Singular Cardinals Problem

Mohammad Golshani

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ZFC axioms

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- *ZFC* =Ordinary Mathematics.
- But most of the talk goes much beyond *ZFC*!!!.

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- *CH* says there are no cardinals between $\aleph_0$ and $2^{\aleph_0}$, i.e., $2^{\aleph_0} = \aleph_1$.

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- *CH* says there are no cardinals between $\aleph_0$ and $2^{\aleph_0}$, i.e., $2^{\aleph_0} = \aleph_1$.
- The continuum problem appeared as the first problem in Hilbert's problem list in 1900.

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- Given any infinite cardinal κ, we can ask the same question for 2^κ.
- Then the generalized Continuum hypothesis (*GCH*) says that:

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- *GCH* first appeared in some works of Peirce, Hausdorff, Tarski and Sierpinski.

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- The basic problem is to determine the behavior of the power function.
- Some related questions are:
  (Continuum problem - Hilbert's first problem): Is *CH* (the assertion: $2^{\aleph_0} = \aleph_1$) true?
- (Generalized continuum problem): Is *GCH* (the assertion: for all infinite cardinals $\kappa, 2^{\kappa} = \kappa^{+}$) true?

Some topics that appear in this talk

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- It will allow us to construct the required inner model in a transfinite way.
- Passing from one level to the next level, we do construct it in a control and unified way.
- It will allow us to be able to control sets we are adding in each step, and so control the size of power sets.

Consistency of *GCH*

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Consistency of GCH

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- He used the method to construct a model L of $ZFC + GCH$, thus showing that GCH is consistent with ZFC.
- Thus adding GCH to mathematics does not lead to a contradiction.
- But it does not say that GCH is provable in mathematics!.

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- He used the method to show that $2^{\aleph_0} = \aleph_2$, and hence $\neg CH$, is consistent with ZFC.
- The method was extended by Robert Solovay (in the same year) to show that $2^{\aleph_0} = \kappa$, for any cardinal κ with $cf(\kappa) > \aleph_0$, is consistent with ZFC.

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5. $V[G]$ includes V and has G as a new element.
6. $V[G]$ is the smallest transitive model of ZFC with the above properties.

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Easton's theorem (1970) says that these two properties are all things we can prove in *ZFC* about the power function on regular cardinals !$^\infty$

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Thus mathematics says nothing (except two trivial facts) about power of regular cardinals.

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To prove his theorem, Easton created the theory of class forcing, where the poset is not necessarily a set.

Easton's theorem

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The situation in this case is much more complicated, as it is not even clear if $V[G] \models ZFC$.

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Singular cardinals hypothesis (*SCH*)

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  2. $cf(2^\kappa) > \kappa$.
- Call this assumption: singular cardinals hypothesis (*SCH*).
- Thus if *SCH* were a theorem of *ZFC*, then the power function would be determined by knowing its behavior on all regular cardinals and the cofinality function.

Singular cardinals hypothesis (SCH)

- Gitik-Magidor: Fortunately, for the career of the authors, but probably unfortunately for mathematics, the situation turned out to be much more complicated.

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- Gitik-Magidor: Fortunately, for the career of the authors, but probably unfortunately for mathematics, the situation turned out to be much more complicated.
- In order to go further, we need to introduce large cardinals!

Large cardinals

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- But we use them in the arguments, and in fact we use much bigger large cardinals!!!
- We also show their existence is necessary for the results!!!!

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4. Strong cardinals.
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The existence of a large cardinal of type (i), implies the consistency of the existence of a proper class of cardinals of type $(i-1)$.

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Using large cardinals, we can violate *SCH*:

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2. (Woodin-Early 1980) Using a strong cardinal,
3. (Gitik-1989) Using a measurable cardinal κ with $o(\kappa) = \kappa^{++}$.

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So we can ask:

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- Can *GCH* first fail at a singular cardinal κ?

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- (Shelah-1983) *SCH* can fail at $\aleph_\omega$ (with $2^{\aleph_\omega} < \aleph_{\omega_1}$) (using one supercompact cardinal),
- (Gitik-Magidor-1992) *GCH* can first fail at $\aleph_\omega$ (with $2^{\aleph_\omega} = \aleph_{\alpha+1}$, for any $\alpha < \omega_1$)(using a strong cardinal).

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- If yes, how large they should be?
- And how can we prove this?

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  3. If that large cardinal does not exist, then $\mathcal{K}$ approximates V nicely.
- Core models can be used to show that large cardinals are needed to get the failure of SCH!!!

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- It follows immediately that if SCH fails, then $0^\sharp$ exists (and hence there is a proper class of inaccessible cardinals in L).
- The work of Dodd-Jensen has started the theory of core models.
- In particular they showed that if SCH fails, then there is an inner model with a measurable cardinal.

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  4. There exists a measurable cardinals κ with $o(\kappa) = \kappa^{++}$.

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Global failure of *GCH*

- In all of the above constructions, just one singular cardinal is considered.
- What if we consider the power function on all cardinals?
- The problem becomes very complicated, and there are very few general results.

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- (James Cummings (1992)) *GCH* can hold at successors but fail at limits (using a strong cardinals),
- (Carmi Merimovich (2006)) We can have $\forall\kappa, 2^\kappa = \kappa^{+n}$, for any fixed natural number $n \geq 2$ (using a strong cardinals),

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- (Sy Friedman) Can we force GCH to fail everywhere without collapsing cardinals and changing cofinalities?
- Theorem(Friedman-G (2013)) Starting from a strong cardinal, we can find a pair (V_1, V_2) of models of ZFC with the same cardinals and cofinalities, such that GCH holds in V_1 and fails everywhere in V_2,

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- (Sy Friedman) Can we force GCH to fail everywhere without collapsing cardinals and changing cofinalities?
- Theorem(Friedman-G (2013)) Starting from a strong cardinal, we can find a pair (V_1, V_2) of models of ZFC with the same cardinals and cofinalities, such that GCH holds in V_1 and fails everywhere in V_2,
- Thus answer to Friedman's question is yes.

Adding a single real

- Given V and a real R, let $V[R]$ be the smallest model of ZFC which includes V and has R as an element (if such a model exists).

Adding a single real

- Given V and a real R, let $V[R]$ be the smallest model of ZFC which includes V and has R as an element (if such a model exists).
- Question(R. Jensen- R. Solovay (1967)) Can we force the failure of CH just by adding a single real, i.e., can we have V and R as above such that $V \models CH$ but CH fails in $V[R]$?

Adding a single real

- Given V and a real R, let $V[R]$ be the smallest model of ZFC which includes V and has R as an element (if such a model exists).
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- Theorem(Shelah-Woodin (1984)) Assuming the existence of λ-many measurable cardinals, we can find V and a real R such that $V \models GCH$ and $V[R] \models 2^{\aleph_0} \geq \lambda$!!!

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- Thus the answer to the question is yes!!!

Getting *ZFC* results

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- None of the above results work for singular cardinals of countable cofinality.

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- For example, he proved a result similar to Galvin-Hajnal for $\aleph_\omega$: if $\aleph_\omega$ is strong limit, then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$.

PCF theory

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- Given a set of A of regular cardinals, let:
  $$PCF(A) = \{cf(\prod A/U) : U \text{ is an ultrafilter on } A\}.$$

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- So by previous results we will have a complete solution of the power function at $\aleph_\omega$.

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- The following is one of the most important open questions in set theory:
- Is it consistent that $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} > \aleph_{\omega_1}$?

Thank you for your attention!!!