Singular Cardinals Problem

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The underlying theory we consider is \textit{ZFC}.
ZFC axioms

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- $ZFC = \text{Ordinary Mathematics}$.
The underlying theory we consider is \textit{ZFC}: \textit{ZFC} = Ordinary Mathematics.

But most of the talk goes much beyond ZFC!!!
The power set function

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- $CH$ says there are no cardinals between $\aleph_0$ and $2^{\aleph_0}$, i.e., $2^{\aleph_0} = \aleph_1$.
- The continuum problem appeared as the first problem in Hilbert’s problem list in 1900.
There is no reason to restrict ourselves to \( \aleph_0 \).
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- Given any infinite cardinal $\kappa$, we can ask the same question for $2^\kappa$.
- Then the **generalized Continuum hypothesis (GCH)** says that:
  $$\forall \kappa, 2^\kappa = \kappa^+.$$  
- **GCH** first appeared in some works of Peirce, Hausdorff, Tarski and Sierpinski.
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- Some related questions are:
  - (Continuum problem - Hilbert’s first problem): Is CH (the assertion \( 2^{\aleph_0} = \aleph_1 \)) true?
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- The power set (or the continuum) function is defined by \( \kappa \mapsto 2^\kappa \).
- The basic problem is to determine the behavior of the power function.
- Some related questions are:
  1. (Continuum problem - Hilbert's first problem): Is CH (the assertion: \( 2^{\aleph_0} = \aleph_1 \)) true?
  2. (Generalized continuum problem): Is GCH (the assertion: for all infinite cardinals \( \kappa, 2^\kappa = \kappa^+ \)) true?
Some topics that appear in this talk

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Inner models

- We are just interested in those inner models which are constructed by some law.
- It will allow us to construct the required inner model in a transfinite way.
- Passing from one level to the next level, we do construct it in a control and unified way.
- It will allow us to be able to control sets we are adding in each step, and so control the size of power sets.
Consistency of $\textit{GCH}$

- The theory of \textit{inner models} was introduced by \textit{Godel}.
Consistency of $GCH$

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- He used the method to construct a model $L$ of $\text{ZFC} + GCH$, thus showing that $GCH$ is consistent with $\text{ZFC}$. 
Consistency of *GCH*

- The theory of **inner models** was introduced by **Godel**.
  
- He used the method to construct a model $L$ of $\textit{ZFC + GCH}$, thus showing that $\textit{GCH}$ is consistent with $\textit{ZFC}$.

- Thus adding $\textit{GCH}$ to mathematics does not lead to a contradiction.
Consistency of $GCH$

- The theory of **inner models** was introduced by **Godel**.
- He used the method to construct a model $L$ of $\text{ZFC} + GCH$, thus showing that $GCH$ is consistent with $\text{ZFC}$.
- Thus adding $GCH$ to mathematics does not lead to a contradiction.
- But it does not say that $GCH$ is provable in mathematics!
Forcing

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He used the method to show that $2^\aleph_0 = \aleph_2$, and hence $\neg CH$, is consistent with $ZFC$.

The method was extended by Robert Solovay (in the same year) to show that $2^\aleph_0 = \kappa$, for any cardinal $\kappa$ with $\text{cf}(\kappa) > \aleph_0$, is consistent with $ZFC$. 
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5. $V[G]$ includes $V$ and has $G$ as a new element.
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5. $V[G]$ includes $V$ and has $G$ as a new element.
6. $V[G]$ is the smallest transitive model of $ZFC$ with the above properties.
Easton’s theorem

Recall that:
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\[ \kappa < \lambda \Rightarrow 2^\kappa \leq 2^\lambda, \]
Easton’s theorem

Recall that:

1. $\kappa < \lambda \Rightarrow 2^\kappa \leq 2^\lambda$,
2. $\forall \kappa, \text{cf}(2^\kappa) > \kappa$. 

Forcing
Easton’s theorem

Recall that:

- $\kappa < \lambda \Rightarrow 2^\kappa \leq 2^\lambda$,
- $\forall \kappa, cf(2^\kappa) > \kappa$.

**Easton’s theorem (1970)** says that these two properties are all things we can prove in $ZFC$ about the power function on regular cardinals!
Easton’s theorem

Recall that:

\[ \kappa \prec \lambda \Rightarrow 2^\kappa \leq 2^\lambda, \]
\[ \forall \kappa, \text{cf}(2^\kappa) > \kappa. \]

Thus mathematics says nothing (except two trivial facts) about power of regular cardinals.
Easton’s theorem

Recall that:

- $\kappa < \lambda \Rightarrow 2^\kappa \leq 2^\lambda$,
- $\forall \kappa$, $cf(2^\kappa) > \kappa$.

To prove his theorem, Easton created the theory of class forcing, where the poset is not necessarily a set.
Easton’s theorem

Recall that:
\[ \kappa < \lambda \Rightarrow 2^\kappa \leq 2^\lambda, \]
\[ \forall \kappa, cf(2^\kappa) > \kappa. \]

The situation in this case is much more complicated, as it is not even clear if \( V[G] \models ZFC \).
Singular cardinals hypothesis \((SCH)\)

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- For $\kappa$ singular, $2^\kappa$ is the least cardinal such that:
  1. $\forall \lambda < \kappa, 2^\lambda \leq 2^\kappa$,
  2. $\text{cf}(2^\kappa) > \kappa$.
- Call this assumption: singular cardinals hypothesis (SCH).
- Thus if SCH were a theorem of ZFC, then the power function would be determined by knowing its behavior on all regular cardinals and the cofinality function.
Singular cardinals hypothesis ($SCH$)

- **Gitik-Magidor**: Fortunately, for the career of the authors, but probably unfortunately for mathematics, the situation turned out to be much more complicated.
Singular cardinals hypothesis (SCH)

- **Gitik-Magidor**: Fortunately, for the career of the authors, but probably unfortunately for mathematics, the situation turned out to be much more complicated.
- In order to go further, we need to introduce large cardinals!
Large cardinals

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- But we use them in the arguments, and in fact we use much bigger large cardinals!!
Large cardinals

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- We also show their existence is necessary for the results!!!!
Some large cardinals that appear in the arguments:
Large cardinals

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4. Strong cardinals.
5. Supercompact cardinals.
Large cardinals

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2. Measurable cardinals.
3. Measurable cardinals of Mitchell order, say, $o(\kappa) = \lambda$.
4. Strong cardinals.
5. Supercompact cardinals.

The existence of a large cardinal of type $(i)$, implies the consistency of the existence of a proper class of cardinals of type $(i - 1)$. 
Consistent failure of \( SCH \)

Using large cardinals, we can violate \( SCH \):
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1. (Silver-1970) Using a supercompact cardinal,
2. (Woodin-Early 1980) Using a strong cardinal,
Consistent failure of $SCH$

Using large cardinals, we can violate $SCH$:

1. (Silver-1970) Using a supercompact cardinal,
2. (Woodin-Early 1980) Using a strong cardinal,
3. (Gitik-1989) Using a measurable cardinal $\kappa$ with $\alpha(\kappa) = \kappa^{++}$. 
Consistent failure of $SCH$

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So we can ask:
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So we can ask:

- Can $\kappa$ be small, say $\aleph_\omega$?
Consistent failure of \textit{SCH}

In all of the above models:

1. The cardinal $\kappa$ in which \textit{SCH} fails is very big, for example it is a limit of measurable cardinals,

2. There are many cardinals below $\kappa$ in which \textit{GCH} fails.

So we can ask:

- Can $\kappa$ be small, say $\aleph_\omega$?
- Can \textit{GCH} first fail at a singular cardinal $\kappa$?
Consistent failure of \textit{SCH}

- (Silver-1974) \textit{GCH} can not first fail at a singular cardinal of uncountable cofinality (the first unexpected \textit{ZFC} result),
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- **(Shelah-1983)** $SCH$ can fail at $\aleph_\omega$ (with $2^{\aleph_\omega} < \aleph_{\omega_1}$) (using one supercompact cardinal),
- **(Gitik-Magidor-1992)** $GCH$ can first fail at $\aleph_\omega$ (with $2^{\aleph_\omega} = \aleph_{\alpha+1}$, for any $\alpha < \omega_1$) (using a strong cardinal).
Do we need large cardinals to get the failure of \( SCH \)?
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- Do we need large cardinals to get the failure of \textit{SCH}?
- If yes, how large they should be?
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- If yes, how large they should be?
- And how can we prove this?
Core models

- Core model theory comes into play!
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- Core models can be used to show that large cardinals are needed to get the failure of $SCH$!!!
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- The work of **Dodd-Jensen** has started the theory of core models.
- In particular they showed that if \( SCH \) fails, then there is an inner model with a measurable cardinal.
Core models

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**Theorem (Gitik-Woodin):** The following are equiconsistent:

1. $SCH$ fails,
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4. There exists a measurable cardinals $\kappa$ with $o(\kappa) = \kappa^{++}$. 
Global failure of $GCH$

- In all of the above constructions, just one singular cardinal is considered.
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- What if we consider the power function on all cardinals?
Global failure of $GCH$

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- What if we consider the power function on all cardinals?
- The problem becomes very complicated, and there are very few general results.
Global failure of $GCH$

- (Foreman-Woodin (1990)) $GCH$ can fail everywhere (i.e., $\forall \kappa, 2^\kappa > \kappa^+$) (using a supercompact cardinal, and a little more),
Global failure of \textit{GCH}

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- (Carmi Merimovich (2006)) We can have $\forall \kappa, 2^\kappa = \kappa^{+n}$, for any fixed natural number $n \geq 2$ (using a strong cardinals),
In all of the above models cofinalities are changed (and in the last two models cardinals are also collapsed),
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- **Theorem** *(Friedman-G (2013))* Starting from a strong cardinal, we can find a pair $(V_1, V_2)$ of models of $ZFC$ with the same cardinals and cofinalities, such that $GCH$ holds in $V_1$ and fails everywhere in $V_2$, 
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- Thus answer to Friedman’s question is yes.
Adding a single real

- Given $V$ and a real $R$, let $V[R]$ be the smallest model of $ZFC$ which includes $V$ and has $R$ as an element (if such a model exists).
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- **Question** (R. Jensen - R. Solovay (1967)) Can we force the failure of CH just by adding a single real, i.e., can we have $V$ and $R$ as above such that $V \models CH$ but CH fails in $V[R]$?
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- **Question** (R. Jensen- R. Solovay (1967)) Can we force the failure of $\textit{CH}$ just by adding a single real, i.e., can we have $V$ and $R$ as above such that $V \models \textit{CH}$ but $\textit{CH}$ fails in $V[R]$?

- **Theorem** (Shelah-Woodin (1984)) Assuming the existence of $\lambda$-many measurable cardinals, we can find $V$ and a real $R$ such that $V \models \textit{GCH}$ and $V[R] \models 2^{\aleph_0} \geq \lambda$!!!
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Thus the answer to the question is yes!!!
Silver’s theorem says that there are some non-trivial $ZFC$ results for singular cardinals of uncountable cofinality.
Getting \textit{ZFC} results

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- After Silver, \textit{Galvin-Hajnal} proved more \textit{ZFC} results about power of singular cardinals of uncountable cofinality.
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After Silver, Galvin-Hajnal proved more \( \text{ZFC} \) results about power of singular cardinals of uncountable cofinality.

For example, they showed that: if \( \forall \alpha < \omega_1, 2^{\aleph_\alpha} < \aleph_{\omega_1} \), then \( 2^{\aleph_{\omega_1}} < \aleph(2^{\omega_1})^+ \).
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None of the above results work for singular cardinals of countable cofinality.
Getting $ZFC$ results

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For example, he proved a result similar to Galvin-Hajnal for $\aleph_\omega$: if $\aleph_\omega$ is strong limit, then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$. 
In late 1980th, Shelah created a technique, called **PCF theory** which shows that **ZFC** is very strong!!!
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Given a set of $A$ of regular cardinals, let:

$$PCF(A) = \{ \text{cf}(\prod A/U) : U \text{ is an ultrafilter on } A \}.$$
PCF theory

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  3. $A \subseteq B \Rightarrow \text{PCF}(A) \subseteq \text{PCF}(B)$,
  4. If $\text{PCF}(A)$ is progressive, then $\text{PCF}(\text{PCF}(A)) = \text{PCF}(A)$. 
PCF theory

- How PCF theory is related to cardinal arithmetic?
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How PCF theory is related to cardinal arithmetic?

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(Shelah) If $A$ is a progressive set of regular cardinals, then $|\text{PCF}(A)| < |A|^{+4}$!!!
It follows that if $\aleph_\omega$ is a strong limit cardinal, then:
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- The conjecture implies if $\aleph_\omega$ is a strong limit cardinal, then $2^{\aleph_\omega} < \aleph_{\omega_1}$.

- So by previous results we will have a complete solution of the power function at $\aleph_\omega$. 

(Gitik-201?) Assuming the existence of suitably large cardinals, it is consistent that the PCF conjecture fails.
PCF theory

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The following is one of the most important open questions in set theory:
PCF theory

- (Gitik-201?) Assuming the existence of suitably large cardinals, it is consistent that the PCF conjecture fails.
- Gitik’s result holds for some very large singular cardinal.
- It is not known if we can extend his proof for $\aleph_\omega$.
- The following is one of the most important open questions in set theory:
- Is it consistent that $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} > \aleph_{\omega_1}$?
Thank you for your attention!!!