

THE SPECIAL ARONSZAJN TREE PROPERTY

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ABSTRACT. Assuming the existence of a proper class of supercompact cardinals, we force a generic extension in which, for every regular cardinal κ , there are κ^+ -Aronszajn trees, and all such trees are special.

1. INTRODUCTION

Aronszajn trees are of fundamental importance in combinatorial set theory, and two of the most interesting problems about them, are the problem of their existence (the Tree Property), and the problem of their specialization (the Special Aronszajn Tree Property).

Given a regular cardinal κ , a κ -Aronszajn tree is a tree of height κ , where all of its levels have size less than κ and it has no cofinal branches of size κ . The *Tree Property* at κ is the assertion “there are no κ -Aronszajn trees”.

By a theorem of König, the tree property holds at \aleph_0 , while by a result of Aronszajn, the tree property fails at \aleph_1 . The problem of the tree property at higher cardinals is more complicated and is independent of ZFC. An interesting and famous question of Magidor asks if the tree property can hold at all regular cardinals bigger than \aleph_1 , and though the problem is widely open, there are many works towards its consistency.

In this paper, we are interested in the problem of specializing Aronszajn trees at the successor of regular cardinals.

Definition 1.1. *A λ^+ -Aronszajn tree T , on a successor cardinal λ^+ , is special, if there exists a function $f: T \rightarrow \lambda$ such that if $x <_T y$, then $f(x) \neq f(y)$.*

The first author's research has been supported by a grant from IPM (No. 971030417).

The specialization function, f , witnesses the fact that T has no cofinal branches. Thus, if T is special, then it remains Aronszajn in any larger model of ZFC in which λ^+ is a cardinal.

For an uncountable regular cardinal κ , let $\text{SATP}(\kappa)$, the *Special Aronszajn Tree Property* at κ , be the assertion “there are κ -Aronszajn trees and all such trees are special”. By Baumgartner-Malitz-Reinhardt [1], $\text{MA} + \neg\text{CH}$ implies $\text{SATP}(\aleph_1)$. Laver-Shelah [7] extended this result to get $\text{SATP}(\kappa^+)$, for κ regular, starting from a weakly compact cardinal bigger than κ .

In this paper, we force the Special Aronszajn Tree Property at many successors of regular cardinals. First, we consider the case of forcing the Special Aronszajn Tree Property at both \aleph_1 and \aleph_2 , and prove the following theorem.

Theorem 1.2. *Assume there exists a weakly compact cardinal. Then there is a generic extension of the universe in which the Special Aronszajn Tree Property holds at both \aleph_1 and \aleph_2 .*

Then we consider the problem of specializing Aronszajn trees at infinitely many successive cardinals, and prove the following theorem.

Theorem 1.3. *Assume there are infinitely many supercompact cardinals. Then there is a forcing extension of the universe in which the Special Aronszajn Tree Property holds at all \aleph_n 's, $0 < n < \omega$.*

The above result can be extended to get the Special Aronszajn Tree Property at all $\aleph_{\alpha+n}$'s, where α is any limit ordinal and $1 < n < \omega$. Finally, we use a class-sized iterated forcing construction to get the following result.

Theorem 1.4. *Assume there are class many supercompact cardinals with no inaccessible limit. Then there is a ZFC-preserving class forcing extension of the universe, in which the Special Aronszajn Tree Property holds at the successor of every regular cardinal.*

Our forcing notions are design to specialize trees at a double successor cardinal, in a way that allow us to specialize trees at many cardinals simultaneously. Using Baumgartner’s forcing, we can also specialize all \aleph_1 -trees. The possibility of specialization of Aronszajn trees at the successor of a singular cardinal or the successor of an inaccessible cardinal remains open.

It is clear that if T is a special κ -Aronszajn tree, then T is not κ -Suslin; so the problem of making all κ -Aronszajn trees special is tightly connected to the κ -Suslin hypothesis, which asserts that there are no κ -Suslin trees. Let the *Generalized Suslin Hypothesis* be the assertion “the κ -Suslin hypothesis holds at all uncountable regular cardinals κ ”. An old and major open question in set theory is if this statement can be consistent. As a corollary of Theorem 1.4, we obtain the following partial answer to it.

Corollary 1.5. *Assume there are class many supercompact cardinals with no inaccessible limit. Then there is a ZFC-preserving class forcing extension of the universe, in which the Generalized Suslin Hypothesis holds at the successor of every regular cardinal.*

The paper is organized as follows. In Section 2 we prove Theorem 1.2. To do this, we first introduce Baumgartner’s forcing for specializing \aleph_1 -Aronszajn trees, and discuss some of its basic properties. Then we introduce a new forcing notion, which specializes names for \aleph_2 -Aronszajn trees, and show that it shares many properties in common with the Laver-Shelah forcing for specializing \aleph_2 -Aronszajn trees. Finally we show how the above results can be combined to define a forcing iteration which gives the proof of Theorem 1.2. In Section 3, we restate the main technical lemmas of Section 2 in a general way which is suitable for the purposes of Section 4 and Section 5. In Section 4 we prove Theorem 1.3, and finally, in Section 5, we show how to iterate the forcing notion of section 4 to prove Theorem 1.4.

Our notations are mostly standard. For facts about forcing and large cardinals we refer the reader to [5].

We force downwards and we always assume that our forcing notions are separative, namely for pair of conditions p, q in a forcing notion \mathbb{P} , $p \leq q$ means that p is stronger than q and $p \Vdash q \in \dot{G}$ where $\dot{G} = \{\langle p, \check{p} \rangle \mid p \in \mathbb{P}\}$ is the canonical name for the generic filter. Also if \mathbb{P} is a forcing notion in the ground model V , when writing $V[G_{\mathbb{P}}]$, we assume $G_{\mathbb{P}}$ is a \mathbb{P} -generic filter over V .

2. THE SPECIAL ARONSZAJN TREE PROPERTY AT \aleph_1 AND \aleph_2

In this section we prove Theorem 1.2. In Subsection 2.1, we review Baumgartner's forcing for specializing \aleph_1 -Aronszajn trees, then in Subsection 2.2, we introduce a forcing notion for specializing names of \aleph_2 -Aronszajn trees. The forcing is a variant of the Laver-Shelah forcing [7], where instead of specializing \aleph_2 -Aronszajn trees, we specialize names of \aleph_2 -Aronszajn trees. In Subsection 2.3, we define the main forcing iteration \mathbb{P} , and in Subsection 2.4, we prove the basic properties of \mathbb{P} . The main technical part is to show that \mathbb{P} satisfies the κ -chain condition, where κ is the weakly compact cardinal we start with. Finally in Subsection 2.5 we complete the proof of Theorem 1.2.

2.1. Baumgartner's forcing for specializing \aleph_1 -Aronszajn trees. In this subsection we briefly review Baumgartner's forcing for specializing \aleph_1 -Aronszajn trees, and refer to [2] for more details on the results of this subsection.

Definition 2.1. *Let T be an \aleph_1 -Aronszajn tree. The Baumgartner's forcing for specializing T , $\mathbb{B}(T)$, is the set of all partial functions $f : T \rightarrow \omega$ such that*

- (1) $\text{dom}(f) \subseteq T$ is finite.
- (2) If $s, t \in \text{dom}(f)$ and $s <_T t$, then $f(s) \neq f(t)$.

The order on $\mathbb{B}(T)$ is the reverse inclusion.

Let us state the basic properties of the forcing notion $\mathbb{B}(T)$.

Lemma 2.2. (a) $\mathbb{B}(T)$ is c.c.c.

(b) In the generic extension by $\mathbb{B}(T)$, the tree T is specialized; in fact if G is $\mathbb{B}(T)$ -generic over the ground model V , then $F = \bigcup G$ is a specializing function from T to ω .

Proof. We sketch the proof for completeness.

(a) We present the proof given in [3]. Suppose towards a contradiction that $\mathbb{B}(T)$ has an uncountable antichain A . We can assume, without loss of generality, that the elements of A all have the same size N . For each $f \in A$, let $\text{dom}(f) = \{t_1^f, \dots, t_N^f\}$. Furthermore, using the Δ -System Lemma, we may assume that there is some r such that for every pair of elements f, g in A , $\text{dom}(f) \cap \text{dom}(g) = r$ and $f \upharpoonright r = g \upharpoonright r$. Note that as A is an antichain, for distinct pairs $f, g \in A$, there exist $k, l < N$ such that t_k^f and t_l^g are compatible in T and $f(t_k^f) = g(t_l^g)$.

Let U be a non-principal uniform ultrafilter on A . Then for each $f \in A$, there exist $k, l < N$ such that

$$A_{f,k,l} = \{g \in A \mid t_k^f \text{ and } t_l^g \text{ are compatible and } f(t_k^f) = g(t_l^g)\} \in U.$$

Let $A' \subseteq A$ be an uncountable subset of A such that these k, l are the same for each $f \in A'$. Let $f, g \in A'$. Then for each $h \in A_{f,k,l} \cap A_{g,k,l}$, t_l^h is compatible with both t_k^f and t_k^g , and since $A_{f,k,l} \cap A_{g,k,l}$ is in U and in particular uncountable, we can find an h as above, such that $t_l^h >_T t_k^f, t_k^g$. As T is a tree, we have t_k^f and t_k^g are compatible in T . It follows that

$$b = \{s \in T \mid \exists f \in A', s \leq_T t_k^f\}$$

is an uncountable branch in T , a contradiction to the assumption that T is Aronszajn.

(b) Is easy, and follows by simple density arguments. \square

Definition 2.3. *Baumgartner's forcing for specializing all \aleph_1 -Aronszajn trees, \mathbb{P} , is defined as the finite support iteration*

$$\mathbb{P} = \langle \langle \mathbb{P}_\alpha \mid \alpha \leq 2^{\aleph_1} \rangle, \langle \dot{Q}_\alpha \mid \alpha < 2^{\aleph_1} \rangle \rangle$$

of forcing notions where

- (1) For each $\alpha < 2^{\aleph_1}$, $\Vdash_{\mathbb{P}_\alpha} \dot{Q}_\alpha = \mathbb{B}(\dot{T}_\alpha)$, for some \mathbb{P}_α -name \dot{T}_α which is forced by $1_{\mathbb{P}_\alpha}$ to be an \aleph_1 -Aronszajn tree.
- (2) If \dot{T} is a \mathbb{P} -name for an \aleph_1 -Aronszajn tree, then for some $\alpha < 2^{\aleph_1}$, \dot{T} is a \mathbb{P}_α -name and $\Vdash_{\mathbb{P}_\alpha} \dot{T} = \dot{T}_\alpha$.

Let us mention some basic properties of \mathbb{P} .

Lemma 2.4. (a) \mathbb{P} is c.c.c.

- (b) In the generic extension by \mathbb{P} , $2^{\aleph_0} = (2^{\aleph_1})^V$ and all \aleph_1 -Aronszajn trees are specialized.

Proof. (a) Follows from Lemma 2.2(a) and the Solovay-Tennenbaum theorem that the finite support iteration of c.c.c. forcing notions is c.c.c.

- (b) Follows from Lemma 2.2(b) and Definition 2.3(2). \square

In the above definition of \mathbb{P} , we used some underlying bookkeeping method which was used in order to pick the names \dot{T}_α . We will need a minor generalization of this. Let \mathcal{T} be a function such that for every c.c.c. forcing notion \mathbb{R} , $\mathcal{T}(\mathbb{R})$ is an \mathbb{R} -name for an \aleph_1 -Aronszajn tree. We do not require that every name for an \aleph_1 -Aronszajn tree is enumerated by \mathcal{T} . Let

$$\mathbb{P}_\gamma(\mathcal{T}) = \langle \langle \mathbb{P}_\alpha(\mathcal{T}) \mid \alpha \leq \gamma \rangle, \langle \dot{Q}_\alpha(\mathcal{T}) \mid \alpha < \gamma \rangle \rangle$$

be the finite support iteration of forcing notions of length γ , where for each $\alpha < \gamma$, $\Vdash_{\mathbb{P}_\alpha(\mathcal{T})} \dot{Q}_\alpha(\mathcal{T}) = \mathbb{B}(\mathcal{T}(\mathbb{P}_\alpha(\mathcal{T})))$.

Note that for every \mathcal{T} as above and every ordinal γ , $\mathbb{P}_\gamma(\mathcal{T})$ is c.c.c., as a finite support iteration of c.c.c. forcing notions.

The following lemma will be used in the course of proving Theorem 1.2.

Lemma 2.5. *Let \mathcal{T} be as above. Let S be a tree of height ω_1 and let γ be an ordinal. Then $\mathbb{P}_\gamma(\mathcal{T})$ does not introduce new branches to S .*

Proof. Let us show that $\mathbb{P}_\gamma(\mathcal{T}) \times \mathbb{P}_\gamma(\mathcal{T})$ is c.c.c. Let \mathcal{T}' be the following function:

- If $\alpha < \gamma$, then $\mathcal{T}'(\mathbb{P}_\alpha(\mathcal{T}')) = \mathcal{T}(\mathbb{P}_\alpha(\mathcal{T}))$. In particular, $\mathbb{P}_\alpha(\mathcal{T}') \cong \mathbb{P}_\alpha(\mathcal{T})$, for all $\alpha \leq \gamma$.
- If $\gamma \leq \alpha < \gamma + \gamma$, and if $\beta < \gamma$ is such that $\alpha = \gamma + \beta$, then $\mathcal{T}'(\mathbb{P}_\alpha(\mathcal{T}')) = \mathcal{T}(\mathbb{P}_\beta(\mathcal{T}))$.

Note that if $\alpha = \gamma + \beta$, where $\beta < \gamma$, then $\Vdash_{\mathbb{P}_\alpha(\mathcal{T}')} \text{“}\mathcal{T}'(\mathbb{P}_\alpha(\mathcal{T}')) \text{ is a special Aronszajn tree, in particular it is Aronszajn”}$. It then follows that the forcing iteration $\mathbb{P}_{\gamma+\gamma}(\mathcal{T}')$ is c.c.c., and by the definition of \mathcal{T}' , one can easily verify that $\mathbb{P}_\gamma(\mathcal{T}) \times \mathbb{P}_\gamma(\mathcal{T}) \cong \mathbb{P}_\gamma(\mathcal{T}) * \dot{\mathbb{P}}_\gamma(\mathcal{T}) \cong \mathbb{P}_{\gamma+\gamma}(\mathcal{T}')$. The lemma follows from [10, Lemma 1.3]. \square

The following definition appears in the literature under various names and notations. For an example in which the following concept is used extensively, see [?].

Definition 2.6. *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \beta < \delta, \alpha \leq \delta \rangle$ be a $< \mu$ -support iteration of forcing notions, and let $I \subseteq \delta$. We define \mathbb{P}_I , by induction on $\text{otp}(I)$, to be the $< \mu$ -support iteration $\mathbb{P}_I = \langle \mathbb{P}_{I \cap \alpha}, \dot{\mathbb{Q}}_{I \cap \beta} \mid \beta \in I, \alpha \in I \cup \{\text{sup}(I) + 1\} \rangle$ of forcing notions, such that:*

- (1) *If $\dot{\mathbb{Q}}_\beta$ is forced by the weakest condition of \mathbb{P}_β to be equivalent to a specific $\mathbb{P}_{I \cap \beta}$ -name, then $\dot{\mathbb{Q}}_{I \cap \beta}$ is such a $\mathbb{P}_{I \cap \beta}$ -name.*
- (2) *Otherwise $\Vdash_{\mathbb{P}_{I \cap \beta}} \text{“}\dot{\mathbb{Q}}_{I \cap \beta} \text{ is the trivial forcing”}$.*

We say that \mathbb{P}_I is a sub-iteration of \mathbb{P} if the second case does not occur.

Note that \mathbb{P}_I is always a regular subforcing of \mathbb{P} .

Lemma 2.7. *Let $\mathbb{P}_\delta(\mathcal{T})$ be an iteration of Baumgartner's forcing as above, and let $I \subseteq \delta$ be a set of indices such that \mathbb{P}_I is a subiteration of \mathbb{P}_δ . Let S be a tree of height ω_1 in the generic extension by \mathbb{P}_I . Then the quotient forcing $\mathbb{P}_\delta/\mathbb{P}_I$ does not add a new branch to S .*

Proof. The quotient forcing $\mathbb{P}_\delta/\mathbb{P}_I$ is equivalent to a finite support iteration of Baumgartner's forcing, and thus we just need to verify that \mathbb{P}_I did not add branches to Aronszajn trees. This follows from Lemma 2.5. \square

2.2. Specializing names for \aleph_2 -Aronszajn trees. In this subsection, we define a forcing notion for specializing names of \aleph_2 -Aronszajn trees.

Definition 2.8. *Let V be the ground model, κ be an inaccessible cardinal in V and suppose that $\mathbb{P} * \dot{\mathbb{Q}}$ is a two step iterated forcing which is κ -c.c. and makes $\kappa = \aleph_2$. Let \dot{T} be a $\mathbb{P} * \dot{\mathbb{Q}}$ -name for a κ -Aronszajn tree. We may assume that \dot{T} is forced to be a tree on $\kappa \times \omega_1$ and that the α -th level of it is forced to be $\{\alpha\} \times \omega_1$. Let $\mathbb{B}_\mathbb{Q}(\dot{T})$ be the following forcing notion as it is defined in $V[G_\mathbb{P}]$:*

Conditions in $\mathbb{B}_\mathbb{Q}(\dot{T})$ are partial functions $f : \kappa \times \omega_1 \rightarrow \omega_1$ such that:

- (1) $\text{dom}(f) \subseteq \kappa \times \omega_1$ is countable.
- (2) If $s, t \in \text{dom}(f)$ and $f(s) = f(t)$ then $\Vdash_{\mathbb{Q}}^{V[G_\mathbb{P}]} \text{“}\check{s} \perp_{\dot{T}} \check{t}\text{”}$.

The ordering is reverse inclusion.

Lemma 2.9. *Work in $V[G_\mathbb{P}]$.*

- (a) *The forcing notion $\mathbb{B}_\mathbb{Q}(\dot{T})$ is \aleph_1 -closed.*
- (b) *In the generic extension by $\mathbb{B}_\mathbb{Q}(\dot{T})$, there is a function $F : \kappa \times \omega_1 \rightarrow \omega_1$ which is a specializing function of every generic interpretation of \dot{T} by a \mathbb{Q} -generic filter over $V[G_\mathbb{P}]$.*

In general, $\mathbb{B}_\mathbb{Q}(\dot{T})$ may fail to satisfy the κ -c.c. However as we will see in the proof of Theorem 1.2, under some suitable assumptions, $\mathbb{B}_\mathbb{Q}(\dot{T})$ will satisfy the κ -c.c., which is the crucial part of the argument.

2.3. Definition of the main forcing. In this subsection, we define our main forcing notion, which will be used in the proof of Theorem 1.2. Assume that GCH holds and let κ be a weakly compact cardinal. Let also $\delta > \kappa$ be a regular cardinal and fix a function $\Phi : \delta \rightarrow H(\delta)$ such that for each $x \in H(\delta)$, $\Phi^{-1}(x)$ is unbounded in δ .

Remark 2.10. *For the proof of Theorem 1.2, it suffices to take $\delta = \kappa^+$, but we present a more general result that will be used for the proof of Theorems 1.3 and 1.4*

We define by induction on $\alpha \leq \delta$ two iterations of forcing notions

$$\mathbb{P}_\delta^1 = \langle \langle \mathbb{P}_\alpha^1 \mid \alpha \leq \delta \rangle, \langle \dot{\mathbb{Q}}_\alpha^1 \mid \alpha < \delta \rangle \rangle$$

and

$$\mathbb{P}_\delta^2 = \langle \langle \mathbb{P}_\alpha^2 \mid \alpha \leq \delta \rangle, \langle \dot{\mathbb{Q}}_\alpha^2 \mid \alpha < \delta \rangle \rangle.$$

Suppose that $\alpha < \delta$ and we have defined the forcing notions \mathbb{P}_β^1 and \mathbb{P}_β^2 for all $\beta < \alpha$. Let us define \mathbb{P}_α^1 and \mathbb{P}_α^2 .

Definition of \mathbb{P}_α^2 . The forcing notion \mathbb{P}_α^2 is defined in V as follows.

Set $\mathbb{Q}_0^2 = \text{Col}(\aleph_1, < \kappa)$.

If α is a limit ordinal and $\text{cf}(\alpha) > \omega$, let \mathbb{P}_α^2 be the direct limit of the forcing notions $\mathbb{P}_\beta^2, \beta < \alpha$. If α is a limit ordinal and $\text{cf}(\alpha) = \omega$, let \mathbb{P}_α^2 be the inverse limit of the forcing notions $\mathbb{P}_\beta^2, \beta < \alpha$.

Now suppose that $\alpha = \beta + 1$ is a successor ordinal. If $\Phi(\beta)$ is a $\mathbb{P}_\beta^2 * \dot{\mathbb{P}}_\beta^1$ -name for a κ -Aronszajn tree, then let \mathbb{Q}_β^2 be a \mathbb{P}_β^2 -name such that

$$\Vdash_{\mathbb{P}_\beta^2} \text{“}\dot{\mathbb{Q}}_\beta^2 = \mathbb{B}_{\mathbb{P}_\beta^1}(\Phi(\beta))\text{”}.$$

Otherwise, let \mathbb{Q}_β^2 be a name for the trivial forcing notion.

Definition of \mathbb{P}_α^1 . The forcing notion \mathbb{P}_α^1 is defined in the generic extension of V by \mathbb{P}_α^2 . Let $V[G_\alpha^2]$ be the generic extension of V by \mathbb{P}_α^2 and work in it.

If α is a limit ordinal, then let \mathbb{P}_α^1 be the direct limit of the forcing notions $\mathbb{P}_\beta^1, \beta < \alpha$.

Let $\alpha = \beta + 1$ be a successor ordinal. If $\Phi(\beta)$ is a $\mathbb{P}_\alpha^2 * \dot{\mathbb{P}}_\beta^1$ -name for an \aleph_1 -Aronszajn tree, then let \mathbb{Q}_β^1 be such that

$$\Vdash_{\mathbb{P}_\alpha^2 * \dot{\mathbb{P}}_\beta^1} \dot{\mathbb{Q}}_\beta^1 = \mathbb{B}(\Phi(\beta)).$$

Otherwise, let \mathbb{Q}_β^1 be the trivial forcing notion.

Definition of the main forcing notion. Finally we define the main forcing notion that will be used in the proof of Theorem 1.2. For each $\alpha \leq \delta$ set $\mathbb{P}_\alpha = \mathbb{P}_\alpha^2 * \dot{\mathbb{P}}_\alpha^1$ and let $\mathbb{P} = \mathbb{P}_\delta$.

We will show that in the generic extension by \mathbb{P} , all Aronszajn trees on \aleph_1 and \aleph_2 are special, and there is an \aleph_2 -Aronszajn tree.

It is important to note that although \mathbb{P}_α^2 and \mathbb{P}_α^1 are defined recursively together, \mathbb{P}_α^2 does not depend on the generic filter of \mathbb{P}_α^1 and specializes any possible \mathbb{P}_α^1 -name for an Aronszajn tree, regardless of whether this tree happened to be special or non-special in the generic extension by \mathbb{P}_α^1 (see Lemma 2.9(b)).

2.4. Properties of the forcing notion \mathbb{P} . In this subsection we state and prove some basic properties of the forcing notions defined above.

Lemma 2.11. *For every $\alpha \leq \delta$, the forcing notion \mathbb{P}_α^2 is \aleph_1 -closed.*

Proof. \mathbb{P}_α^2 is a countable support iteration of \aleph_1 -closed forcing notions, and hence is \aleph_1 -closed. □

Then next lemma resembles Lemma 2.2.

Lemma 2.12. (a) *For every $\alpha \leq \delta$, $\Vdash_{\mathbb{P}_\alpha^2}$ “ $\dot{\mathbb{P}}_\alpha^1$ is c.c.c.”.*

(b) *For every $\alpha \leq \delta$, $\Vdash_{\mathbb{P}_\alpha^2}$ “ $\dot{\mathbb{P}}_\alpha^1$ is c.c.c.”.*

Proof. Let us show, by induction on $\alpha \leq \delta$, that \mathbb{P}_α^1 is c.c.c. in the generic extension by \mathbb{P}_γ^2 , for all $\gamma \in [\alpha, \delta]$.

For a limit ordinal α , \mathbb{P}_α^1 is the direct limit of the forcing notions \mathbb{P}_β^1 , $\beta < \alpha$, and thus it is c.c.c.

Let $\alpha = \beta + 1$ be a successor ordinal.

Then either $\mathbb{P}_\alpha^1 = \mathbb{P}_\beta^1$ and there is nothing to prove, or else, $\mathbb{P}_\alpha^1 = \mathbb{P}_\beta^1 * \mathbb{B}(\dot{T})$ where $\dot{T} = \Phi(\beta)$ is a $\mathbb{P}_\alpha^2 * \mathbb{P}_\beta^1$ -name for an \aleph_1 -Aronszajn tree. We need to show that the forcing $\mathbb{B}(\dot{T})$ is c.c.c. in the generic extension by \mathbb{P}_γ^2 , for $\gamma \in [\alpha, \kappa^+]$. Since the conditions in Baumgartner's forcing are finite, this forcing is absolute between any model of set theory that contains the evaluation of the name \dot{T} . Thus, it is sufficient to show that the tree $T = \dot{T}[G_{\mathbb{P}_\alpha^2 * \mathbb{P}_\beta^1}]$, which is Aronszajn in the generic extension by $\mathbb{P}_\alpha^2 * \mathbb{P}_\beta^1$, remains Aronszajn in the generic extension by $\mathbb{P}_\gamma^2 * \mathbb{P}_\beta^1$, for every $\gamma \in [\alpha, \delta]$.

Work in the generic extension by \mathbb{P}_α^2 and let $\gamma \in [\alpha, \delta]$. In this model the tree T is introduced by the forcing \mathbb{P}_β^1 , which is c.c.c. (by the inductive assumption). Let \mathbb{R} be the quotient forcing $\mathbb{P}_\gamma^2 / \mathbb{P}_\alpha^2$. This forcing is \aleph_1 -closed in the generic extension by \mathbb{P}_α^2 , as a countable support iteration of \aleph_1 -closed forcing notions. By the induction hypothesis, \mathbb{P}_α^1 is c.c.c. in the generic extension by \mathbb{P}_α^2 . Thus, we can apply [9, Lemma 3.2] over the generic extension by \mathbb{P}_α^2 , and conclude that forcing with \mathbb{R} over the larger generic extension by $\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1$ does not introduce new branches to the \aleph_1 -tree T . The lemma follows. \square

The next lemma is the main step towards completing the proof of Theorem 1.2.

Lemma 2.13. \mathbb{P}_α^2 is κ -Knaster for each $\alpha \leq \delta$. In particular, \mathbb{P}_δ^2 satisfies the κ -c.c.

Before we dive into the details, let us sketch the main ideas of the proof.

The proof consists of two steps. First, we will show that for every κ -Aronszajn tree T , that appears in the iteration, for many $\lambda < \kappa$, the relation between elements

above the λ -th level of T and elements below the λ -th level of the tree is undetermined by the restriction of the forcing to some nicely chosen model \mathcal{M}_λ (we will make this statement more precise in the proof ahead). From this, we will conclude that for densely many conditions p and for many $\lambda < \kappa$, there are extensions of p into two stronger conditions p', p'' , such that the restrictions of p' and p'' to \mathcal{M}_λ are the same, i.e., $p' \upharpoonright \mathcal{M}_\lambda = p'' \upharpoonright \mathcal{M}_\lambda$, and for every element t in the domain of p' or p'' above λ , p' forces that $\sigma' \leq t$, p'' forces that $\sigma'' \leq t$ and σ', σ'' are incompatible. The witnesses σ', σ'' , will depend also on \mathbb{P}_δ^1 . We call p' and p'' a separating pair for p .

The second step is, given a sequence of κ many conditions in \mathbb{P}_δ^2 , $\langle p_i \mid i < \kappa \rangle$, to extend each p_i to a separating pair p'_i, p''_i as above and then, using a Δ -System argument, to fix the incompatibility witnesses in some diagonal way. Then, we will show that every p_i and p_j are compatible and in fact, $p'_i \cup p''_j$ is a condition.

The proof imitates the proof of Laver-Shelah's theorem for specializing all \aleph_2 -Aronszajn trees [7], but with one additional difficulty - the separating pairs in our construction deal also with the conditions in \mathbb{P}_δ^1 .

Let us now return to the course of the proof.

Proof. We prove by induction on $\beta \leq \delta$ that \mathbb{P}_β^2 satisfies the κ -Knaster property. It is clear that $\mathbb{P}_1^2 \simeq \text{Col}(\aleph_1, < \kappa)$ is κ -Knaster. Now suppose that $\beta \leq \delta$ and each $\mathbb{P}_\alpha^2, \alpha < \beta$, is κ -Knaster. We show that \mathbb{P}_β^2 is also κ -Knaster.

If $\text{cf}(\beta) > \kappa$, then \mathbb{P}_β^2 is easily seen to be κ -Knaster, as any subset of \mathbb{P}_β^2 of size κ is included in some \mathbb{P}_α^2 , for some $\alpha < \beta$, so, by the induction hypothesis, it contains a subset of size κ of pairwise compatible elements in \mathbb{P}_α^2 and hence each pair of elements in this subset will be compatible in \mathbb{P}_β^2 as well.

Now suppose that $\text{cf}(\beta) \leq \kappa$ and let $\langle \beta_i \mid i < \text{cf}(\beta) \rangle$ be an increasing sequence cofinal in β .

Let $\theta > \delta$ be a sufficiently large regular cardinal and let $\mathcal{M} \preceq H(\theta)$ be such that

- $|\mathcal{M}| = \kappa$ and ${}^{<\kappa}\mathcal{M} \subseteq \mathcal{M}$.
- $V_\kappa \cup \{\beta_i \mid i < \text{cf}(\beta)\} \subseteq \mathcal{M}$.
- $\kappa, \Phi, \delta, \beta, \langle \beta_i \mid i < \text{cf}(\beta) \rangle, \langle \mathbb{P}_\beta^1 \mid \beta \leq \delta \rangle, \langle \mathbb{P}_\beta^2 \mid \beta \leq \delta \rangle, \dots \in \mathcal{M}$.

Let $\bar{\mathcal{M}}$ be the transitive collapse of \mathcal{M} with $\pi : \mathcal{M} \rightarrow \bar{\mathcal{M}}$ being the transitive collapse map. For each $x \in \mathcal{M}$ we write x^* for $\pi(x)$. Note that since $\kappa + 1 \subseteq \mathcal{M}$, for $A \subseteq \kappa$, $A \in \bar{\mathcal{M}}$, if and only if $A \in \mathcal{M}$ and $A^* = A$.

By [4], there exists a transitive model \mathcal{N} , closed under $< \kappa$ -sequences, and an elementary embedding $j : \bar{\mathcal{M}} \rightarrow \mathcal{N}$ with critical point κ such that $j, \bar{\mathcal{M}} \in \mathcal{N}$. Let

$$\mathcal{F} = \{A \subseteq \kappa \mid A \in \mathcal{M} \text{ and } \kappa \in j(A)\}.$$

Then \mathcal{F} is an \mathcal{M} -normal κ -complete \mathcal{M} -ultrafilter on κ . Let also \mathcal{S} be the collection of \mathcal{F} -positive sets, i.e.,

$$\mathcal{S} = \{D \subseteq \kappa \mid \forall A \in \mathcal{F}, D \cap A \neq \emptyset\}.$$

Let us remark that every member of \mathcal{F} is positive with respect to the weakly compact filter, which is a normal filter in V .

Let us define the sequence $\langle \mathcal{M}_\lambda \mid \lambda < \kappa \rangle$ as follows. Let $\phi : V_\kappa \leftrightarrow \mathcal{M}$ be a bijection and for each $\lambda < \kappa$ set:

$$\mathcal{M}_\lambda = \phi[V_\lambda].$$

It is evident that $\{\lambda < \kappa \mid \mathcal{M}_\lambda \cap \kappa = \lambda\} \in \mathcal{F}$.

Since $\mathcal{M} \prec H(\theta)$, if \mathbb{P}_β^2 is not κ -Knaster, then $\mathcal{M} \models$ “ \mathbb{P}_β^2 is not κ -Knaster”. In particular, there is a sequence of conditions $\langle p_\alpha \mid \alpha < \kappa \rangle \in \mathcal{M}$ witnessing it and since $\kappa \subseteq \mathcal{M}$, $p_\alpha \in \mathcal{M}$ for all $\alpha < \kappa$. We conclude that $\mathbb{P}_\beta^2 \cap \mathcal{M}$ is not κ -Knaster. Thus, let us concentrate in showing that $\mathbb{P}_\beta^2 \cap \mathcal{M}$ is κ -Knaster.

Let us assume that $\lambda < \kappa$ is an inaccessible cardinal, ${}^{<\lambda}\mathcal{M}_\lambda \subseteq \mathcal{M}_\lambda$ and that $\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda$ is a regular subforcing of \mathbb{P}_β^2 (later, during the proof, we will show that such λ 's exist). For such a cardinal λ and $p \in \mathbb{P}_\beta^2 \cap \mathcal{M}$, we denote by $p \upharpoonright \mathcal{M}_\lambda$, the following condition in $\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda$. Let $p = \langle p_\alpha \mid \alpha < \beta \rangle$. Then $p \upharpoonright \mathcal{M}_\lambda$ is the

condition $\langle p'_\alpha \mid \alpha < \beta \rangle$, where p'_α is the trivial condition is $\alpha \notin \mathcal{M}_\lambda$ and otherwise $p'_\alpha = p_\alpha \upharpoonright \mathcal{M}_\lambda$. Namely, $p \upharpoonright \mathcal{M}_\lambda$ is obtained from p by removing all coordinates which do not appear in \mathcal{M}_λ and restricting the specialization functions to values from \mathcal{M}_λ . By the closure of \mathcal{M}_λ , $p \upharpoonright \mathcal{M}_\lambda \in \mathcal{M}_\lambda$.

Let $G \subseteq \mathbb{P}_\beta^2 \cap \mathcal{M}$ be a generic filter. Then in $V[G]$ there is a natural generic filter,

$$G \cap \mathcal{M}_\lambda := \{p \upharpoonright \mathcal{M}_\lambda \mid p \in G\} = \{p \in G \mid p \in \mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda\} \subseteq \mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda.$$

Let us denote, temporarily, the quotient forcing $(\mathbb{P}_\beta^2 \cap \mathcal{M}) / (G \cap \mathcal{M}_\lambda)$ by \mathbb{R} . It is possible that for a condition $p \in \mathbb{P}_\beta^2 \cap \mathcal{M}$, $p \upharpoonright \mathcal{M}_\lambda \not\Vdash "p \in \mathbb{R}"$. Nevertheless, it is impossible that $p \upharpoonright \mathcal{M}_\lambda \Vdash "p \notin \mathbb{R}"$, and thus there is an extension $q \leq p \upharpoonright \mathcal{M}_\lambda$ for which $q \Vdash p \in \mathbb{R}$ or equivalently for every $r \leq q$ in $\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda$, r is compatible with p . By modifying $p \upharpoonright \mathcal{M}_\lambda$ we can ensure that $p \upharpoonright \mathcal{M}_\lambda \Vdash p \in \mathbb{R}$. This situation is denoted by $*_\lambda(p, p \upharpoonright \mathcal{M}_\lambda)$ in [7]. In this paper, we will say in this case that p is λ -compatible.

Before diving into the main technical lemma, let us use the following analysis of names of branches in the trees $\Phi(\alpha)$.

Notation 2.14. For forcing notions \mathbb{P} and \mathbb{Q} , we use $\mathbb{P} < \mathbb{Q}$ to mean that \mathbb{P} is a regular sub-forcing of \mathbb{Q} .

Claim 2.15. Let $\alpha \in \beta \cap \mathcal{M}$ be an ordinal such that $T_\alpha = \Phi(\alpha)$ is a $\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1$ -name for an Aronszajn tree. Let $\lambda < \kappa$ be an inaccessible cardinal such that:

- (1) $\mathcal{M}_\lambda \cap \kappa = \lambda$.
- (2) ${}^{<\lambda}\mathcal{M}_\lambda \subseteq \mathcal{M}_\lambda$.
- (3) $\mathbb{P}_\alpha^2 \cap \mathcal{M}_\lambda < \mathbb{P}_\alpha^2 \cap \mathcal{M}$.
- (4) $\Vdash_{\mathbb{P}_\alpha^2} \mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda < \mathbb{P}_\alpha^1 \cap \mathcal{M}$ and moreover, it is a sub-iteration.

Then every cofinal branch in $T_\alpha \cap (\lambda \times \omega)$ in $\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1$ exists in $\mathbb{P}_\alpha^2 * (\mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda)$.

Proof. The quotient $(\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) / (\mathbb{P}_\alpha^2 * (\mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda))$ is a finite support iteration of Baumgartner's forcing, and in the generic extension by $\mathbb{P}_\alpha^2 * (\mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda)$, λ has cofinality ω_1 . Thus, by Lemma 2.5, no new cofinal branch to $T_\alpha \cap (\lambda \times \omega)$ is added by this forcing. \square

Let us remark that since Baumgartner's forcing is c.c.c., requirement (4) in the lemma follows immediately from the other requirements. Moreover, since the forcing $\mathbb{P}_\alpha^1 / (\mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda)$ is c.c.c. in the generic extension by $\mathbb{P}_\alpha^2 * (\mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda)$, for a given name for a branch \dot{b} , one can find in the ground model countably many $\mathbb{P}_\alpha^2 * (\mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda)$ -names $\{\dot{b}_n \mid n < \omega\}$ for branches, such that the weakest condition of the quotient forcing, forces that \dot{b} is evaluated as one of them.

The main technical tool is the following separation claim.

Claim 2.16. *Assume $\alpha \in \beta \cap \mathcal{M}$, \mathbb{P}_α^2 is κ -Knaster and $T_\alpha = \Phi(\alpha)$ is a $\mathbb{P}_\alpha^2 * \dot{\mathbb{P}}_\alpha^1$ -name for a κ -Aronszajn tree. Then there exists a measure one set $B \in \mathcal{F}$ such that for every $\lambda \in B$:*

- (1) $\alpha \in \mathcal{M}_\lambda$.
- (2) $\mathcal{M}_\lambda \cap \kappa = \lambda$ and λ is inaccessible.
- (3) \mathcal{M}_λ is closed under $< \lambda$ -sequences.
- (4) $\mathbb{P}_\alpha^2 \cap \mathcal{M}_\lambda \leq \mathbb{P}_\alpha^2 \cap \mathcal{M}$ and is λ -c.c.
- (5) $\mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda$ is (equivalent to) an $\mathbb{P}_\alpha^2 \cap \mathcal{M}_\lambda$ -name.
- (6) $(\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) \cap \mathcal{M}_\lambda \leq (\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) \cap \mathcal{M}$. Moreover, $\mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda$ is a sub-iteration of $\mathbb{P}_\alpha^1 \cap \mathcal{M}$.
- (7) $(\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) \cap \mathcal{M}_\lambda$ forces that $T_\alpha \cap (\lambda \times \omega_1)$ is an Aronszajn tree.

For every such λ we have:

- (8) $\Vdash_{\mathbb{P}_\alpha^2} \mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda \leq \mathbb{P}_\alpha^1$.
- (9) For every pair of $(\mathbb{P}_\alpha^2 \cap \mathcal{M}) * (\mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda)$ -names of cofinal branches $\dot{\tau}, \dot{\theta}$ in the first λ levels of T_α and $p \in \mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda$, and for every λ -compatible $q', q'' \in \mathbb{P}_\beta^2 \cap \mathcal{M}$ with $p = q' \upharpoonright \mathcal{M}_\lambda = q'' \upharpoonright \mathcal{M}_\lambda$, there are λ -compatible conditions

$p', p'' \in \mathbb{P}_\beta^2 \cap \mathcal{M}$, and a countable sequence $\langle (\bar{p}_n, \xi_n, \theta_n, \tau_n) \mid n < \omega \rangle \in \mathcal{M}_\lambda$ such that:

- (a) $p' \leq q', p'' \leq q''$ and $p' \upharpoonright \mathcal{M}_\lambda = p'' \upharpoonright \mathcal{M}_\lambda$.
- (b) $p' \upharpoonright \mathcal{M}_\lambda \Vdash_{\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda} \text{“} \bar{p}_n \in \dot{\mathbb{P}}_\beta^1 \cap \mathcal{M}_\lambda \text{”}$.
- (c) $\xi_n < \lambda, \theta_n, \tau_n \in \{\xi_n\} \times \omega_1$ and $\theta_n \neq \tau_n$.
- (d) $(p' \upharpoonright \alpha, \bar{p}_n \upharpoonright \alpha) \Vdash \text{“} \check{\tau}_n \leq_{T_\alpha} \check{\tau} \text{”}$ and $(p'' \upharpoonright \alpha, \bar{p}_n \upharpoonright \alpha) \Vdash \text{“} \check{\theta}_n \leq_{T_\alpha} \check{\theta} \text{”}$.
- (e) $p' \upharpoonright \mathcal{M}_\lambda \Vdash_{\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda} \text{“} \{\bar{p}_n \mid n < \omega\} \text{ is a maximal antichain in } \dot{\mathbb{P}}_\beta^1 \text{”}$.

Proof. By the hypotheses of Claim 2.16, \mathbb{P}_α^2 has the κ -c.c. Let B_α be the set of all inaccessible cardinals $\lambda < \kappa$ that satisfy the requirements (1)-(6) of the lemma.

Let us verify that $\kappa \in j(B_\alpha)$, and hence $B_\alpha \in \mathcal{F}$. First, note that since the sequence $\langle \mathcal{M}_\lambda \mid \lambda < \kappa \rangle$ is continuous, $j(\mathcal{M})_\kappa = \bigcup_{\lambda < \kappa} j(\mathcal{M}_\lambda) = j^{\text{“}}\mathcal{M}$.

- (1) $j(\alpha) \in j^{\text{“}}\mathcal{M}$, since $\alpha \in \mathcal{M}$ by the assumption of the lemma.
- (2) $j^{\text{“}}\mathcal{M} \cap j(\kappa) = \kappa$.
- (3) $j^{\text{“}}\mathcal{M}$ is closed under $< \kappa$ -sequences. This is true since \mathcal{M} is closed under $< \kappa$ -sequences.
- (4) $j(\mathbb{P}_\alpha^2) \cap j^{\text{“}}\mathcal{M} = j^{\text{“}}(\mathbb{P}_\alpha^2 \cap \mathcal{M})$ and in particular, it is isomorphic to $\mathbb{P}_\alpha^2 \cap \mathcal{M}$ and is κ -c.c. From this fact, together with the closure of $j^{\text{“}}\mathcal{M}$ we conclude that it is a regular subforcing of $j(\mathbb{P}_\alpha^2 \cap \mathcal{M})$.
- (5) This is the same as in the previous assertion.
- (6) Using the previous item and the chain condition of the forcing.
- (7) As in the previous assertion, $j(\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) \cap j^{\text{“}}\mathcal{M}$ is isomorphic to $(\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) \cap \mathcal{M}$. By the chain condition of the forcing $\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1$, $(\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) \cap \mathcal{M} \triangleleft \mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1$. Thus, we conclude that $j^{\text{“}}T_\alpha$ which is exactly the name of $j(T_\alpha) \cap (\kappa \times \omega_1)$, is a name with respect to the regular subforcing $j^{\text{“}}((\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) \cap \mathcal{M})$. Clearly, the subforcing forces it to be an Aronszajn tree.

Next, let us show that the elements of B_α satisfy the clauses (7) and (8) of the lemma. Thus suppose that $\lambda \in B_\alpha$, and fix names $\dot{\theta}$ and $\dot{\tau}$ for branches, and conditions p, q' and q'' as in the statement of the lemma.

It then follows from the choice of λ that, for any $(\mathbb{P}_\alpha^2 \cap \mathcal{M}) * (\mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda)$ -generic filter G over V , the branches $\dot{\theta}^G, \dot{\tau}^G \notin V[G_{(\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) \cap \mathcal{M}_\lambda}]$, where $G_{(\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) \cap \mathcal{M}_\lambda} = G \cap ((\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) \cap \mathcal{M}_\lambda)$.

We now claim that there are densely many pairs of conditions $(p', \bar{p}), (p'', \bar{p}) \in \mathbb{P}_\beta^2 / (\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda) * (\mathbb{P}_\beta^1 \cap \mathcal{M}_\lambda)$ such that $(p' \upharpoonright \alpha, \bar{p} \upharpoonright \alpha), (p'' \upharpoonright \alpha, \bar{p} \upharpoonright \alpha) \in \mathbb{P}_\alpha^2 * \dot{\mathbb{P}}_\alpha^1$ force incompatible values for the branches below $\dot{\theta}$ and $\dot{\tau}$, p' and p'' are λ -compatible and $p' \upharpoonright \mathcal{M}_\lambda = p'' \upharpoonright \mathcal{M}_\lambda$.

If not, we can find conditions $(p', \bar{p}), (p'', \bar{p})$ so that for any extensions $q' \leq p'$ and $q'' \leq p''$ which are λ -compatible and $q' \upharpoonright \mathcal{M}_\lambda = q'' \upharpoonright \mathcal{M}_\lambda$, and any $\bar{q} \leq \bar{p}$, the conditions $(q' \upharpoonright \alpha, \bar{q} \upharpoonright \alpha), (q'' \upharpoonright \alpha, \bar{q} \upharpoonright \alpha) \in \mathbb{P}_\alpha^2 * \dot{\mathbb{P}}_\alpha^1$ can not force incompatible values for the branches $\dot{\theta}$ and $\dot{\tau}$.

Let $G = G_{(\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) \cap \mathcal{M}_\lambda}$ be V -generic for $(\mathbb{P}_\alpha^2 * \mathbb{P}_\alpha^1) \cap \mathcal{M}_\lambda$, and let H_1, H_2 be mutually generic filters for the forcing $\mathbb{P}_\alpha^2 / (\mathbb{P}_\alpha^2 \cap \mathcal{M}_\lambda) * (\mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda)$ over the model $V[G]$. Let us assume that $(p' \upharpoonright \alpha) \upharpoonright \mathcal{M}_\lambda \in G$, $p' \upharpoonright \alpha \in H_1$ and $p'' \upharpoonright \alpha \in H_2$.

By the assumption, $\dot{\tau}^{G*H_1} = \dot{\theta}^{G*H_2}$. In particular,

$$\dot{\tau}^{G*H_1} \in V[G][H_1] \cap V[G][H_2],$$

and by the mutual genericity of H_1 and H_2 - it is in $V[G]$, which is impossible.

Thus we can find a pair of conditions $(p'_0, \bar{p}_0), (p''_0, \bar{p}_0) \in ((\mathbb{P}_\beta^2 \cap \mathcal{M}) / (\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda)) * (\dot{\mathbb{P}}_\beta^1 \cap \mathcal{M}_\lambda)$ with $p'_0 \leq p', p''_0 \leq p''$ and $p'_0 \upharpoonright \mathcal{M}_\lambda = p''_0 \upharpoonright \mathcal{M}_\lambda$ together with $\xi_0 < \lambda$ and elements in $\theta_0, \tau_0 \in \{\xi_0\} \times \omega_1$ such that

- $(p'_0 \upharpoonright \alpha, \bar{p}_0 \upharpoonright \alpha) \Vdash \check{\theta}_0 \in \dot{\theta}$.
- $(p''_0 \upharpoonright \alpha, \bar{p}_0 \upharpoonright \alpha) \Vdash \check{\tau}_0 \in \dot{\tau}$.

Let us repeat the process. Suppose that $\nu < \omega_1$ and we have defined the pairs $(p'_n, \bar{p}_n), (p''_n, \bar{p}_n) \in (\mathbb{P}_\beta^2 \cap \mathcal{M}) / (\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda) * (\dot{\mathbb{P}}_\beta^1 \cap \mathcal{M}_\lambda)$ together with ξ_n and $\theta_n, \tau_n \in \lambda \times \omega_1$ such that

- The sequences $\langle p'_n \mid n < \nu \rangle$ and $\langle p''_n \mid n < \nu \rangle$ are decreasing and for each n , p'_n and p''_n are λ -compatible.
- $p'_n \upharpoonright \mathcal{M}_\lambda = p''_n \upharpoonright \mathcal{M}_\lambda$.
- $p'_n \upharpoonright \mathcal{M}_\lambda \Vdash_{\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda} \text{“} \bar{p}_n \in \dot{\mathbb{P}}_\beta^1 \cap \mathcal{M}_\lambda \text{”}$.
- For $m < n < \nu$, $p'_n \upharpoonright \mathcal{M}_\lambda \Vdash_{\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda} \text{“} \bar{p}_m \text{ and } \bar{p}_n \text{ are incompatible”}$.
- $\xi_n < \lambda, \theta_n, \tau_n \in \{\xi_n\} \times \omega_1$ and $\theta_n \neq \tau_n$.
- $(p'_n \upharpoonright \alpha, \bar{p}_n \upharpoonright \alpha) \Vdash \text{“} \check{\theta}_n \in \dot{\theta} \text{”}$.
- $(p''_n \upharpoonright \alpha, \bar{p}_n \upharpoonright \alpha) \Vdash \text{“} \check{\tau}_n \in \dot{\tau} \text{”}$.

Let $q'_\nu = \bigcup_{n < \nu} p'_n$ and $q''_\nu = \bigcup_{n < \nu} p''_n$. Then $q'_\nu, q''_\nu \in (\mathbb{P}_\beta^2 \cap \mathcal{M}) / (\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda)$ and $q'_\nu \upharpoonright \mathcal{M}_\lambda = q''_\nu \upharpoonright \mathcal{M}_\lambda$. If

$$q'_\nu \upharpoonright \mathcal{M}_\lambda \Vdash \text{“} \{\bar{p}_n \mid n < \nu\} \text{ is a maximal antichain”},$$

then we stop the construction. Otherwise find a condition \bar{q}_ν which is forced to be incompatible with all \bar{p}_n 's, $n < \nu$, and let $(p'_\nu, \bar{p}_\nu), (p''_\nu, \bar{p}_\nu) \in ((\mathbb{P}_\beta^2 \cap \mathcal{M}) / (\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda)) * (\dot{\mathbb{P}}_\beta^1 \cap \mathcal{M}_\lambda)$, $\xi_\nu < \lambda$ and $\theta_\nu, \tau_\nu \in \{\xi_\nu\} \times \omega_1$ be such that

- $(p'_\nu, \bar{p}_\nu) \leq (\bar{q}'_\nu, \bar{q}_\nu)$ and $(p''_\nu, \bar{p}_\nu) \leq (\bar{q}''_\nu, \bar{q}_\nu)$.
- $p'_\nu \upharpoonright \mathcal{M}_\lambda = p''_\nu \upharpoonright \mathcal{M}_\lambda$.
- $(p'_\nu \upharpoonright \alpha, \bar{p}_\nu \upharpoonright \alpha) \Vdash \text{“} \check{\theta}_\nu \in \dot{\theta} \text{”}$.
- $(p''_\nu \upharpoonright \alpha, \bar{p}_\nu \upharpoonright \alpha) \Vdash \text{“} \check{\tau}_\nu \in \dot{\tau} \text{”}$.

By Lemmas 2.12 and 2.11, this process terminates after at most countably many steps. At the end of the process, we get a countable ordinal ϑ , sequences $\langle p'_n \mid n < \vartheta \rangle$ and $\langle p''_n \mid n < \vartheta \rangle$ of conditions in $(\mathbb{P}_\beta \cap \mathcal{M}) / (\mathbb{P}_\beta \cap \mathcal{M}_\lambda)$, and sequences $\{\bar{p}_n \mid n < \vartheta\}$ and $\langle (\xi_n, \theta_n, \tau_n) \mid n < \vartheta \rangle$ such that

- The sequences $\langle p'_n \mid n < \vartheta \rangle$ and $\langle p''_n \mid n < \vartheta \rangle$ are decreasing and $p'_n \upharpoonright \mathcal{M}_\lambda = p''_n \upharpoonright \mathcal{M}_\lambda$. Let $p' = \bigcup_{n < \vartheta} p'_n$ and $p'' = \bigcup_{n < \vartheta} p''_n$.

- $p' \upharpoonright \mathcal{M}_\lambda \Vdash_{\mathbb{P}_\beta^2 \cap \mathcal{M}_\lambda} \langle \bar{p}_n \mid n < \omega \rangle$ is a maximal antichain in $\dot{\mathbb{P}}_\beta^1$.
- For all $n < \vartheta$, $\theta_n, \tau_n \in \{\xi_n\} \times \omega_1$ and $\theta_n \neq \tau_n$.
- For all $n < \vartheta$, $(p'_n \upharpoonright \alpha, \bar{p}_n \upharpoonright \alpha) \Vdash \check{\tau}_n \in \check{\tau}$ and $(p''_n \upharpoonright \alpha, \bar{p}_n \upharpoonright \alpha) \Vdash \check{\theta}_n \in \check{\theta}$.

Then p', p'' together with the sequence $\langle (\bar{p}_n, \xi_n, \theta_n, \tau_n) \mid n < \vartheta \rangle$ are as required. \square

Let us call the sequence $\langle (\bar{p}_n, \xi_n, \theta_n, \tau_n) \mid n < \omega \rangle$ a λ -*separating witness* for the branches θ, τ relative to p', p'' .

Let λ be as in the claim. Let $p', p'' \in \mathbb{P}_\beta^2 \cap \mathcal{M}$ be arbitrary λ -compatible conditions, with $p' \upharpoonright \mathcal{M}_\lambda = p'' \upharpoonright \mathcal{M}_\lambda$. For every $\alpha \in \text{dom}(p') \cap \mathcal{M}_\lambda$ and every element $\theta \in \text{dom}(p'(\alpha))$ above λ , there are at most countably $(\mathbb{P}_\alpha^2 \cap \mathcal{M}) * (\mathbb{P}_\alpha^2 \cap \mathcal{M}_\lambda)$ -names for the branch $\{t \in T_\alpha \mid t \leq \theta, \text{Lev}_{T_\alpha}(t) < \lambda\}$, by the Claim 2.15 and the discussion following it.

If θ, τ are elements in the tree T_α with $\text{Lev}_{T_\alpha}(\theta), \text{Lev}_{T_\alpha}(\tau) \geq \lambda$, then we may apply Claim 2.16 for the countably many possible pairs of names for the branches below λ that θ and τ contributes, and obtain countably many separation pairs. Let us call this countable collection of separating witness, a λ -*separating witness* for θ and τ .

Thus, by the repeated use of Claim 2.16, countably many times, there is a measure one set $A \in \mathcal{F}$ such that for every condition $p \in \mathbb{P}_\beta^2 \cap \mathcal{M}$, and every $\lambda \in A$, $\mathcal{M}_\lambda \cap \kappa = \kappa$, there are conditions $p', p'' \leq p$ such that

- $p' \upharpoonright \mathcal{M}_\lambda = p'' \upharpoonright \mathcal{M}_\lambda$.
- For every $\alpha \in \beta \cap \mathcal{M}_\lambda$, any pair of elements above λ in $\text{dom}(p'(\alpha)) \times \text{dom}(p''(\alpha))$ has a λ -separating witness in \mathcal{M}_λ relative to $p' \upharpoonright \alpha, p'' \upharpoonright \alpha$.

We call this pair (p', p'') a λ -*separating pair*. Note that $\text{dom}(p)$ might contain elements from $\mathcal{M} \setminus \mathcal{M}_\lambda$ which are not treated.

Now let $\langle p_\lambda \mid \lambda < \kappa \rangle \in \mathcal{M}$ be a sequence of conditions in \mathbb{P}_β^2 .¹

¹Recall that if there exists a sequence of condition $\langle p_\lambda \mid \lambda < \kappa \rangle$ which contradicts the κ -Knaster property of \mathbb{P}_β^2 , then there is such a sequence in \mathcal{M} as well, by elementarity.

For every $\lambda \in A$, p_λ can be extended to a λ -separating pair $(p'_\lambda, p''_\lambda) \in \mathcal{M}$. Let $s_\lambda \in \mathcal{M}_\lambda$ be the list of separating witnesses.

The function that sends λ to $(s_\lambda, p' \upharpoonright \mathcal{M}_\lambda)$ is regressive. By the normality of the weakly compact filter (recall that every member of \mathcal{F} is positive with respect to the weakly compact filter), and by further shrinking if necessary, we may assume that on a positive set $D \in \mathcal{S}$, all s_λ and $p' \upharpoonright \mathcal{M}_\lambda, \lambda \in D$, are fixed.

Moreover, we may assume that for every $\lambda \in D$ and $\lambda' \in D$ above λ , $p'_\lambda, p''_\lambda \in \mathcal{M}_{\lambda'}$ (in particular, the domain of p'_λ, p''_λ as well as the domain of $p'_\lambda(\alpha), p''_\lambda(\alpha)$ are subsets of $\mathcal{M}_{\lambda'}$). By additional shrinking of D , if needed, we may assume that the sets $\text{supp}(p'_\lambda) \cup \text{supp}(p''_\lambda)$ form a Δ -system with root Λ , in the sense that, if $\lambda < \lambda'$ are in D , and $\alpha \in (\text{supp}(p'_\lambda) \cup \text{supp}(p''_\lambda)) \setminus \Lambda$, then $\alpha \notin \text{supp}(p'_{\lambda'}) \cup \text{supp}(p''_{\lambda'})$. Without loss of generality, $\Lambda \subseteq \mathcal{M}_\lambda$ for $\lambda = \min D$.

We claim that for any $\lambda < \lambda'$ in D , p_λ is compatible with $p_{\lambda'}$, and moreover this compatibility is witnessed by the condition q , which is defined by $q(\alpha) = p'_\lambda(\alpha) \cup p''_{\lambda'}(\alpha)$ for every $\alpha < \beta$. It is enough to show that q is a condition. Clearly, $\text{dom}(q)$ is at most countable. Therefore, it is enough to show that $q \upharpoonright \gamma$ forces that $q(\gamma)$ is a condition for all $\gamma < \beta$. We prove this by induction on $\gamma < \beta$.

For $\gamma = 0$, $q(0) \in \text{Col}(\aleph_1, < \kappa)$, since it is the union of two conditions that have the same intersection with \mathcal{M}_λ , and have disjoint domains above it.

Assume that $q \upharpoonright \gamma$ is a condition. We may assume that $T_\gamma = \Phi(\gamma)$ is a $\mathbb{P}_\gamma^2 * \dot{\mathbb{P}}_\gamma^1$ -name for a κ -Aronszajn tree, as otherwise the forcing at stage γ is trivial. We may also assume that $\gamma \in \Lambda$, since otherwise either $\gamma \notin \text{supp}(p'_\lambda)$ or $\gamma \notin \text{supp}(p''_{\lambda'})$. In order to show that $q \upharpoonright \gamma \Vdash$ “ $q(\gamma)$ is a condition”, we have to show that if $t, t' \in \text{dom}(q(\gamma))$ and $q(\gamma)(t) = q(\gamma)(t')$, then $q \upharpoonright \gamma \Vdash_{\mathbb{P}_\gamma^2} \text{“} \mathbf{1}_{\mathbb{P}_\gamma^1} \Vdash_{\mathbb{P}_\gamma^1} \check{t} \perp_{T_\gamma} \check{t}' \text{”}$.

We may suppose that both of t and t' are above λ , as otherwise we can use the fact $p'_\lambda \upharpoonright \mathcal{M}_\lambda = p''_{\lambda'} \upharpoonright \mathcal{M}'_\lambda$ and the fact that $\gamma \in \mathcal{M}_\lambda$, to conclude the result.

Recall that $(p'_\lambda, p''_\lambda)$ is a separating pair. Let \dot{b}_t be one of the countably many possible names for branches below λ of elements below t and let $\dot{b}_{t'}$ be a corresponding

name for t' . The separating witness $\langle \bar{p}_n, \tau_n, \theta_n \mid n < \omega \rangle$ was stabilized for elements in D , and thus $(p'_\lambda \upharpoonright \gamma, \bar{p}_n \upharpoonright \gamma) \Vdash \check{\tau}_n \in \dot{b}_t$ and $(p''_\lambda \upharpoonright \gamma, \bar{p}_n \upharpoonright \gamma) \Vdash \check{\theta}_n \in \dot{b}_{t'}$, where $\tau_n \neq \theta_n$. By the induction hypothesis, $q \upharpoonright \gamma$ is a condition and it is stronger than $p'_\lambda \upharpoonright \gamma$ and $p''_\lambda \upharpoonright \gamma$. Let us denote, temporarily by \tilde{t} the element in the λ -th level of T_γ above \dot{b}_t and by \tilde{t}' the element in the λ' -th level of T_γ above $\dot{b}_{t'}$. We obtained that for all $n < \omega$, $(q \upharpoonright \gamma, \bar{p}_n) \Vdash \check{t} \perp_{T_\gamma} \tilde{t}'$. Now if $q \upharpoonright \gamma \not\Vdash_{\mathbb{P}_\gamma^2} \check{1}_{\mathbb{P}_\gamma^1} \Vdash_{\mathbb{P}_\gamma^1} \check{t} \perp \tilde{t}'$, then there is a condition $q' \leq q \upharpoonright \gamma$ and $\bar{p} \in \mathbb{P}_\gamma^1$ such that $(q', \bar{p}) \Vdash \check{t} = \tilde{t}'$. But \bar{p} is compatible with \bar{p}_n , for some $n < \omega$. As q' is stronger than $q \upharpoonright \gamma$, $(q', \bar{p}_n \upharpoonright \gamma) \Vdash \check{t} = \tilde{t}'$. It then follows that

$$(q', \bar{p}_n \upharpoonright \gamma) \Vdash \check{\theta}_n \leq_{T_\gamma} \tilde{t}' = \check{t} \quad \& \quad (q', \bar{p}_n \upharpoonright \gamma) \Vdash \check{\tau}_n \leq_{T_\gamma} \tilde{t}'.$$

This is in contradiction with the choice of θ_n and τ_n . Since this is true for all possible $\tilde{t} \leq_{T_\gamma} t$ and $\tilde{t}' \leq_{T_\gamma} t'$, we conclude that they are forced to be incompatible.

If γ is a limit ordinal and $q \upharpoonright \bar{\gamma}$ is a condition for all $\bar{\gamma} < \gamma$, then $q \upharpoonright \gamma$ is a condition as well. Lemma 2.13 follows. \square

The next lemma follows from Lemmas 2.12 and 2.13

- Lemma 2.17.** (a) For every $\alpha \leq \delta$, $\mathbb{P}_\alpha^2 * \dot{\mathbb{P}}_\alpha^1$ satisfies the κ -c.c.
 (b) $\mathbb{P} = \mathbb{P}_\delta^2 * \dot{\mathbb{P}}_\delta^1$ satisfies the κ -c.c.

Putting the above lemmas together, we obtain the following result.

Lemma 2.18. Suppose G is \mathbb{P} -generic over V . Then

- (a) $\aleph_1^{V[G]} = \aleph_1$, $\aleph_2^{V[G]} = \kappa$ and $\aleph_3^{V[G]} = \kappa^+$.
 (b) $V[G] \models \aleph_0 = 2^{\aleph_0} = 2^{\aleph_1} = \delta$.

2.5. Completing the proof of Theorem 1.2. In this subsection we complete the proof of Theorem 1.2. The next lemma follows from Lemma 2.17

Lemma 2.19. Suppose $X \in V[G_\mathbb{P}]$ and $X \subseteq \kappa$. Then $X \in V[G_{\mathbb{P}_\alpha^2 * \dot{\mathbb{P}}_\alpha^1}]$, for some $\alpha < \delta$.

We start by showing that the special Aronszajn tree property holds in $V[G_{\mathbb{P}}]$.

Lemma 2.20. \mathbb{P} forces SATP(\aleph_1).

Proof. Let T be an \aleph_1 -Aronszajn tree and let \dot{T} be a \mathbb{P} -name for it. Then for some $\alpha < \delta$ it is in fact a \mathbb{P}_α -name and $\dot{T} = \Phi(\alpha)$. Then

$$\Vdash_{\mathbb{P}_{\alpha+1}^2 * \dot{\mathbb{P}}_{\alpha+1}^1} \text{“}\dot{T} \text{ is specialized”},$$

and hence there exists $F \in V[G_{\mathbb{P}_{\alpha+1}^2 * \dot{\mathbb{P}}_{\alpha+1}^1}]$ which is a specializing function for T . As $V[G_{\mathbb{P}}] \supseteq V[G_{\mathbb{P}_{\alpha+1}^2 * \dot{\mathbb{P}}_{\alpha+1}^1}]$ and these models have the same cardinals, F is also a specializing function for T in $V[G_{\mathbb{P}}]$. \square

In order to show that the forcing notion \mathbb{P} specializes all \aleph_2 -Aronszajn trees, we need the following lemma which is an analogue of Lemma 2.9(b).

Lemma 2.21. Suppose $\alpha < \delta$ and $\Phi(\alpha)$ is a $\mathbb{P}_\alpha^2 * \dot{\mathbb{P}}_\alpha^1$ -name for a κ -Aronszajn tree. Then in the extension by $\mathbb{P}_{\alpha+1}^2$, there exists a function $F : \kappa \times \omega_1 \rightarrow \omega_1$ which is a specializing function of every generic interpretation of $\Phi(\alpha)$ by a \mathbb{P}_α^1 -generic filter.

Lemma 2.22. \mathbb{P} forces SATP(κ).

Proof. First, there is an \aleph_2 -Aronszajn tree in the generic extension, as the forcing $\text{Col}(\omega_1, < \kappa)$ adds a special \aleph_2 -Aronszajn tree and cardinals are preserved in the rest of the iteration.

Let T be a κ -Aronszajn tree and let \dot{T} be a \mathbb{P} -name for it. Then for some $\alpha < \delta$ it is in fact a \mathbb{P}_α -name and $\dot{T} = \Phi(\alpha)$. By Lemma 2.21, there exists $F \in V[G_{\mathbb{P}_{\alpha+1}^2 * \dot{\mathbb{P}}_\alpha^1}]$ which specializes T . As $V[G_{\mathbb{P}}]$ is a cardinal preserving extension of $V[G_{\mathbb{P}_{\alpha+1}^2 * \dot{\mathbb{P}}_\alpha^1}]$, F also witnesses that T is specialized in $V[G_{\mathbb{P}}]$. The lemma follows. \square

3. SPECIALIZING NAMES OF HIGHER ARONSZAJN TREES: AN ABSTRACT APPROACH

Let us note that in the proof of Theorem 1.2, we did not use the way the forcing notions $\mathbb{P}_\alpha^1, \alpha \leq \delta$ were defined, but only the fact that they satisfy the c.c.c. and

that the forcing notions $\mathbb{P}_\alpha^1, \alpha \leq \delta$, do not add new branches to trees of height \aleph_1 . In this section we present the above situation in an abstract way that will be used for the next sections of this paper.

Thus suppose that $\mu < \kappa < \delta$ are regular cardinals. Let Φ and Ψ be two functions such that:

- $\Phi : \delta \rightarrow H(\delta)$ is such that for each $x \in H(\delta)$, $\Phi^{-1}(x)$ is unbounded in δ .
- $\Psi : \delta \rightarrow H(\delta)$ is such that for each $\alpha < \delta$, $\Psi(\alpha)$ is a forcing notion.

Let

$$\langle\langle \mathbb{P}_\alpha^2 \mid \alpha \leq \delta \rangle\rangle, \langle\langle \dot{\mathbb{Q}}_\alpha^2 \mid \alpha < \delta \rangle\rangle.$$

be a forcing iteration of length δ , defined as follows:

Set $\mathbb{Q}_0^2 = \text{Col}(\mu, < \kappa)$.

If α is a limit ordinal and $\text{cf}(\alpha) \geq \mu$, let \mathbb{P}_α^2 be the direct limit of the forcing notions $\mathbb{P}_\beta^2, \beta < \alpha$. If α is a limit ordinal and $\text{cf}(\alpha) < \mu$, let \mathbb{P}_α^2 be the inverse limit of the forcing notions $\mathbb{P}_\beta^2, \beta < \alpha$.

Now suppose that $\alpha = \beta + 1$ is a successor ordinal. Let us assume that $\Psi(\beta)$ is such that $\Psi(\beta) = \mathbb{P}_\beta^2 * \dot{\mathbb{P}}_\beta^1$ for some \mathbb{P}_β^2 -name $\dot{\mathbb{P}}_\beta^1$, where \mathbb{P}_β^1 is an iteration of length $\leq \beta$ with $< \zeta$ -supports, for some $\zeta < \mu$, of forcing notions of size $< \kappa$. Moreover, let us assume that $\Phi(\beta)$ is a $\Psi(\beta)$ -name for a κ -Aronszajn tree with the universe $\kappa \times \mu$. Then let $\dot{\mathbb{Q}}_\beta^2$ be a \mathbb{P}_β^2 -name, such that in the generic extension $V[G_{\mathbb{P}_\beta^2}]$, the forcing notion \mathbb{Q}_β^2 is defined as follows:

- Conditions in \mathbb{Q}_β^2 are partial functions $f : \kappa \times \mu \rightarrow \mu$ such that:
 - (1) $\text{dom}(f) \subseteq \kappa \times \mu$ has size $< \mu$.
 - (2) If $s, t \in \text{dom}(f)$ and $f(s) = f(t)$ then $\Vdash_{\mathbb{P}_\beta^1} \text{“}\dot{s} \perp_{\Phi(\beta)} \dot{t}\text{”}$.
- For $f, g \in \mathbb{Q}_\beta^2$, $f \leq g$ if and only if $f \supseteq g$.

Otherwise, let $\dot{\mathbb{Q}}_\beta^2$ be a name for the trivial forcing notion.

It is obvious that the forcing notions $\mathbb{P}_\alpha^2, \alpha \leq \delta$ are μ -directed closed.

Let us recall all of those properties which were used in the proof of Claim 2.16.

Definition 3.1. We say that the triple (Φ, Ψ, δ) is (μ, κ) -suitable, if the following conditions hold. First, let $\mathbb{P}_\alpha^2, \dot{\mathbb{P}}_\alpha^1$, be defined as above using Φ and Ψ . Also, let $\langle \mathcal{M}_\lambda \mid \lambda < \kappa \rangle$ be a continuous chain of elementary submodels of the universe of size $< \kappa$ which contain all the relevant information. Let $\mathcal{M} = \bigcup_{\lambda < \kappa} \mathcal{M}_\lambda$.

- (1) $\mu < \kappa$ are regular cardinals and $\Phi, \Psi: \delta \rightarrow H(\delta)$ are as above.
- (2) For each $\alpha \leq \delta$ and $\gamma \in [\alpha, \delta]$, $\Vdash_{\mathbb{P}_\gamma^2}$ “ $\dot{\mathbb{P}}_\alpha^1$ is μ -c.c.”.
- (3) For each $\lambda < \kappa$ and $\alpha \in \mathcal{M}_\lambda \cap \delta$, if

- (a) $\mathbb{P}_\alpha^2 \cap \mathcal{M}_\lambda \triangleleft \mathbb{P}_\alpha^2 \cap \mathcal{M}$.
- (b) $\Vdash_{\mathbb{P}_\alpha^2 \cap \mathcal{M}}$ “ $\mathbb{P}_\alpha^1 \cap \mathcal{M}_\lambda \triangleleft \mathbb{P}_\alpha^1 \cap \mathcal{M}$ and moreover, it is a sub-iteration”.
- (c) $\Phi(\alpha)$ is a $\mathbb{P}_\alpha^2 * \dot{\mathbb{P}}_\alpha^1$ -name for a κ -Aronszajn tree.
- (d) $\Phi(\alpha) \cap \mathcal{M}_\lambda$ is a $(\mathbb{P}_\alpha^2 \cap \mathcal{M}_\lambda) * (\dot{\mathbb{P}}_\alpha^1 \cap \mathcal{M}_\lambda)$ -name for a λ -Aronszajn tree.

Then forcing with $(\mathbb{P}_\alpha^2 \cap \mathcal{M} * \dot{\mathbb{P}}_\alpha^1) / (\mathbb{P}_\alpha^2 \cap \mathcal{M} * (\dot{\mathbb{P}}_\alpha^1 \cap \mathcal{M}_\lambda))$ does not add any new branches to $\Phi(\alpha) \cap \mathcal{M}_\lambda$.

Lemma 3.2. Suppose that (Φ, Ψ, δ) are (μ, κ) -suitable and κ is weakly compact. Then \mathbb{P}_α^2 is κ -Knaster for every $\alpha \leq \delta$.

The following lemma is parallel to Lemma 2.7, but for the forcing \mathbb{P}_δ^2 , in the abstract context.

Lemma 3.3. Assume that \mathbb{P}_δ^2 is derived from a (μ, κ) -suitable triple (Φ, Ψ, δ) and κ is weakly compact. Let $I \subseteq \delta$ be a set of ordinals such that \mathbb{P}_I^2 is a sub-iteration. Let also S be a tree of height κ in the generic extension by \mathbb{P}_I^2 . Then, $\mathbb{P}_\delta^2 / \mathbb{P}_I^2$ does not add a new cofinal branch to S .

Proof. The forcing $\mathbb{P}_I^2 * ((\mathbb{P}_\delta^2 / \mathbb{P}_I^2) \times (\mathbb{P}_\delta^2 / \mathbb{P}_I^2))$ is forcing equivalent to \mathbb{P}_γ^2 , for some ordinal γ , by modifying Φ and Ψ . Thus, it is κ -c.c. In particular, the forcing $(\mathbb{P}_\delta^2 / \mathbb{P}_I^2) \times (\mathbb{P}_\delta^2 / \mathbb{P}_I^2)$ is forced to be κ -c.c and thus by [9], $(\mathbb{P}_\delta^2 / \mathbb{P}_I^2)$ does not add cofinal branches to a tree of height κ . \square

In the next sections we will use the mechanism of this section in order to specialize trees at many cardinals simultaneously. Thus, we will need to verify that when using Ψ to guess forcing notions that specialize trees, the rest of the iteration does not destroy their chain condition.

Lemma 3.4. *Let $\mu < \kappa$ be regular cardinals and let (Φ, Ψ, δ) be (μ, κ) -suitable. Let $I \subseteq \delta$ be a set of ordinals such that \mathbb{P}_I^2 is a sub-iteration of \mathbb{P}_δ^2 . Let S be a μ -Aronszajn tree which is introduced by a μ -c.c. forcing notion \mathbb{R} in the generic extension by \mathbb{P}_I^2 . Then $\mathbb{P}^2/\mathbb{P}_I^2$ does not introduce new branches to S .*

Proof. Note that $\mathbb{P}^2/\mathbb{P}_I^2$ is μ -closed in the generic extension by \mathbb{P}_I^2 and $2^\mu \geq \kappa$. Thus, we can apply [9, Lemma 3.2] and conclude that it cannot add a cofinal branch to S . \square

We will use this lemma inductively in order to justify the preservation of the chain condition of the specialization forcings in generic extensions.

4. THE SPECIAL ARONSZAJN TREE PROPERTY AT ω -MANY SUCCESSIVE CARDINALS

In this section we prove Theorem 1.3. The proof is essentially based on a modification of the proof of Theorem 1.2, using the abstract approach as described in Section 3, where instead of considering two successive cardinals we consider ω -many of them.

Thus let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of supercompact cardinals, $\delta = (\sup_{n < \omega} \kappa_n)^{++}$ and let $\mu < \kappa_0$ be a regular cardinal ².

Let us recall Laver's indestructibility supercompactness lemma, in the form that will be used in this paper.

²For the proof of Theorem 1.3 it suffices to take $\mu = \aleph_0$, but here we will prove a stronger statement that will be used in the next section for the proof of Theorem 1.4.

Lemma 4.1 (Laver, [6]). *Assume η is a regular cardinal and $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals above η . Then there exists an η -directed closed forcing notion $\mathbb{L}(\eta, \langle \kappa_n \mid n < \omega \rangle)$ which makes the supercompactness of each κ_n indestructible under κ_n -directed closed forcing notions. Further, if $2^{\kappa_n} = \kappa_n^+$ in the ground model, then it holds in the generic extension by $\mathbb{L}(\eta, \langle \kappa_n \mid n < \omega \rangle)$.*

By the above lemma, we may also assume that for each n , κ_n is indestructible under κ_n -directed closed forcing notions and that $2^{\kappa_n} = \kappa_n^+$. For notational reasons, it is convenient to denote $\kappa_{-1} = \mu$.

For each regular cardinal $\eta < \delta$, set $S_\eta^\delta = \{\alpha < \delta \mid \text{cf}(\alpha) = \eta\}$. Let also $\Phi : \delta \rightarrow H(\delta)$ be such that for each $x \in H(\delta)$ and $n < \omega$, $\Phi^{-1}(x) \cap S_{\kappa_n^+}^\delta$ is unbounded in δ .

We define an iteration

$$\mathbb{P}_\delta = \langle \langle \mathbb{P}_\alpha \mid \alpha \leq \delta \rangle, \langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle \rangle$$

of length δ as follows. During the iteration, we also define the auxiliary forcing notions $\mathbb{P}_\alpha(< \kappa_n)$, $\mathbb{P}_\alpha(\kappa_n)$ and $\mathbb{P}_\alpha(> \kappa_n)$, for $n < \omega$, $\alpha \leq \delta$ in such a way that

$$\mathbb{P}_\alpha \cong \mathbb{P}_\alpha(> \kappa_n) * \dot{\mathbb{P}}_\alpha(\kappa_n) * \dot{\mathbb{P}}_\alpha(< \kappa_n),$$

where

- (a) $\mathbb{P}_\alpha(> \kappa_n)$ is κ_n -directed closed.
- (b) $\Vdash_{\mathbb{P}_\alpha(> \kappa_n)} \dot{\mathbb{P}}_\alpha(\kappa_n)$ is κ_n -c.c. and κ_{n-1} -directed closed”.
- (c) $\Vdash_{\mathbb{P}_\alpha(> \kappa_n) * \dot{\mathbb{P}}_\alpha(\kappa_n)} \dot{\mathbb{P}}_\alpha(< \kappa_n)$ is κ_{n-1} -c.c. and μ -directed closed”.

Set $\mathbb{Q}_0 = \prod_{n < \omega} \text{Col}(\kappa_{n-1}, < \kappa_n)$ be the full-support product of the forcing notions $\text{Col}(\kappa_{n-1}, < \kappa_n)$, $n < \omega$. Let also

- (1) $\mathbb{P}_1(< \kappa_n) = \prod_{m < n} \text{Col}(\kappa_{m-1}, < \kappa_m)$.
- (2) $\mathbb{P}_1(\kappa_n) = \text{Col}(\kappa_{n-1}, < \kappa_n)$.
- (3) $\mathbb{P}_1(> \kappa_n) = \prod_{m > n} \text{Col}(\kappa_{m-1}, < \kappa_m)$.

Now suppose that $\alpha \leq \delta$, and that we have defined the forcing notions \mathbb{P}_β and $\mathbb{P}_\beta(< \kappa_n)$, $\mathbb{P}_\beta(\kappa_n)$, $\mathbb{P}_\beta(> \kappa_n)$ for $n < \omega$ and $\beta < \alpha$. We define \mathbb{P}_α , $\mathbb{P}_\alpha(< \kappa_n)$, $\mathbb{P}_\alpha(\kappa_n)$ and $\mathbb{P}_\alpha(> \kappa_n)$ as follows. A condition p is in \mathbb{P}_α if and only if

- (1) p has domain α and $\text{supp}(p) \subseteq \bigcup_{n < \omega} S_{\kappa_n^+}^\delta$, where $\text{supp}(p)$ denotes the support of p .
- (2) For each $n < \omega$, $|\text{supp}(p) \cap S_{\kappa_n^+}^\delta| < \kappa_{n-1}$.
- (3) If $\beta \in \text{supp}(p) \cap S_{\kappa_n^+}^\delta$, then $\mathbb{P}_{\beta+1}(> \kappa_n) \cong \mathbb{P}_\beta(> \kappa_n)$.
- (4) If $\beta \in \text{supp}(p) \cap S_{\kappa_n^+}^\delta$ and $\Phi(\beta)$ is a $\mathbb{P}_{\beta+1}(> \kappa_n) * \dot{\mathbb{P}}_\beta(\kappa_n) * \dot{\mathbb{P}}_\beta(< \kappa_n)$ -name for a κ_n -Aronszajn tree, then it is forced by $\mathbb{P}_{\beta+1}(> \kappa_n)$ that $\dot{\mathbb{Q}}_\beta$ consists of those partial functions $f : \Phi(\beta) \rightarrow \kappa_{n-1}$ with domain of size $< \kappa_{n-1}$, such that for every $t, s \in \text{dom}(f)$ with $f(t) = f(s)$, we have $\Vdash_{\mathbb{P}_\beta(< \kappa_n)} t \perp_{\Phi(\beta)} s$.
Otherwise $\dot{\mathbb{Q}}_\beta$ is forced to be the trivial forcing notion.

For $n < \omega$, $\mathbb{P}_\alpha(> \kappa_n)$ is defined as

$$\mathbb{P}_\alpha(> \kappa_n) = \{p \in \mathbb{P}_\alpha \mid \text{supp}(p) \subseteq \bigcup_{m > n} S_{\kappa_m^+}^\delta\}.$$

It is then clear that $\mathbb{P}_\alpha(> \kappa_n)$ is a regular subforcing of \mathbb{P}_α . Working in $\mathbb{P}_\alpha(> \kappa_n)$, the forcing notion $\mathbb{P}_\alpha(\kappa_n)$ is defined as

$$\mathbb{P}_\alpha(\kappa_n) = \{p \in \mathbb{P}_\alpha \mid \text{supp}(p) \subseteq S_{\kappa_n^+}^\delta\}.$$

Finally, the forcing notion $\mathbb{P}_\alpha(< \kappa_n)$ is defined in the generic extension by $\mathbb{P}_\alpha(> \kappa_n) * \dot{\mathbb{P}}_\alpha(\kappa_n)$ by

$$\mathbb{P}_\alpha(< \kappa_n) = \{p \in \mathbb{P}_\alpha \mid \text{supp}(p) \subseteq \bigcup_{m < n} S_{\kappa_m^+}^\delta\}.$$

Note that the map

$$p \mapsto (p \upharpoonright \bigcup_{m > n} S_{\kappa_m^+}^\delta, p \upharpoonright S_{\kappa_n^+}^\delta, p \upharpoonright \bigcup_{m < n} S_{\kappa_m^+}^\delta)$$

defines a dense embedding from \mathbb{P}_α to $\mathbb{P}_\alpha(> \kappa_n) * \dot{\mathbb{P}}_\alpha(\kappa_n) * \dot{\mathbb{P}}_\alpha(< \kappa_n)$ and hence $\mathbb{P}_\alpha \cong \mathbb{P}_\alpha(> \kappa_n) * \dot{\mathbb{P}}_\alpha(\kappa_n) * \dot{\mathbb{P}}_\alpha(< \kappa_n)$.

Let us argue that clauses (a)-(c) continue to hold at α . Clause (a) is evident. Clauses (b) and (c) follow from the next lemma.

Lemma 4.2. *Work in the generic extension $V[G_{\mathbb{P}_\alpha(>\kappa_n)}]$ by $\mathbb{P}_\alpha(>\kappa_n)$. Then $\mathbb{P}_\alpha(\kappa_n)$ is κ_{n-1} -directed closed and κ_n -c.c. and $\Vdash_{\mathbb{P}_\alpha(\kappa_n)}$ “ $\mathbb{P}_\alpha(<\kappa_n)$ is μ -closed and κ_{n-1} -c.c.”*

Proof. Work in $V[G_{\mathbb{P}_\alpha(>\kappa_n)}]$. It is clear that $\mathbb{P}_\alpha(\kappa_n)$ is κ_{n-1} -directed closed and $\Vdash_{\mathbb{P}_\alpha(\kappa_n)}$ “ $\mathbb{P}_\alpha(<\kappa_n)$ is μ -closed”.

As the forcing notion $\mathbb{P}_\alpha(>\kappa_n)$ is κ_n -directed closed, the cardinals $\kappa_m, m \leq n$, remain supercompact in $V[G_{\mathbb{P}_\alpha(>\kappa_n)}]$. Suppose also $G_{\mathbb{P}_\alpha(\kappa_n)}$ is $\mathbb{P}_\alpha(\kappa_n)$ -generic over $V[G_{\mathbb{P}_\alpha(>\kappa_n)}]$.

Working in $V[G_{\mathbb{P}_\alpha(>\kappa_n)}][G_{\mathbb{P}_\alpha(\kappa_n)}]$, each $\kappa_m, m < n$, remains supercompact, and the forcing notion $\mathbb{P}_\alpha(<\kappa_n)$ can be seen as a finite iteration

$$\mathbb{P}_\alpha(<\kappa_n) \cong \mathbb{P}_\alpha(\kappa_{n-1}) * \cdots * \dot{\mathbb{P}}_\alpha(\kappa_0),$$

where for each $m < n$,

- (1) $\mathbb{P}_\alpha(\kappa_{n-1}) * \cdots * \dot{\mathbb{P}}_\alpha(\kappa_{m+1})$ is κ_m -directed closed;
- (2) It is forced by $\mathbb{P}_\alpha(\kappa_{n-1}) * \cdots * \dot{\mathbb{P}}_\alpha(\kappa_{m+1})$ that the forcing notion $\mathbb{P}_\alpha(\kappa_m)$ specializes $\mathbb{P}_\alpha(\kappa_{m-1}) * \cdots * \dot{\mathbb{P}}_\alpha(\kappa_0)$ -names of κ_m -Aronszajn trees.

By (1), κ_m remains supercompact and hence weakly compact in the generic extension by $\mathbb{P}_\alpha(\kappa_{n-1}) * \cdots * \dot{\mathbb{P}}_\alpha(\kappa_{m+1})$, so using Lemma 3.2 and by induction on $m < n$,

$$\Vdash_{\mathbb{P}_\alpha(\kappa_{n-1}) * \cdots * \dot{\mathbb{P}}_\alpha(\kappa_{m+1})} \text{“} \mathbb{P}_\alpha(\kappa_m) \text{ is } \kappa_{m-1}\text{-directed closed and } \kappa_m\text{-c.c.} \text{”}.$$

In particular $\Vdash_{\mathbb{P}_\alpha(\kappa_n)}$ “ $\mathbb{P}_\alpha(<\kappa_n)$ is κ_{n-1} -c.c.”.

Note that in order to apply Lemma 3.2, we had to make sure that whenever some name for a κ_n -Aronszajn tree is chosen in step $\gamma < \alpha$, then it is going to remain Aronszajn after forcing with $\mathbb{P}_\alpha(>\kappa_n)/\mathbb{P}_\gamma(>\kappa_n)$. This is true by the arguments of Lemma 3.4, working inductively to show that the chain condition requirements hold.

As $\mathbb{P}_\alpha(> \kappa_n)$ is κ_n -directed closed, κ_n remains supercompact and hence weakly compact in $V[G_{\mathbb{P}_\alpha(> \kappa_n)}]$. So again by Lemma 3.2 the forcing notion $\mathbb{P}_\alpha(\kappa_n)$ is κ_n -c.c. \square

Let

$$\langle\langle G_\alpha \mid \alpha \leq \delta \rangle, \langle H_\alpha \mid \alpha < \delta \rangle\rangle$$

be \mathbb{P}_δ -generic over V . Thus for each $\alpha \leq \delta$, G_α is \mathbb{P}_α -generic over V , and if $\alpha < \delta$, then H_α is $\dot{\mathbb{Q}}_\alpha[G_\alpha]$ -generic over $V[G_\alpha]$.

It is clear that

Lemma 4.3. (a) $\Vdash_{\mathbb{P}_1} \text{“}\forall n < \omega, \kappa_n = \mu^{+n+1} \text{ and } 2^{\kappa_n} = \kappa_n^+ \text{”}$.

(b) $\Vdash_{\mathbb{P}_1} \text{“for all } n > 0, \text{ there are special } \kappa_n\text{-Aronszajn trees”}$.

Proof. We have $\mathbb{P}_1 \cong \mathbb{Q}_0 = \prod_{n < \omega} \text{Col}(\kappa_{n-1}, < \kappa_n)$, and by our assumption $2^{\kappa_n} = \kappa_n^+$, for $n < \omega$. Clause (a) follows immediately. Clause (b) follows from (a) and the Specker’s theorem [8]. \square

The next lemma can be proved easily using a Δ -System argument.

Lemma 4.4. *For every $\alpha \leq \delta$, the forcing \mathbb{P}_α is δ -c.c.*

The next lemma follows from the above arguments.

Lemma 4.5. *The models $V[G_1]$ and $V[G_\delta]$ have the same cardinals and cofinalities.*

In particular, $V[G_\delta] \models \text{“for each } n < \omega, \kappa_n = \mu^{+n+1} \text{ and } \delta = \mu^{+\omega+2} \text{”}$. Furthermore

$$V[G_\delta] \models \text{“}\forall n < \omega, 2^\mu = 2^{\kappa_n} = \delta \text{”}.$$

By Lemmas 4.3 and 4.5, we can conclude that:

Lemma 4.6. $V[G_\delta] \models \text{“For each } n < \omega, \text{ there are } \kappa_n\text{-Aronszajn trees”}$.

The next lemma completes the proof of Theorem 1.3.

Lemma 4.7. *In $V[G_\delta]$, and for each $n < \omega$, all κ_n -Aronszajn trees are special.*

Proof. Suppose $n < \omega$ and T is a κ_n -Aronszajn tree in $V[G_\delta]$. Let $\dot{T} \in H(\delta)$ be a name for T . Then by our choice of Φ , the set

$$\{\alpha \in S_{\kappa_n^+}^\delta \mid \Phi(\alpha) = \dot{T}\}$$

is unbounded in δ , and hence by Lemma 4.4, we can find some $\alpha \in S_{\kappa_n^+}^\delta$ such that $\Phi(\alpha) = \dot{T}$, and $\Phi(\alpha)$ is a $\mathbb{P}_{\alpha+1}(> \kappa_n) * \dot{\mathbb{P}}_\alpha(\kappa_n) * \dot{\mathbb{P}}_\alpha(< \kappa_n)$ -name for a κ_n -Aronszajn tree. By our definition of the forcing at step α , we can find a function $F : T \rightarrow \kappa_{n-1}$ which is a specializing function for T in $V[G_{\alpha+1}]$. As the models $V[G_\delta] \supseteq V[G_{\alpha+1}]$ have the same cardinals, F witnesses that T is special in $V[G_\delta]$. \square

5. THE SPECIAL ARONSZAJN TREE PROPERTY AT SUCCESSOR OF EVERY REGULAR CARDINAL

In this section we prove Theorem 1.4. Recall from Section 4, that we essentially proved the following lemma:

Lemma 5.1. *Assume α is a limit ordinal and $\kappa_1 < \dots < \kappa_n < \dots$ are indestructible supercompact cardinals above \aleph_α . Then there is an $\aleph_{\alpha+1}$ -directed closed forcing notion $\mathbb{P}(\alpha, \langle \kappa_n \mid 1 < n < \omega \rangle)$ of size $\delta = (\sup_{n < \omega} \kappa_n)^{++}$ such that the following hold in a generic extension by $\mathbb{P}(\alpha, \langle \kappa_n \mid 1 < n < \omega \rangle)$:*

- (a) *For each $1 < n < \omega$, $\aleph_{\alpha+n} = \kappa_n$ and $\delta = \aleph_{\alpha+\omega+2}$.*
- (b) *$\forall 1 \leq n < \omega$, $2^{\aleph_{\alpha+n}} = \delta$.*
- (c) *The Special Aronszajn Tree Property holds at all $\aleph_{\alpha+n}$'s, $1 < n < \omega$.*

Now suppose that $\langle \kappa_\xi \mid 0 < \xi \in ON \rangle$ is an increasing and continuous sequence of cardinals, such that $\kappa_{\xi+1}$ is a supercompact cardinal, for every ordinal ξ , and set $\kappa_0 = \aleph_0$. We also assume that no limit point of the sequence is an inaccessible cardinal. Let

$$\langle \langle \mathbb{P}_\alpha \mid \alpha \in ON, \alpha = 0 \text{ or } \lim(\alpha) \rangle, \langle \dot{\mathbb{Q}}_\alpha \mid \alpha \in ON, \alpha = 0 \text{ or } \lim(\alpha) \rangle \rangle$$

be the reverse Easton iteration of forcing notions such that

- (1) $\mathbb{P}_0 = \{1_{\mathbb{P}_0}\}$ is the trivial forcing.
- (2) $\Vdash_{\mathbb{P}_0} \dot{Q}_0 = \mathbb{L}(\aleph_1, \langle \kappa_n \mid 0 < n < \omega \rangle) * \dot{\mathbb{P}}(0, \langle \kappa_n \mid 0 < n < \omega \rangle)$.
- (3) For each limit ordinal $\alpha > 0$,

$$\Vdash_{\mathbb{P}_\alpha} \dot{Q}_\alpha = \mathbb{L}(\kappa_\alpha^+, \langle \kappa_{\alpha+n} \mid 0 < n < \omega \rangle) * \dot{\mathbb{P}}(\alpha, \langle \kappa_{\alpha+n} \mid 0 < n < \omega \rangle)$$

Note that at each step α , the forcing notion \mathbb{P}_α has size less than $\kappa_{\alpha+1}$, so cardinals $\kappa_{\alpha+n}$, $0 < n < \omega$, remain supercompact in the generic extension by \mathbb{P}_α . Therefore, the forcing notion \dot{Q}_α is well-defined in $V[G_{\mathbb{P}_\alpha}]$.

Finally let \mathbb{P} be the direct limit of the above forcing construction and let G be \mathbb{P} -generic over V .

Lemma 5.2. *The following hold in $V[G]$:*

- (a) $\forall \xi \in ON, \aleph_\xi = \kappa_\xi$.
- (b) For each limit ordinal α and each $1 < n < \omega$, $2^{\aleph_{\alpha+n}} = \kappa_{\alpha+\omega+2} = \aleph_{\alpha+\omega+2}$.

Let us show that in the generic extension by \mathbb{P} , the Special Aronszajn Tree Property holds at the successor of every regular cardinal. Thus assume α is a limit ordinal (the case $\alpha = 0$ is similar). We can write the forcing notion \mathbb{P} as $\mathbb{P} = \mathbb{P}_\alpha * \dot{Q}_\alpha * \dot{\mathbb{P}}_{(\alpha, \infty)}$, where, the forcing notion $\dot{\mathbb{P}}_{(\alpha, \infty)}$ is defined in $V[G_{\mathbb{P}_\alpha * \dot{Q}_\alpha}]$, in the same way that we defined \mathbb{P} , using the forcing notions $\mathbb{P}_\beta, \dot{Q}_\beta$, where $\alpha < \beta$ is a limit ordinal. In particular, we have

$$\Vdash_{\mathbb{P}_\alpha * \dot{Q}_\alpha} \dot{\mathbb{P}}_{(\alpha, \infty)} \text{ is } \kappa_{\alpha+\omega+1}\text{-closed}.$$

By Lemma 5.1,

$$\Vdash_{\mathbb{P}_\alpha * \dot{Q}_\alpha} \bigwedge_{1 < n < \omega} \text{SATP}(\aleph_{\alpha+n}).$$

Since $\Vdash_{\mathbb{P}_\alpha * \dot{Q}_\alpha}$ “the forcing notion $\dot{\mathbb{P}}_{(\alpha, \infty)}$ does not add any new $\kappa_{\alpha+\omega}$ -sequences”, we have

$$\Vdash_{\mathbb{P}} \bigwedge_{1 < n < \omega} \text{SATP}(\aleph_{\alpha+n}).$$

The result follows immediately. \square

We close the paper with the following question, which is an analogue of Magidor's question regarding the Tree Property.

Question 5.3. *Is it consistent, relative to the existence of large cardinals, that Special Aronszajn Tree Property holds for all uncountable regular cardinals ?*

Let us also remark that the following question is still open:

Question 5.4. *Let λ be successor of a singular cardinal. Is $\text{SATP}(\lambda)$ consistent? I.e., is it consistent that there is a λ -Aronszajn tree, and every λ -Aronszajn tree is special?*

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