SPECIAL ARONSZAJN TREE PROPERTY

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Abstract. Assuming the existence of a proper class of supercompact cardinals, we force that for every regular cardinal κ, there are κ^+-Aronszajn trees and all such trees are special.

1. Introduction

Aronszajn trees are of fundamental importance in combinatorial set theory and two of the most interesting problems about them are the problem of their existence (the Tree Property) and the problem of their specialization.

Given a regular cardinal κ, the *Tree Property* at κ is the assertion "there are no κ-Aronszajn trees". By a theorem of König, the tree property holds at $\aleph_0$. By a result of Aronszajn, the tree property fails at $\aleph_1$. The problem of the tree property at higher cardinals is more complicated and is independent of ZFC. An interesting and famous question of Magidor asks if the tree property can hold at all regular cardinals bigger than $\aleph_1$ and though the problem is widely open, there are works towards its consistency.

Definition 1.1. *A λ^+-Aronszajn tree T on a successor cardinal λ^+ is* special *if there is a function $f\colon T \to \lambda$ such that if $x \leq_T y$ then $f(x) \neq f(y)$.*

The specialization function, f, witnesses the fact that T has no branch. Thus, if T is special then it remains Aronszajn in any larger model of ZFC in which λ^+ is a cardinal.

For an uncountable regular cardinal κ, let SATP(κ), the *Special Aronszajn Tree Property* at κ, be the assertion "there are κ-Aronszajn trees and all such trees are special". By Baumgartner-Malitz-Reinhardt [1], $MA + \neg CH$ implies SATP($\aleph_1$). Laver-Shelah [5] extended this result to get SATP(κ^+), for κ regular, starting from a weakly compact cardinal bigger than κ.

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In this paper we force the Special Aronszajn Tree Property at many successive cardinals. First we consider the case of forcing the Special Aronszajn Tree Property at both $\aleph_1$ and $\aleph_2$ and prove the following.

Theorem 1.2. *Assume there exists a weakly compact cardinal. Then there is a generic extension of the universe in which the Special Aronszajn Tree Property holds at both* $\aleph_1$ *and* $\aleph_2$*.*

Then we extend the result to consider infinitely many successive cardinals.

Theorem 1.3. *Assume there are infinitely many supercompact cardinals. Then there is a forcing extension in which the Special Aronszajn Tree Property holds at all* $\aleph_n$*'s,* $0 < n < \omega$.

The above result can be extended to get the Special Aronszajn Tree Property at all $\aleph_{\alpha+n}$'s, where α is any limit ordinal and $1 < n < \omega$. Finally we iterate the proof of Theorem 1.3 to get the following.

Theorem 1.4. *Assume there are class many supercompact cardinals. Then there is a* ZFC*-preserving class forcing extension of the universe in which the Special Aronszajn Tree Property holds at successor of all regular cardinals.*

It is clear that if T is a special κ-Aronszajn tree, then T is not *Suslin*; so the problem of making all κ-Aronszajn trees special is tightly connected to the κ-Suslin hypothesis, which asserts there are no Suslin trees on cardinal κ. Let the *Generalized Suslin Hypothesis* be the assertion "the κ-Suslin hypothesis holds at all uncountable regular cardinals κ". An old question in set theory asks if this statement can be consistent. As a corollary of Theorem 1.4, we obtain the following partial answer to it.

Corollary 1.5. *Assume there are class many supercompact cardinals. Then there is a* ZFC*-preserving class forcing extension of the universe in which the Generalized Suslin Hypothesis holds at successor of all regular cardinals.*

In Section 2 we prove 1.2, in Section 3 we prove 1.3 and finally in Section 4 we prove 1.4.

Out notations are mostly standard. For facts about forcing and large cardinals we refer the reader to [3]. We force downwards and we always assume that our forcing notions are separative, namely for pair of conditions p, q in a forcing notion $\mathbb{P}$, $p \leq q$ means that p is stronger than q and $p \Vdash q \in \dot{G}$ where $\dot{G} = \{\langle p, \check{p}\rangle \mid p \in \mathbb{P}\}$ is the canonical name for the generic filter.

2. Special Aronszajn Tree Property at $\aleph_1$ and $\aleph_2$

In this section we prove Theorem 1.2.

2.1. **Baumgartner forcing for specializing $\aleph_1$-Aronszajn trees.** In this subsection we briefly review Baumgartner's forcing for specializing $\aleph_1$-Aronszajn trees.

Definition 2.1. *Let T be an $\aleph_1$-Aronszajn tree. The Baumgartner forcing for specializing T, $\mathbb{B}(T)$, is defined as the set of all partial functions $f : T \to \omega$ such that*

(1) $\mathrm{dom}(f) \subseteq T$ *is finite.*
(2) *If $s, t \in \mathrm{dom}(f)$ and $s <_T t$, then $f(s) \neq f(t)$.*

The order on $\mathbb{B}(T)$ is the reverse inclusion.

Let us state the basic properties of the forcing $\mathbb{B}(T)$.

Lemma 2.2. (a) $\mathbb{B}(T)$ *is c.c.c.*

(b) *In the generic extension by $\mathbb{B}(T)$, the tree T is specialized; in fact if G is $\mathbb{B}(T)$-generic over the ground model V, then $F = \bigcup G$ is a specializing function from T to ω.*

Proof. We sketch the proof for completeness.

(a) We follow [2]. Suppose towards a contradiction that $\mathbb{B}(T)$ has an uncountable antichain A. We can assume, without loss of generality, that the elements of A all have the same size N, and for each $f \in A$ let $\mathrm{dom}(f) = \{t^f_1, \ldots, t^f_N\}$. Furthermore, using the Δ-system lemma, we may assume that there is some r such that for every pair of elements f, g in A, $\mathrm{dom}\, f \cap \mathrm{dom}\, g = r$ and $f \restriction r = g \restriction r$. Note that as A is an antichain, for distinct pairs $f, g \in A$, there exist $k, l < N$ such that t^f_k and t^g_l are compatible in T and $f(t^f_k) = g(t^g_l)$.

Let U be a non-principal uniform ultrafilter on A. Then for each f there exist $k, l < N$ such that

$$A_{f,k,l} = \{g \in A \mid t^f_k \text{ and } t^g_l \text{ are compatible and } f(t^f_k) = g(t^g_l)\} \in U.$$

By narrowing A down, we may assume without loss of generality, that these k, l are the same for each f. Let $f, g \in A$. Then for each $h \in A_{f,k,l} \cap A_{g,k,l}$, t^h_l is compatible with both t^f_k and t^g_k, and since $A_{f,k,l} \cap A_{g,k,l}$ is uncountable, we can find an h as above such that $t^h_l >_T t^f_k,\ t^g_k$. As T is a tree, we have t^f_k and t^g_k are compatible in T. It follows that

$$b = \{s \in T \mid \exists f \in A, s \leq_T t^f_k\}$$

is an uncountable branch in T, a contradiction to the assumption that T is Aronszajn.

(b) Is easy, and follows by simple density arguments. □

Definition 2.3. *Let V be the ground model. Baumgartner's forcing for specializing all $\aleph_1$-Aronszajn trees, $\mathbb{P}$, is defined as the finite support iteration*

$$\mathbb{P} = \langle\langle \mathbb{P}_\alpha \mid \alpha \leq 2^{\aleph_1}\rangle, \langle \dot{\mathbb{Q}}_\alpha \mid \alpha < 2^{\aleph_1}\rangle\rangle$$

of forcing notions where

(1) *For each $\alpha < 2^{\aleph_1}, \Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{B}(\dot{T}_\alpha)$", for some $\mathbb{P}_\alpha$-name $\dot{T}_\alpha$ which is forced by $\mathbb{P}_\alpha$ to be an $\aleph_1$-Aronszajn tree.*

(2) *If $\dot{T}$ is a $\mathbb{P}$-name for an $\aleph_1$-Aronszajn tree, then for some $\alpha < 2^{\aleph_1}, \dot{T}$ is a $\mathbb{P}_\alpha$-name and $\Vdash_\alpha$ "$\dot{T} = \dot{T}_\alpha$".*

Let us mention some basic properties of $\mathbb{P}$.

Lemma 2.4. (a) *$\mathbb{P}$ is c.c.c.*

(b) *In the generic extension by $\mathbb{P}, 2^{\aleph_0} = (2^{\aleph_1})^V$ and all $\aleph_1$-Aronszajn trees are specialized.*

Proof. (a) Follows from 2.2(a) and the Solovay-Tennenbaum theorem that the finite support iteration of c.c.c. forcing notions is c.c.c.

(b) Follows from 2.2(b) and 2.3(2). □

2.2. **Specializing names for $\aleph_2$-Aronszajn trees.** In this section we define a forcing notion for specializing names of $\aleph_2$-Aronszajn trees.

Definition 2.5. *Let V be the ground model, κ be an inaccessible cardinal in V and suppose that $\mathbb{P} * \dot{\mathbb{Q}}$ is a two step iteration forcing which is κ-c.c. and makes $\kappa = \aleph_2$. Let $\dot{T}$ be a*

$\mathbb{P} * \dot{\mathbb{Q}}$-name for a κ-Aronszajn tree. We may assume that $\dot{T}$ is forced to be a tree on $\kappa \times \omega_1$ and that the α-th level of it is forced to be $\{\alpha\} \times \omega_1$. Let $\mathbb{B}_{\mathbb{Q}}(\dot{T})$ be the following forcing notion as it is defined in $V^{\mathbb{P}}$:

Conditions in $\mathbb{B}_{\mathbb{Q}}(\dot{T})$ are partial functions $f : \kappa \times \omega_1 \to \omega_1$ such that:

(1) $\operatorname{dom}(f) \subseteq \kappa \times \omega_1$ *is countable.*

(2) *If $s, t \in \operatorname{dom}(f)$ and $f(s) = f(t)$ then $\Vdash^{V^{\mathbb{P}}}_{\mathbb{Q}}$ "$\check{s} \perp_{\dot{T}} \check{t}$".*

The ordering is reverse inclusion.

Lemma 2.6. *Work in $V^{\mathbb{P}}$.*

(a) *The forcing notion $\mathbb{B}_{\mathbb{Q}}(\dot{T})$ is $\aleph_1$-closed.*

(b) *In the generic extension by $\mathbb{B}_{\mathbb{Q}}(\dot{T})$, there is a function $F : \kappa \times \omega_1 \to \omega_1$ which is a specializing function in all further generic extensions by $\mathbb{Q}$.*

In general, $\mathbb{B}_{\mathbb{Q}}(\dot{T})$ may fail to satisfy the κ-c.c. As we will see later, in our case for the proof of 1.2, $\mathbb{B}_{\mathbb{Q}}(\dot{T})$ will satisfy the κ-c.c., which is the crucial part of the argument.

2.3. **Definition of the main forcing.** In this section we define our main forcing notion, which will be used in the proof of 1.2. Assume GCH holds and let κ be a weakly compact cardinal.

We define countable support iteration

$$\langle \langle \mathbb{P}_\alpha \mid \alpha \leq \kappa^+ \rangle, \langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa^+ \rangle \rangle$$

together with another sequence

$$\langle \mathbb{B}_\alpha \mid 0 < \alpha \leq \kappa^+ \rangle$$

of forcing notions and two sequences

$$\langle \dot{T}^1_\alpha \mid 0 < \alpha < \kappa^+ \rangle, \langle \dot{T}^2_\alpha \mid 0 < \alpha < \kappa^+ \rangle$$

of names for trees, where $\dot{T}^1_\alpha$ is a name for an Aronszajn tree on ω_1 and $\dot{T}^2_\alpha$ is a name for an Aronszajn tree on κ, in such a way that

(1) $\mathbb{P}_0 = \{1_{\mathbb{P}_0}\}$ is the trivial forcing notion.

(2) For each $\alpha < \kappa^+, \mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.

(3) $\mathbb{Q}_0 = \operatorname{Col}(\aleph_1, < \kappa)$.

(4) For all $\alpha < \kappa^+$, $\mathbb{B}_{\alpha+1} = \mathbb{B}_\alpha * \mathbb{B}(\dot{T}^1_\alpha)$.

(5) For $\alpha \leq \kappa^+$ limit, $\mathbb{B}_\alpha$ is the direct limit of $\mathbb{B}_\beta$, $\beta < \alpha$.

(6) Each $\dot{T}^2_\alpha$ is a $\mathbb{P}_\alpha * \mathbb{B}_\alpha$-name for a κ-Aronszajn tree. Each $\dot{T}^1_\alpha$ is a $\mathbb{P}_\alpha * \mathbb{B}_\alpha$-name for an ω_1-Aronszajn tree.

(7) $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{B}}_{\mathbb{B}_\alpha}(\dot{T}^2_\alpha)$".

(8) If $\mathbb{P}_{\kappa^+} * \dot{\mathbb{B}}_{\kappa^+}$ forces that $\dot{T}$ is a κ-Aronszajn tree, then for some $\alpha < \kappa^+$, $\Vdash_{\mathbb{P}_\alpha * \dot{\mathbb{B}}_\alpha}$ "$\dot{T} = \dot{T}^2_\alpha$". Similarly, if $\dot{T}$ is a name for a ω_1-Aronszajn tree then for some $\alpha < \kappa^+$, $\Vdash_{\mathbb{P}_\alpha * \dot{\mathbb{B}}_\alpha} \dot{T} = \dot{T}^1_\alpha$.

Remark 2.7. *As we will see in 2.9, the forcing $\mathbb{P}_{\kappa^+}$, and hence $\mathbb{P}_{\kappa^+} * \dot{\mathbb{B}}_{\kappa^+}$ satisfies the κ-c.c. This allows us, using a suitable bookkeeping, to arrange the forcing construction to satisfy clause 8 above.*

It is important to note that although $\mathbb{P}_{\kappa^+}$ and $\mathbb{B}_{\kappa^+}$ are defined recursively together, $\mathbb{P}_{\kappa^+}$ does not depend on the generic filter of $\mathbb{B}_{\kappa^+}$ and it specializes any possible $\mathbb{B}_{\kappa^+}$-name for an Aronszajn tree, regardless of whether this tree happened to be special or non-special in the generic extension by $\mathbb{B}_{\kappa^+}$.

2.4. **Basic properties of $\mathbb{P}_{\kappa^+}$.** In this section we present the basic properties of our forcing notion $\mathbb{P}_{\kappa^+}$. The next lemma is clear.

Lemma 2.8. *$\mathbb{P}_{\kappa^+}$ is $\aleph_1$-closed.*

Using the closure of $\mathbb{P}_{\kappa^+}$, we may restrict ourselves to a dense subset in which every condition is a countable set of elements from the ground model (and not merely a name).

We now prove the following lemma which is the main step towards completing the proof of 1.2.

Lemma 2.9. *$\mathbb{P}_{\kappa^+}$ satisfies the κ-c.c. and even the κ-Knaster property.*

The rest of this subsection is devoted to the proof of the above lemma. The proof consists of two steps. First, we will show that for every κ-Aronszajn tree, T, that appears in the iteration, for many $\alpha < \kappa$, the relation between elements above the α-th level of T and elements below the α-th level of the tree is undetermined by the restriction of the forcing to V_α. From this we will conclude that for every condition p, for many $\alpha < \kappa$ there are extensions of p into two stronger conditions p', p'' such that the restriction of p' and p'' to V_α is the same and for every element t in the domain of p' or p'' above α, p' forces that $\sigma' \leq t$, p'' forces that $\sigma'' \leq t$ and σ', σ'' are incompatible. Those witnesses, σ', σ'', will depend also on $\mathbb{B}_{\kappa^+}$.

The second step is, given a sequence of κ many conditions in $\mathbb{P}$, $\langle p_i \mid i < \kappa \rangle$, to extend each p_i to a pair p'_i, p''_i as above and then, using a Δ-system argument, to fix the incompatibility witnesses in some diagonal way. Then, we will show that every p_i and p_j are compatible and in fact $p'_i \cup p''_j$ is a condition.

The proof imitates the proof of Laver-Shelah's theorem for specializing all ω_2-trees, [5], but with one additional difficulty - the separating pairs in our construction deal also with the conditions in $\mathbb{B}_{\kappa^+}$.

Let us now return to the course of the proof. Recall that κ is a weakly compact cardinal, so let $\mathcal{F}$ be the weakly compact filter on κ. Recall that if $A \subseteq V_\kappa$ and Φ is a Π^1_1-statement, and $\langle V_\kappa, A, \in \rangle \models \Phi$ then the set $\{\lambda < \kappa \mid \langle V_\lambda, A \cap V_\lambda, \in \rangle \models \Phi\}$ is in $\mathcal{F}$. $\mathcal{F}$ is κ-complete and normal.

We prove by induction on α that $\mathbb{P}_\alpha$ satisfies the κ-Knaster property. It is clear that $\mathbb{P}_1 \simeq \mathrm{Col}(\aleph_1, < \kappa)$ is κ-Knaster. Now suppose that $\alpha \leq \kappa^+$ and each $\mathbb{P}_\beta, \beta < \alpha$, is κ-Knaster. We show that $\mathbb{P}_\alpha$ is also κ-Knaster. For simplicity of notations, we will first assume that $\alpha \leq \kappa$. At the end of the proof we will explain how to extend the arguments for the general case.

The main technical tool is the following separation claim.

Claim 2.10. *Assume $\mathbb{P}_\alpha$ is κ-c.c. Then there exists a measure one set $A \in \mathcal{F}$ such that for every $\lambda \in A$, if θ, τ are elements in T_α of level greater of equal to λ and $p \in \mathbb{P}_\alpha \cap V_\lambda$, then for every q', q'' with $p = q' \cap V_\lambda = q'' \cap V_\lambda$ there are $p', p'' \in \mathbb{P}_\alpha$ and a countable sequence $\langle (b_n, \theta_n, \tau_n) \mid n < \omega \rangle$ such that:*

(1) *$p' \leq q'$, $p'' \leq q''$ and $p' \cap V_\lambda = p'' \cap V_\lambda$.*
(2) *$b_n \in \mathbb{B}_\alpha \cap V_\lambda$.*
(3) *$\theta_n, \tau_n \in \lambda \times \omega_1$ and $\theta_n \neq \tau_n$.*

(4) $(b_n, p') \Vdash$ "$\check{\tau}_n \leq_{\dot{T}_\alpha} \check{\tau}$" *and* $(b_n, p'') \Vdash$ "$\check{\theta}_n \leq_{\dot{T}_\alpha} \check{\theta}$".

(5) $\{b_n \mid n < \omega\}$ *is a maximal antichain in* $\mathbb{B}_\alpha$.

Notation 2.11. *For forcing notions* $\mathbb{P}$ *and* $\mathbb{Q}$, *we use* $\mathbb{P} \lessdot \mathbb{Q}$ *to mean that* $\mathbb{P}$ *is a regular sub-forcing of* $\mathbb{Q}$.

Proof. By the hypothesis of the lemma, $\mathbb{P}_\alpha$ has the κ-c.c. and therefore by a Π^1_1-reflection argument and using our choice of $\dot{T}_\alpha$, the set

$$\{\lambda < \kappa \mid \mathbb{P}_\alpha \cap V_\lambda \lessdot \mathbb{P}_\alpha \text{ and it forces } \dot{T}_\alpha \cap V_\lambda \text{ is a } \mathbb{B}_\alpha \cap V_\lambda\text{-name for a } \lambda\text{-Aronszajn tree}\}$$

is in $\mathcal{F}$. Call this set A. We show that A is as required. Suppose that $\lambda \in A$. Let θ, τ be elements in T_α in level greater of equal to λ and let p be a condition in $\mathbb{P}_\alpha \cap V_\lambda$.

The branches in $T_\alpha \cap V_\lambda$ below θ and below τ are both new (relative to the forcing $\mathbb{P}_\alpha \cap V_\lambda$). Hence it is forced that in $V^{\mathbb{B}_\alpha}$ there are densely many pairs of conditions (p'_0, p''_0) in $\mathbb{P}_\alpha / \mathbb{P}_\alpha \cap V_\lambda$ such that p'_0, p''_0 force incompatible values for the branches below θ and τ and $p'_0 \cap V_\lambda = p''_0 \cap V_\lambda$. Pick $b_0 \in \mathbb{B}_\alpha \cap V_\lambda$ such that b_0 decides the values of these two conditions below the pair q', q'' and the elements $\theta_0, \tau_0 \in \lambda \times \omega_1$ which witness the incompatibility.

If every condition in $\mathbb{B}_\alpha$ forces that those two conditions witness the incompatibility of θ and τ, via θ_0 and τ_0, pick any maximal antichain that contains b_0 and halt.

Otherwise, assume that there is a condition in $\mathbb{B}_\alpha$ that forces those two conditions do not witness the incompatibility by θ_0, τ_0. This condition is incompatible with b_0, and we extend it to a condition b_1 that forces a further extension of p'_1 of p'_0 and p''_1 of p''_0 witness the incompatibility of of branches below θ, τ using $\theta_1, \tau_1 \in \lambda \times \omega_1$.

As $\mathbb{B}_\alpha$ is c.c.c., and $\mathbb{P}_\alpha$ is $\aleph_1$-closed, we can continue this process, which terminates after at most countably many steps. At the end of the process, we get a countable ordinal ϑ, sequences $\langle p'_n \mid n < \vartheta\rangle$ and $\langle p''_n \mid n < \vartheta\rangle$ of conditions in $\mathbb{P}_\alpha / \mathbb{P}_\alpha \cap V_\lambda$, a sequence $\{b_n \mid n < \vartheta\}$ of conditions in $\mathbb{B}_\alpha$ and a sequence $\langle(\theta_n, \tau_n) \mid n < \vartheta\rangle$ such that

- The sequences $\langle p'_n \mid n < \vartheta\rangle$ and $\langle p''_n \mid n < \vartheta\rangle$ are decreasing and $p'_n \cap V_\lambda = p''_n \cap V_\lambda$.
- $\{b_n \mid n < \vartheta\}$ is a maximal antichain in $\mathbb{B}_\alpha$.
- For all $n < \vartheta$, $\theta_n, \tau_n \in \lambda \times \omega_1$ and $\theta_n \neq \tau_n$.
- For all $n < \vartheta, (b_n, p'_n) \Vdash$ "$\tau_n \leq_{\dot{T}_\alpha} \tau$" and $(b_n, p''_n) \Vdash$ "$\theta_n \leq_{\dot{T}_\alpha} \theta$".

Let p' extend all $p'_n, n < \vartheta$ and p'' extends all $p''_n, n < \vartheta$. Then p', p'' together with $\langle (b_n, \theta_n, \tau_n) \mid n < \vartheta \rangle$ are as required. □

Let us call the sequence $\langle (b_n, \theta_n, \tau_n) \mid n < \omega \rangle$ a separating witness for θ, τ relative to p', p''. By repeated use of 2.10, for every condition p, we may find a measure one set $A \in \mathcal{F}$ such that for every $\lambda \in A$ there is a pair of conditions p', p'' extending p, such that $p' \cap V_\lambda = p'' \cap V_\lambda$ and for every $\beta < \alpha$ and any pair of elements above λ in $\operatorname{dom} p'(\beta) \times \operatorname{dom} p''(\beta)$ has a separating witness relative to $p' \restriction \beta, p'' \restriction \beta$. Note that all of the separating witnesses are in V_λ. We call this pair (p', p'') a separating pair.

Let's back to the proof of 2.9.

Let $\langle p_\lambda \mid \lambda < \kappa \rangle$ be a sequence of conditions in $\mathbb{P}_\alpha$. By the normality of the filter $\mathcal{F}$, for positive set of $\lambda < \kappa$, we may extend each p_λ to a separating pair $(p'_\lambda, p''_\lambda)$, above λ. Using the normality of the filter $\mathcal{F}$ again, we can assume that on a positive set D, $p'_\lambda \cap V_\lambda = p''_\lambda \cap V_\lambda$ and the separating witnesses are the same for all such λ's; call it $\langle (b_n, \theta_n, \tau_n) \mid n < \omega \rangle$. Narrowing D down, we may assume that for every $\beta < \alpha$, $\{\operatorname{dom} p_\lambda(\beta) \mid \lambda \in D\}$ is a Δ-system, and have the same values of the root of the system. We claim that for any $\lambda < \lambda'$ in D, p_λ is compatible with $p_{\lambda'}$, and moreover it is witnessed by the condition q, which is defined by $q(\beta) = p'_\lambda(\beta) \cup p''_{\lambda'}(\beta)$ for every $\beta < \alpha$. It is enough to show that q is a condition. Clearly, $\operatorname{dom} q$ is at most countable. Therefore, it is enough to show that $q \restriction \gamma$ is a condition for all $\gamma < \alpha$.

We prove it by induction on $\gamma < \alpha$. For $\gamma = 1$, $q(0) \in \operatorname{Col}(\omega_1, < \kappa)$ since it is an union of two conditions that have the same intersection with V_λ and have disjoint domains above it.

Assume that $q \restriction \gamma$ is a condition. We have to show that if $t, t' \in \operatorname{dom}(q(\gamma))$ then $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma} 1_{\mathbb{B}_\gamma} \Vdash_{\mathbb{B}_\gamma} \check{t} \perp_{T_\gamma} \check{t}'$. If either $t \in V_\lambda$ or $t' \in V_\lambda$ then it follows from the fact that p_λ is a condition or that $p_{\lambda'}$ is a condition. Recall that $(p'_\lambda, p''_\lambda)$ is a separating pair, and thus $p'_\lambda \restriction \gamma \Vdash b_n \Vdash \check{\tau}_n \leq \check{t}$ and $p''_\lambda \restriction \gamma \Vdash b_n \Vdash \check{\theta}_n \leq \check{t}'$, for some $\tau_n \neq \theta_n$. By the induction hypothesis, $q \restriction \gamma$ is a condition. But it forces that b_n forces that t, t' are incompatible in the tree relation. Now if $q \restriction \gamma \nVdash$ " $\Vdash_{\mathbb{B}_\gamma} \check{t} \perp \check{t}'$", then there is a condition $q' \leq q \restriction \gamma$ and $b \in \mathbb{B}_\gamma$ such that $(q', b) \Vdash \check{t} \leq_{T_\gamma} \check{t}'$. But b is compatible with b_n (for some $n < \omega$) and q' is stronger than q and thus also force that b_n separates t and t' - a contradiction.

Let us now treat the general case. Let $\alpha < \kappa^+$ be any ordinal. Let P be a predicate on V_κ, coding the iteration up to α. Clearly, all the previous arguments (about the existence of separating pairs), go without difficulties to the case in which the language contains an additional predicate. In particular, for every condition $p \in \mathbb{P}_\alpha$ there is a measure one set of cardinals $\lambda < \kappa$ such that $\langle V_\lambda, \in, P\rangle$ is an elementary submodel of $\langle V_\kappa, \in, P\rangle$ and $P \cap V_\lambda$ codes a regular subforcing of $\mathbb{P}_\alpha$.

2.5. **Completing the proof of 1.2.** We show that the generic extension by $\mathbb{P}_{\kappa^+} * \dot{\mathbb{B}}_{\kappa^+}$ is as required.

- The fact that SATP($\aleph_1$) holds in the extension by $\mathbb{P}_{\kappa^+} * \dot{\mathbb{B}}_{\kappa^+}$ follows from the definition of $\mathbb{B}_{\kappa^+}$, clause (8) of the definition of main forcing construction and 2.2(b).
- The fact that SATP($\aleph_2$) holds in the extension by $\mathbb{P}_{\kappa^+} * \dot{\mathbb{B}}_{\kappa^+}$ follows from clause (8) of the definition of main forcing construction and 2.6(b).

Let us note that the same arguments give us the following, more general, statement:

Lemma 2.12. *Let κ be a weakly compact cardinal and $\mu < \kappa$. Let $\mathbb{B}_\alpha$ be a sequence, such that:*

(1) *For every $\alpha < \beta$, $\mathbb{B}_\alpha$ is a regular subforcing of $\mathbb{B}_\beta$.*
(2) *For every α, $\mathbb{B}_\alpha$ is μ.c.c.*
(3) *For every α such that $\operatorname{cf}\alpha \geq \mu$, $\mathbb{B}_\alpha$ is the direct limit of $\langle \mathbb{B}_\gamma \mid \gamma < \alpha\rangle$.*
(4) $|\mathbb{B}_{\alpha+1}/\mathbb{B}_\alpha| < \kappa$.

And assume that $\mathbb{P}_\alpha$ is defined as an iteration for the specialization of all $\mathbb{B}_\alpha$-names of κ-Aronszajn trees, where $\mathbb{B}_{\alpha+1}$ can depend on the generic filter of $\mathbb{P}_\alpha$, then $\mathbb{P}_\alpha$ is κ-c.c. and μ-closed.

Note that while the forcings $\mathbb{B}_\alpha$ can depend on the generic for $\mathbb{P}_\alpha$, the definition of $\mathbb{P}_\alpha$ does not depend on the generic filter for $\mathbb{B}_\alpha$. The conditions in the α step in $\mathbb{P}$, $\mathbb{Q}_\alpha$, are partial specialization functions, of cardinality $< \mu$, which are forced to specialize a given $\mathbb{B}_\alpha$-name for a κ-Aronszajn tree.

3. Special Aronszajn Tree Property at ω-many successive cardinals

In this section we prove Theorem 1.3. The proof is essentially based on a modification of the proof of Theorem 1.2, where instead of considering two successive cardinals we consider ω-many of them.

Thus let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of supercompact cardinals, $\lambda = (\sup_{n<\omega} \kappa_n)^{++}$ and let $\mu < \kappa_0$ be a regular cardinal [1]. We may also assume that for each n, κ_n is Laver indestructible and $2^{\kappa_n} = \kappa_n^+$. For notational reasons, it is convenient to denote $\kappa_{-1} = \mu$.

We define an iteration

$$\langle \langle \mathbb{P}_\alpha \mid \alpha \leq \lambda, \rangle, \langle \dot{\mathbb{Q}}_\beta \mid \beta < \lambda \rangle \rangle$$

of length λ together with a function $g \colon \lambda \to \omega$. If $g(\alpha) = n$, then the α step in the iteration will add a specialization function for a tree of height κ_n.

$\mathbb{Q}_0$ is the full-support product $\prod_{n<\omega} \mathrm{Col}(\kappa_{n-1}, < \kappa_n)$. For each $\alpha \leq \lambda$:

1. Any $p \in \mathbb{P}_\alpha$ has domain α and for each $n < \omega$, $|\operatorname{supp}(p) \cap g^{-1}(\{n\})| < \kappa_{n-1}$, where $\operatorname{supp}(p)$ denotes the support of p.
2. For every $n < \omega$, $\mathbb{P}_\alpha \cong \mathbb{P}_\alpha(> \kappa_n) * \dot{\mathbb{P}}_\alpha(\kappa_n) * \dot{\mathbb{P}}_\alpha(< \kappa_n)$, where
   - (a) $\mathbb{P}_\alpha(> \kappa_n)$ is κ_n-directed closed.
   - (b) $\dot{\mathbb{P}}_\alpha(\kappa_n)$ is forced to be κ_n-c.c. and κ_{n-1}-directed closed.
   - (c) $\dot{\mathbb{P}}_\alpha(< \kappa_n)$ is forced to be κ_{n-1}-c.c. and μ-directed closed.
3. If $\alpha < \lambda$, then there are $n = g(\alpha) < \omega$ and a $\mathbb{P}_\alpha$-name, $\dot{T}_\alpha$, such that 1_α forces $\dot{T}_\alpha$ is a κ_n-Aronszajn tree.
4. If $\alpha < \lambda$ and $n, \dot{T}_\alpha$ are as above, then $\mathbb{Q}_\alpha$ consists of those partial functions f with domain of size $< \kappa_{n-1}$, such that for every $t, s \in \operatorname{dom}(f)$ with $f(t) = f(s)$, we have $\Vdash_{\mathbb{P}_\alpha^{<\kappa_n}} t \perp_{\dot{T}_\alpha} s$.
5. If $n < \omega$ and $\dot{T}$ is a $\mathbb{P}_\lambda$-name for a κ_n-Aronszajn tree, then there is $\alpha < \lambda$ such that $g(\alpha) = n$ and $\Vdash_{\mathbb{P}_\lambda} \dot{T} = \dot{T}_\alpha$.

[1] For the proof of Theorem 1.3 it suffices to take $\mu = \aleph_0$, but here we will prove something stronger that will be used in the next section for the proof of our main theorem.

$\mathbb{P}_\alpha(>\kappa_n)$ is defined as the iteration of all forcing notions $\mathbb{Q}_\beta$, with $g(\beta)>n$. This forcing notion is a regular subforcing of $\mathbb{P}_\alpha$, as each such $\mathbb{Q}_\beta$ is defined in the generic extension by $\mathbb{P}_\beta(>\kappa_n)$. Similarly, $\mathbb{P}_\alpha(\kappa_n)$ is the iteration of all $\mathbb{Q}_\beta$, over $g(\beta)=n$. This iteration is defined over the generic extension by $\mathbb{P}_\alpha(>\kappa_n)$. Each such $\mathbb{Q}_\beta$ is forced to be κ_{n-1}-directed closed forcing notion. As the later components of $\mathbb{P}_\alpha(>\kappa_n)$ are κ_{n-1}-distributive, those forcing notion remain κ_{n-1}-directed closed.

Applying this argument, by induction on α, we can prove that if we let $m=g(\alpha)$, then $\mathbb{Q}_\alpha$ is κ_m-directed closed in the generic extension by $\mathbb{P}_\alpha(>\kappa_m)*\mathbb{P}_\alpha(\kappa_m)$.

Remark 3.1. *By Lemma 3.3, the forcing $\mathbb{P}_\lambda$ is λ-c.c., and so by a suitable book-keeping argument we can manage that clause (5) above to hold.*

Let

$$\langle\langle G_\alpha \mid \alpha\leq\lambda\rangle, \langle H_\alpha \mid \alpha<\lambda\rangle\rangle$$

be $\mathbb{P}_\lambda$-generic over V.

It is clear that

Lemma 3.2. (a) $\Vdash_{\mathbb{Q}_0}$ *"$\forall n<\omega, \kappa_n=\mu^{+n+1}$ and $2^{\kappa_n}=\kappa_n^+$".*

(b) $\Vdash_{\mathbb{Q}_0}$ *"for all $n>0$, there are special κ_n-Aronszajn trees".*

The next lemma can be proved easily using a Δ-system argument.

Lemma 3.3. *The forcing $\mathbb{P}_\lambda$ is λ-c.c.*

Lemma 3.4. *Suppose that $\alpha<\lambda$ and $g(\alpha)=n$ Then $\Vdash_{\mathbb{P}_\alpha(>\kappa_n)}$ "$\dot{\mathbb{Q}}_\alpha$ is κ_n-c.c."*

Proof. By the indestructibility of κ_n and clause (3-a) above, κ_n remains supercompact, and hence weakly compact, in the generic extension by $\mathbb{P}_\alpha(>\kappa_n)$, so we can apply the argument of Lemma 2.12 in order to conclude that $\mathbb{Q}_\alpha$ is κ_n-c.c. in $V^{\mathbb{P}_\alpha(>\kappa_n)}$. □

Using the above lemma and the arguments of the preceding section, we can show that the models $V[G_1]$ and $V[G_\lambda]$ have the same cardinals and cofinalities. in particular,

Lemma 3.5. $V[G_\lambda]\models$ *" for each $n<\omega$, $\kappa_n=\mu^{+n+1}$.*

Finally we have the following which concludes the proof of Theorem 1.3

Lemma 3.6. $V[G_\lambda] \models$ *"for each $n < \omega$, there are κ_n-Aronszajn trees and all such trees are special".*

Proof. By the fact that the models $V[G_1]$ and $V[G_\lambda]$ have the same cardinals, we conclude that each special κ_n-Aronszajn tree in $V[G_1]$ remains κ_n-Aronszajn in the final extension $V[G_\lambda]$ and hence by Lemma 3.2, there are κ_n-Aronszajn trees in $V[G_\lambda]$.

Also using the definition of the forcing notion and clause (6) above, if $\dot{T}$ is a κ_n-Aronszajn tree in $V[G_\lambda]$, and if $\dot{T}$ is a $\mathbb{P}_\lambda$-name for it, then at some stage α, $\dot{T} = \dot{T}_\alpha$ and so T is specialized in $V[G_{\alpha+1}]$ and hence also in $V[G_\lambda]$. □

4. Special Aronszajn Tree Property at successor of every regular cardinal

In this section we prove Theorem 1.4. Recall from Section 3 that, we essentially proved the following lemma:

Lemma 4.1. *Assume α is a limit ordinal and $\kappa_1 < \cdots < \kappa_n < \dots$ are indestructible supercompact cardinals above $\aleph_\alpha$. Then there is an $\aleph_{\alpha+1}$-closed forcing notion $\mathbb{P}(\alpha, \langle \kappa_n \mid 1 < n < \omega \rangle)$ of size $\lambda = (\sup_{n<\omega} \kappa_n)^{++}$ such that the following hold in a generic extension by $\mathbb{P}(\alpha, \langle \kappa_n \mid 1 < n < \omega \rangle)$:*

(a) *For each $1 < n < \omega, \aleph_{\alpha+n} = \kappa_n$ and $\lambda = \aleph_{\alpha+\omega+1}$.*
(b) $\forall\ 1 \leq n < \omega,\ 2^{\aleph_{\alpha+n}} = \lambda$.
(c) *Special Aronszajn Tree Property holds at all $\aleph_{\alpha+n}$'s, $1 < n < \omega$.*

We also need the following lemma which is essentially due to Laver [4].

Lemma 4.2. *Assume η is a regular cardinal and $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals above η. Then there exists an η-directed closed forcing notion $\mathbb{L}(\eta, \langle \kappa_n \mid n < \omega \rangle)$ which makes the supercompactness of each κ_n indestructible under κ_n-directed closed forcing notion.*

Now suppose that $\langle \kappa_\xi \mid 0 < \xi \in ON \rangle$ is an increasing and continuous sequence of supercompact cardinals and their limits and set $\kappa_0 = \aleph_0$. We can assume that no limit point of the sequence is an inaccessible cardinal. Let

$$\langle \langle \mathbb{P}_\alpha \mid \alpha \in ON, \alpha = 0 \text{ or } \lim(\alpha) \rangle, \langle \dot{\mathbb{Q}}_\alpha \mid \alpha \in ON, \alpha = 0 \text{ or } \lim(\alpha) \rangle \rangle$$

be the reverse Easton iteration such that

(1) $\mathbb{P}_0 = \{1_{\mathbb{P}_0}\}$ is the trivial forcing.

(2) $V^{\mathbb{P}_0} \models$"$\mathbb{Q}_0 = \mathbb{L}(\aleph_1, \langle \kappa_n \mid 0 < n < \omega \rangle) * \dot{\mathbb{P}}(0, \langle \kappa_n \mid 0 < n < \omega \rangle)$.

(3) For each limit ordinal $\alpha > 0$,

$$V^{\mathbb{P}_\alpha} \models \text{“}\mathbb{Q}_\alpha = \mathbb{L}(\kappa_\alpha^+, \langle \kappa_{\alpha+n} \mid 0 < n < \omega \rangle) * \dot{\mathbb{P}}(\alpha, \langle \kappa_{\alpha+n} \mid 0 < n < \omega \rangle)\text{”}.$$

Note that at each step α, the forcing notion $\mathbb{P}_\alpha$ has size less than $\kappa_{\alpha+1}$, so cardinals $\kappa_{\alpha+n}, 0 < n < \omega$, remain supercompact in the generic extension by $\mathbb{P}_\alpha$ and so the forcing notion $\mathbb{Q}_\alpha$ is well-defined in $V^{\mathbb{P}_\alpha}$.

Finally let $\mathbb{P}$ be the direct limit of the above forcing construction and let G be $\mathbb{P}$-generic over V.

Lemma 4.3. *The following hold in* $V[G]$*:*

(a) $\forall \xi \in ON, \aleph_\xi = \kappa_\xi$.

(b) *For each limit ordinal* α *and each* $1 < n < \omega$, $2^{\aleph_{\alpha+n}} = \kappa_{\alpha+\omega+1} = \aleph_{\alpha+\omega+1}$.

Let's show that in the generic extension by $\mathbb{P}$, Special Aronszajn Tree Property holds at successor of every regular cardinal. Thus assume α is a limit ordinal (the case $\alpha = 0$ is similar). We can write the forcing $\mathbb{P}$ as $\mathbb{P} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha * \dot{\mathbb{P}}_\infty$, and by 4.1,

$$V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha} \models \text{“} \bigwedge_{1<n<\omega} \mathrm{SATP}(\aleph_{\alpha+n})\text{”}.$$

But

$$V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha} \models \text{“ } \mathbb{P}_\infty \text{ is } \kappa_{\alpha+\omega+1} - \text{closed”},$$

and so $V^{\mathbb{P}} \models$"$\bigwedge_{1<n<\omega} \mathrm{SATP}(\aleph_{\alpha+n})$". The result follows immediately. □

We close the paper with the following question, which is an analogue of Magidor's question for tree property.

Question 4.4. *Is it consistent, relative to the existence of large cardinals, that Special Aronszajn Tree Property holds for all uncountable regular cardinals* ?

We remark that the following question is open:

Question 4.5. *Let* λ *be successor of a singular cardinal. Is* $\mathrm{SATP}(\lambda)$ *consistent? I.e., is it consistent that there is a* λ*-Aronszajn tree, and every* λ*-Aronszajn tree is special?*

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