principles* $(\aleph_2, \aleph_1) \rightarrow (\aleph_3, \aleph_2)$ *and* $(\aleph_3, \aleph_2) \rightarrow (\aleph_2, \aleph_1)$ *fail simultaneously.*

As we showed in Corollary 3.2, the gap-$(n+1)$-cardinal transfer principle implies the gap-n-cardinal transfer principle.

On the other hand if $L[G]$ is a generic extension of L by the Levy collapse of an inaccessible cardinal κ to $\aleph_2$, then it follows from results of Vaught [7], Chang [1] and Jensen [2] that the gap-1-cardinal transfer principle holds in $L[G]$, while by Silver's result stated in the introduction, the gap-2-cardinal transfer principle fails in $L[G]$. We do not know the answer for higher gaps.

Question 4.2. *Assume* $n > 1$. *Is it consistent that the gap-n-cardinal transfer principle holds while the gap-$(n+1)$-cardinal transfer principle fails*?

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