# THE SPECIAL ARONSZAJN TREE PROPERTY AT $\aleph_2$ AND GCH

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Abstract. Starting from the existence of a weakly compact cardinal, we build a generic extension of the universe in which GCH holds and all $\aleph_2$-Aronszajn trees are special and hence there are no $\aleph_2$-Souslin trees. This result answers a longstanding open question from the 1970's.

## 1. introduction

Let $\kappa$ be a regular uncountable cardinal. Let us recall that a $\kappa$-tree is a tree $T$ of height $\kappa$ all of whose levels are smaller than $\kappa$, and that a $\kappa$-tree is called a $\kappa$-Aronszajn tree if it has no $\kappa$-branches. Also, $T$ is called a $\kappa$-Souslin tree if it has no $\kappa$-branches and no antichains of size $\kappa$. When $\kappa = \lambda^+$ is a successor cardinal, a $\kappa$-Aronszajn tree $T$ is said to be *special* if there exists a function $f : T \to \lambda$ such that $f(x) \neq f(y)$ whenever $x, y \in T$ are such that $x <_T y$. We say that $f$ *specializes* $T$. Let us make the following definition:

**Definition 1.1.** (1) *Souslin's Hypothesis at $\kappa$, $\mathrm{SH}_\kappa$, is the statement "there are no $\kappa$-Souslin trees".*

(2) *The special Aronszajn tree property at $\kappa = \lambda^+$, $\mathrm{SATP}_\kappa$, is the statement "there exist $\kappa$-Aronszajn trees and all such trees are special" (see [5]).*

Aronszajn trees were introduced by Aronszajn (see [9]), who proved the existence, in ZFC, of a special $\aleph_1$-Aronszajn tree. Later, Specker ([14]) showed that $2^{<\lambda} = \lambda$ implies the existence of special $\lambda^+$-Aronszajn trees for $\lambda$ regular, and Jensen ([7]) produced special $\lambda^+$-Aronszajn trees for singular $\lambda$ in $L$.

In [3], Baumgartner, Malitz and Reinhardt showed that Martin's Axiom + $2^{\aleph_0} > \aleph_1$ implies $\mathrm{SATP}_{\aleph_1}$, and hence $\mathrm{SH}_{\aleph_1}$ as well. Later, Jensen (see [4] and [12]) produced a model of GCH in which $\mathrm{SATP}_{\aleph_1}$ holds.

The situation at $\aleph_2$ turned out to be more complicated. In [7], Jensen proved that the existence of an $\aleph_2$-Souslin tree follows from each of the

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hypotheses CH + $\diamondsuit(S^2_1)$ and $\square_{\omega_1} + \diamondsuit(S^2_0)$ (where, given $m < n < \omega$, $S^n_m = \{\alpha < \aleph_n \mid \operatorname{cf}(\alpha) = \aleph_m\}$). The second result was improved by Gregory in [6], where he proved that GCH together the existence of a non–reflecting stationary subset of $S^2_0$ yields the existence of an $\aleph_2$-Souslin tree. In [10], Laver and Shelah produced, relative to the existence of a weakly compact cardinal, a model of ZFC + CH in which the special Aronszajn tree property at $\aleph_2$ holds. But in their model $2^{\aleph_1} > \aleph_2$, and the task of finding a model of ZFC+GCH+SATP$_{\aleph_2}$ remained as a major open problem. The earliest published mention of this problem seems to appear in [8].

In this paper we solve the above problem by proving the following theorem.

**Theorem 1.2.** *Suppose $\kappa$ is a weakly compact cardinal. Then there exists a set–generic extension of the universe in which GCH holds, $\kappa = \aleph_2$, and the special Aronszajn tree property at $\aleph_2$ (and hence Souslin's Hypothesis at $\aleph_2$) holds.*

**Remark 1.3.**
1. *Our argument can be easily extended to deal with the successor of any regular cardinal.*
2. *By results of Shelah and Stanley (*[13]*) and of Rinot (*[11]*), our large cardinal assumption is optimal. Specifically:*
   - (a) *It is proved in* [13] *that if $\omega_2$ is not weakly compact in L, then either $\square_{\omega_1}$ holds or there is a non–special $\aleph_2$-Aronszajn tree; in particular, GCH+*SATP$_{\aleph_2}$ *implies that $\omega_2$ is weakly compact in L by one of Jensen's results mentioned above.*
   - (b) *It is proved in* [11] *that if both GCH and Souslin's Hypothesis at $\aleph_2$ hold, then in fact $\square(\omega_2)$ fails; on the other hand, Todorčević (*[15]*) proved that $\neg\square(\omega_2)$ implies that $\omega_2$ is weakly compact in L.*

The rest of the paper is devoted to the proof of Theorem 1.2. We will next give an (inevitably) vague and incomplete description of the forcing witnessing the conclusion of this theorem.

The construction of this forcing combines a natural iteration for specializing $\aleph_2$-Aronszajn trees, due to Laver and Shelah ([10]), with ideas from [2]. More specifically, we build a certain countable support forcing iteration $\langle \mathbb{Q}_\beta \mid \beta \leq \kappa^+\rangle$ with side conditions. The first step of the construction is essentially the Lévy–collapse of the weakly compact cardinal $\kappa$ to become $\omega_2$. At subsequent stages, we consider forcings for specializing $\aleph_2$-Aronszajn trees by countable approximations. Conditions in a given $\mathbb{Q}_\beta$, for $\beta > 0$, will consist of a working part $f_q$, together with a certain side condition. The working part $f_q$ will be a countable function with domain contained in $\beta$ such that for all $\alpha \in \operatorname{dom}(f_q)$,

- $f_q(\alpha)$ is a condition in the Lévy–collapse if $\alpha = 0$, and
- if $\alpha > 0$, $f_q(\alpha)$ is a countable function contained in $(\kappa\times\omega_1)\times\omega_1$.

This function $f_q(\alpha)$, when $\alpha > 0$, will be seen as an approximation to a specializing function for a certain tree $\underset{\sim}{T}_\alpha$ on $\kappa \times \omega_1$, chosen via a given bookkeeping function $\Phi : \kappa^+ \to H(\kappa^+)$.

The side condition will be a countable directed graph $\tau_q$ whose vertices are ordered pairs of the form $(N, \gamma)$, where $N$ is an elementary submodel of $H(\kappa^+)$ such that $|N| = |N\cap\kappa|$ and ${}^{<|N|}N \subseteq N$, and where $\gamma$ is an ordinal in the closure of $N \cap \{\xi+1 \mid \xi < \beta\}$ in the order topology. Given any such $(N, \gamma)$, $\gamma$ is to be seen as a *marker for $N$ in $\tau_q$*, telling us up to which stage is $N$ 'active' as a model. We will tend to call such pairs $(N, \gamma)$ *models with markers*. Whenever $\langle(N_0, \gamma_0),(N_1, \gamma_1)\rangle$ is an edge in $\tau_q$, for a condition $q$, we have that $(N_0, \in)$ and $(N_1, \in)$ are $\in$–isomorphic via a (unique) isomorphism $\Psi_{N_0,N_1}$—with respect to a certain sequence $(\Phi_\alpha)_{\alpha<\beta}$ of increasingly expressive predicates contained in $H(\kappa^+)$—which is such that $\Psi_{N_0,N_1}(\alpha) \leq \alpha$ for every ordinal $\alpha \in N_0$.

In the above situation, the model $N_1$ is to be seen as a 'projection' of $N_0$ for $q$ with respect to all stages $\alpha$ and $\Psi_{N_0,N_1}(\alpha)$ such that $\alpha \in N_0\cap\gamma_0$ and $\Psi_{N_0,N_1}(\alpha) < \gamma_1$. What this means is that the natural restriction of $q \restriction \alpha+1$ to $N_0$—where $q \restriction \alpha+1$ denotes the restriction of $q$ to $\mathbb{Q}_{\alpha+1}$—is to be copied over, via $\Psi_{N_0,N_1}$, into the restriction of $q \restriction \Psi_{N_0,N_1}(\alpha + 1)$ to $N_1$, i.e., we require that

$$\Psi_{N_0,N_1}(f_q(\alpha) \restriction N_0) = f_q(\alpha) \restriction N_0 \subseteq f_q(\Psi_{N_0,N_1}(\alpha)),$$

and similarly for the restriction of $\tau_q \restriction \alpha+1$ to $N_0$ (where the restriction $\tau_q \restriction \alpha+1$ is defined naturally). We are, so to speak, *copying into the past information coming from the future*. The restriction of $q \restriction \Psi_{N_0,N_1}(\alpha+1)$ to $N_1$ may certainly contain more information than that given by the copy of the restriction of $q \restriction \alpha + 1$ to $N_1$.[1] Thanks to the way we are setting up the copying procedure—namely, only copying from the future into the past—, it is straightforward to see that our construction is in fact a forcing iteration, in the sense that $\mathbb{Q}_\alpha$ is a complete suborder of $\mathbb{Q}_\beta$ for all $\alpha < \beta$. This does not need be true in general, in forcing constructions of this sort, if we also allow to copy 'from the past into the future'.[2]

[1]On the other hand, any extra information in $N_1$ carried by $q \restriction \Psi_{N_0,N_1}(\alpha)$ will be compatible with the information contained in the restriction of $q \restriction \alpha + 1$ to $N_0$ (s. the next footnote).

[2]It turns out that, in our specific construction, we could as well have required to also copy information 'from the past into future', i.e., that full symmetry obtains between the relevant stages of $N_0$ and $N_1$ in the above situation (s. Lemma 2.12).

Given an edge $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ in $\tau_q$ and a stage $\alpha \in N_0 \cap \gamma_0$, we would like to require that $\mathbb{Q}_{\alpha+1} \cap N_0$ be a complete suborder of $\mathbb{Q}_{\alpha+1}$; indeed, having this would be useful at one point in the proof that our construction has the $\kappa$-chain condition. This cannot be accomplished, while defining $\mathbb{Q}_{\alpha+1}$, on pain of circularity. However, a certain approximation to the above situation can be stipulated, which we do, and this suffices for our purposes.

Our construction is $\sigma$-closed for all $\beta \leq \kappa^+$. In particular, forcing with $\mathbb{Q}_{\kappa^+}$ preserves $\omega_1$ and CH. The preservation of all higher cardinals proceeds by showing that the construction has the $\kappa$-chain condition. For this, we use the weak compactness of $\kappa$ in an essential way. The fact that the length of our iteration is not longer than $\kappa^+$ seems to be needed in this proof. Finally, the edges occurring in $\tau_q$, for a condition $q$, are crucially used in the proof that our forcing preserves $2^{\aleph_1} = \aleph_2$.

Side conditions are often employed in forcing constructions with the purpose of guaranteeing that certain cardinals are preserved. In the present construction, on the other hand, they are used to ensure that the relevant level of GCH[3] be preserved. This use of side conditions is taken from [2]. In that paper, side conditions are edges coming from a finite set of models with markers. Moreover, the models there are countable. Modulo the changes in the definition, the copying requirement in that construction is the same as in the present construction. This copying requirement is crucially used, in [2], in the proof of CH-preservation. It is worth observing that, while in the construction from [2] a certain account of structure is needed for the models occurring in the side condition,[4] no structure whatsoever (for the underlying set of models) is needed in the present construction. We should point out that even if it preserves $2^{\aleph_1} = \aleph_2$, our construction does add new subsets of $\omega_1$, although only $\aleph_2$-many of them (cf. the construction in [2], where CH is preserved but $\aleph_1$-many new reals are added).

The paper is organized as follows. In Section 2 we define our main forcing construction and prove some if its basic properties. In Section 3 we show that the forcing has the $\kappa$-chain condition. This is the most elaborate proof in the paper.[5] Finally, in Section 4 we complete the proof of Theorem 1.2. The main argument in this section is to show that our forcing preserves $2^{\aleph_1} = \aleph_2$.

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However, the current presentation, only deriving full symmetry for a dense set of conditions, seems to be cleaner.

[3] $2^{\aleph_1} = \aleph_2$.

[4] Using the terminology of [1], they need to come from a symmetric system.

[5] Cf. the proof in [10], where the hardest part is to prove that the forcing is $\kappa$-c.c., or the proof in [2], where the hardest part is to prove that the forcing is proper.

## 2. Definition of the forcing and its basic properties

In this section we define our main forcing and prove some of its basic properties.

Let us fix, for the remainder of this paper, a weakly compact cardinal $\kappa$, and let us assume, without loss of generality, that $2^\mu = \mu^+$ for every cardinal $\mu \geq \kappa$.[6]

Throughout the paper, if $N$ is a set such that $N \cap \kappa$ is an ordinal, we denote this ordinal by $\delta_N$ and call it *the height of* $N$. If $X$ is a set, we let

$$\mathrm{cl}(X) = X \cup \{\alpha \in \mathrm{Ord} \mid \alpha = \sup(X \cap \alpha)\}.$$

If, in addition, $\gamma$ is an ordinal, we let $\gamma_X$ be the highest ordinal $\xi \in \mathrm{cl}(X)$ such that $\xi \leq \gamma$.

Given an ordered pair $q = (f_q, \tau_q)$, where $f_q$ is a function, and given a model $N$, we denote by $q \restriction N$ the ordered pair $(\bar{f}, \tau_q \cap N)$, where $\bar{f}$ is the function with domain $\mathrm{dom}(f_q) \cap N$ such that $\bar{f}(x) = f_q(x) \cap N$ for each $x \in \mathrm{dom}(\bar{f})$.

Let

$$\Phi : \kappa^+ \to H(\kappa^+)$$

be such that for each $x \in H(\kappa^+)$, $\Phi^{-1}(x)$ is an unbounded subset of $\kappa^+$. $\Phi$ exists by $2^\kappa = \kappa^+$.

Let also $(\Phi_\alpha)_{\alpha<\kappa^+}$ be the following sequence of subsets of $H(\kappa^+)$.

- $\Phi_0 = \Phi$
- If $\alpha > 0$, then $\Phi_\alpha$ codes, in some fixed canonical way, the satisfaction predicate for the structure

$$\langle H(\kappa^+), \in, \vec{\Phi}_\alpha \rangle,$$

  where $\vec{\Phi}_\alpha = (\Phi_{\alpha'})_{\alpha'<\alpha}$.

We will call ordered pairs of the form $(N, \gamma)$, where

- $N \subseteq H(\kappa^+)$, $N \cap \kappa \in \kappa$, $|N| = |N \cap \kappa|$, and ${}^{<|N|}N \subseteq N$,
- $\gamma \in \mathrm{cl}(N \cap \kappa^+)$, and
- $(N, \in, \Phi_\alpha \cap N)$ is an elementary submodel of $(H(\kappa^+), \in, \Phi_\alpha)$ for every $\alpha \in N \cap \gamma$,

*models with markers*. We will often use, without mention, the fact that $(N, \gamma) \in N'$ whenever $(N, \gamma)$ and $(N', \gamma')$ are models with markers and $N \in N'$.

[6]In fact, if $\kappa$ is weakly compact, then GCH at every cardinal $\mu \geq \kappa$ can be easily arranged by collapsing cardinals with conditions of size $\kappa$, which will preserve the weak compactness of $\kappa$.

Given models $N_0$ and $N_1$ such that $(N_0, \in) \cong (N_1, \in)$, we will denote the unique $\in$-isomorphism

$$\Psi : (N_0, \in) \to (N_1, \in)$$

by $\Psi_{N_0,N_1}$.

Given any nonzero ordinal $\eta < \kappa^+$, let $e_\eta$ be the first, in some well–order of $H(\kappa^+)$ definable in $(H(\kappa^+), \in, \Phi)$, surjection from $\kappa$ onto $\eta$. Let $\vec{e} = \langle e_\eta \mid 0 < \eta < \kappa^+\rangle$. We will say that a model $N \subseteq H(\kappa^+)$ is *closed under* $\vec{e}$ if $e_\eta(\xi) \in N$ for every nonzero $\eta \in N \cap \kappa^+$ and every $\xi \in \kappa \cap N$.

In the fact below (and elsewhere in the paper), we use the convention, given functions $f_0$, $f_1$, to let the expression $(f_1 \circ f_0)(x)$ denote $f_1(f_0(x))$, provided $x \in \operatorname{dom}(f_0)$ and $f_0(x) \in \operatorname{dom}(f_1)$ (and similarly for the expression $(f_m \circ \ldots \circ f_0)(x)$, given functions $f_0, \ldots, f_m$). The following standard fact will be used in the proof of Lemma 3.3.

**Fact 2.1.** *Suppose $N_0$ and $N_1$ are models closed under $\vec{e}$ of the same height. Then $N_0 \cap N_1 \cap \kappa^+$ is an initial segment of both $N_0 \cap \kappa^+$ and $N_1 \cap \kappa^+$. In particular, if $(N_0^i, \gamma_0^i)$ and $(N_1^i, \gamma_1^i)$ (for $i \leq n$) are models with markers such that for all $i$,*

- $\gamma_0^i$, $\gamma_1^i > 1$,
- *there is an isomorphism $\Psi_{N_0^i,N_1^i}$ between $(N_0^i, \Phi_1)$ and $(N_1^i, \Phi_1)$, and*
- $(\Psi_{N_0^i,N_1^i} \circ \ldots \circ \Psi_{N_0^0,N_1^0})(x) \in N_0^{i+1}$ *if $i + 1 \leq n$,*

*then $(\Psi_{N_0^n,N_1^n} \circ \ldots \circ \Psi_{N_0^0,N_1^0})(x) = x$ for every $x \in N_0^0 \cap N_1^n$.*

*Proof.* Let us first prove the first assertion. Given any nonzero $\eta \in N_0 \cap N_1 \cap \kappa^+$ and any $\alpha \in N_0 \cap \eta$ there is some $\xi \in N_0 \cap \kappa$ such that $e_\eta(\xi) = \alpha$. But since $\eta$ and $\xi$ are both members of $N_1$, we also have that $\alpha = e_\eta(\xi) \in N_1$.

As to the second assertion, let us first consider the case in which $\delta_{N_0^i} = \delta_{N_0^{i'}}$ for all $i$, $i'$. Note that if $x \in N_0^0 \cap N_1^n$ and $\alpha < \kappa^+$ is least such that $\Phi(\alpha) = x$, then $\alpha \in N_0^0 \cap N_1^n$ and

$$\alpha' := (\Psi_{N_0^n,N_1^n} \circ \ldots \circ \Psi_{N_0^0,N_1^0})(\alpha) = \alpha$$

since $\operatorname{ot}(\alpha \cap N_1^n) = \operatorname{ot}(\alpha \cap N_0^0) = \operatorname{ot}(\alpha' \cap N_1^n)$, where the first equality follows from the first assertion and the second equality from the definition of $\alpha'$ as $(\Psi_{N_0^n,N_1^n} \circ \ldots \circ \Psi_{N_0^0,N_1^0})(\alpha)$. It follows that $(\Psi_{N_0^n,N_1^n} \circ \ldots \circ \Psi_{N_0^0,N_1^0})(x) = (\Psi_{N_0^n,N_1^n} \circ \ldots \circ \Psi_{N_0^0,N_1^0})(\Phi(\alpha)) = \Phi(\Psi_{N_0^n,N_1^n} \circ \ldots \circ \Psi_{N_0^0,N_1^0}(\alpha)) = \Phi(\alpha) = x$.

Suppose now that $\delta_{N_0^i} < \delta_{N_0^{i'}}$ for some $i \neq i'$ and let $i^*$ be such that $\delta_{N_0^{i^*}} = \min\{\delta_{N_0^i} \mid i \leq n\}$. Since each $N_\epsilon^i$ is closed under sequences of length less than $|N^i_\epsilon|$ and $|N^{i^{**}}_\epsilon| = |\delta_{N^{i^*}_0}|$ for every $\epsilon$ and every $i^{**}$ such that $\delta_{N^{i^{**}}_0} = \delta_{N^{i^*}_0}$, it is easy to see that there is a sequence $(\langle \bar{N}^i_0, \bar{N}^i_1 \rangle)_{i \leq n}$ of pairs of models with the following properties.

- For all $i \leq n$, $(\bar{N}^i_0, \in, \Phi_1 \cap \bar{N}^i_0)$ and $(\bar{N}^i_1, \in, \Phi_1 \cap \bar{N}^i_1)$ are elementary submodels of $(H(\kappa^+), \in, \Phi_1)$ and
$$(\bar{N}^i_0, \in, \Phi_0 \cap \bar{N}^i_0) \cong (\bar{N}^i_1, \in, \Phi_1 \cap \bar{N}^i_1)$$
- $\delta_{\bar{N}^i_0} = \delta_{\bar{N}^{i'}_0} = \delta_{N^{i^*}_0}$ for all $i$, $i' \leq n$.
- For every $i \leq n$ and every $\epsilon \in \{0, 1\}$, $\bar{N}^i_\epsilon = N^i_\epsilon$ or $\bar{N}^i_\epsilon \in N^i_\epsilon$.
- $x \in \bar{N}^0_0 \cap \bar{N}^n_1$.
- $(\Psi_{N^n_0, N^n_1} \circ \ldots \circ \Psi_{N^0_0, N^0_1})(x) = (\Psi_{\bar{N}^n_0, \bar{N}^n_1} \circ \ldots \circ \Psi_{\bar{N}^0_0, \bar{N}^1_1})(x)$.

But now we are done by the previous case. □

We define a sequence $\langle \mathbb{Q}_\beta \mid \beta \leq \kappa^+ \rangle$ of forcing notions.

For each $\alpha < \kappa^+$, assuming $\mathbb{Q}_\alpha$ has been defined and $\Vdash_{\mathbb{Q}_\alpha}$ CH, we let $\underset{\sim}{T}_\alpha$ be a canonically chosen $\mathbb{Q}_\alpha$-name for a $\kappa$-Aronszajn tree.[7] Further, if $\Phi(\alpha)$ is a $\mathbb{Q}_\alpha$-name for a $\kappa$-Aronszajn tree, then we let $\underset{\sim}{T}_\alpha = \Phi(\alpha)$. For simplicity of exposition we will assume that the universe of $\underset{\sim}{T}_\alpha$ is forced to be $\kappa \times \omega_1$ and that for each $\rho < \kappa$, its $\rho$-th level is $\{\rho\} \times \omega_1$.[8] We will often refer to members of $\kappa \times \omega_1$ as *nodes*.

Now suppose that $\beta \leq \kappa^+$ and that $\mathbb{Q}_\alpha$ has been defined for all $\alpha < \beta$.

It will be convenient to use the following pieces of terminology: given models with markers $(N_0, \gamma_0)$, $(N_1, \gamma_1)$ such that $\gamma_0 \in \mathrm{cl}(N_0 \cap \{\xi + 1 \mid \xi < \beta\})$ and $\gamma_1 \in \mathrm{cl}(N_1 \cap \{\xi + 1 \mid \xi < \beta\})$, we will say that $(N_1, \gamma_1)$ *is a projection of* $(N_0, \gamma_0)$ *(below* $\beta$*)* if and only if $(N_1, \in) \cong (N_0, \in)$ and the unique isomorphism $\Psi_{N_0, N_1}$ from $N_0$ onto $N_1$ has the following properties.

(1) $\Psi_{N_0, N_1}(\xi) \leq \xi$ for all ordinals $\xi \in N_0$.
(2) For each $\alpha \in N_0 \cap \gamma_0$ such that $\alpha' := \Psi_{N_0, N_1}(\alpha) < \gamma_1$, $\Psi_{N_0, N_1}$ is an isomorphism between the structures
$$(N_0, \in, \Phi_\alpha)$$
and
$$(N_1, \in, \Phi_{\alpha'}).$$

Given models with markers $(N, \gamma)$, $(N_0, \gamma_0)$ and $(N_1, \gamma_1)$, if $N \in N_0$ and $(N_0, \in) \cong (N_1, \in)$, then we let $\pi^{\gamma, N}_{N_0, \gamma_0, N_1, \gamma_1}$ denote the supremum of the set of ordinals $\xi + 1$ such that

[7] Note that there is indeed such a name since $\Vdash_{\mathbb{Q}_\alpha}$ CH.

[8] This makes sense given that, as we will see, $\mathbb{Q}_\alpha$ will be a forcing preserving $\omega_1$ and turning $\kappa$ into $\aleph_2$.

- $\xi \in \Psi_{N_0,N_1}(N) \cap \Psi_{N_0,N_1}(\gamma)$,
- $\xi < \gamma_0$, and
- $\Psi_{N_0,N_1}(\xi) < \gamma_1$

We will call ordered pairs of the form $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$, where $\gamma_0 \in \mathrm{cl}(N_0 \cap \{\xi+1 \mid \xi < \beta\})$, $\gamma_1 \in \mathrm{cl}(N_1 \cap \{\xi+1 \mid \xi < \beta\})$, and $(N_1, \gamma_1)$ is a projection of $(N_0, \gamma_0)$, *edges (below $\beta$)*. Given a collection $\tau$ of edges and given an ordinal $\alpha$, we denote by $\tau \restriction \alpha$ the set

$$\{\langle (N_0, \min\{\alpha, \gamma_0\}_{N_0}), (N_1, \min\{\alpha, \gamma_1\}_{N_1})\rangle \mid \langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle \in \tau\}.$$

Note that $\tau \restriction \alpha$ is a collection of edges below $\alpha$.

Given an ordered pair $q = (f_q, \tau_q)$, where $f_q$ is a function and $\tau_q$ is a set of edges, and given an ordinal $\alpha$, we denote by $q \restriction \alpha$ the ordered pair $(f_q \restriction \alpha, \tau_q \restriction \alpha)$.

We are now ready to define $\mathbb{Q}_\beta$.

A condition in $\mathbb{Q}_\beta$ is an ordered pair of the form $q = (f_q, \tau_q)$ with the following properties.

(1) $f_q$ is a countable function such that

$$\mathrm{dom}(f_q) \subseteq \beta$$

and such that the following holds for every $\alpha \in \mathrm{dom}(f_q)$.
   (a) If $\alpha = 0$, then $f_q(\alpha)$ is a condition in $\mathrm{Col}(\omega_1, {<}\kappa)$, the Lévy collapse turning $\kappa$ into $\aleph_2$, i.e., $f_q(0)$ is a countable function with domain included in $\kappa \times \omega_1$ such that $(f_q(0))(\rho, \xi) < \rho$ for all $(\rho, \xi) \in \mathrm{dom}(f_q(0))$.
   (b) If $\alpha > 0$, then $f_q(\alpha) : \kappa \times \omega_1 \to \omega_1$ is a countable function.

(2) $\tau_q$ is a countable set of edges below $\beta$.

(3) The following holds for every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle \in \tau_q$.
   (a) If $\langle (N_0', \gamma_0'), (N_1', \gamma_1')\rangle \in N_0 \cap \tau_q$, then
   $$\langle (\Psi_{N_0,N_1}(N_0'), \gamma_0^*), (\Psi_{N_0,N_1}(N_1'), \gamma_1^*)\rangle \in \tau_q$$
   for some $\gamma_0^* \geq \pi^{\gamma_0', N_0'}_{N_0,\gamma_0,N_1,\gamma_1}$ and $\gamma_1^* \geq \pi^{\gamma_1', N_1'}_{N_0,\gamma_0,N_1,\gamma_1}$.
   (b) The following holds for each nonzero ordinal $\alpha \in \mathrm{dom}(f_q) \cap N_0 \cap \gamma_0$ such that $\Psi_{N_0,N_1}(\alpha) < \gamma_1$.
      (i) $\Psi_{N_0,N_1}(\alpha) \in \mathrm{dom}(f_q)$
      (ii) $f_q(\alpha) \restriction \delta_{N_0} \times \omega_1 \subseteq f_q(\Psi_{N_0,N_1}(\alpha))$

(4) $q \restriction \alpha \in \mathbb{Q}_\alpha$ for all $\alpha < \beta$.

(5) The following holds for every nonzero $\alpha < \beta$.
   (a) If $\alpha \in \mathrm{dom}(f_q)$, then $q \restriction \alpha$ forces that $f_q(\alpha)$ is a partial specializing function for $\underset{\sim}{T}_\alpha$.
   (b) For every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle \in \tau_q$, if $\alpha \in N_0 \cap \gamma_0$, then $\mathbb{Q}_{\alpha+1} \cap N_0 \lessdot \mathbb{Q}^{N_0}_{\alpha+1}$, where $\mathbb{Q}^{N_0}_{\alpha+1}$ is the partial order whose conditions are ordered pairs $p = (f_p, \tau_p)$ such that

(i) $f_p$ is a function such that $\mathrm{dom}(f_p) \subseteq \alpha + 1$,
(ii) if $\alpha \in \mathrm{dom}(f_p)$, then $f_p(\alpha) : \kappa \times \omega_1 \to \omega_1$ is a countable function,
(iii) $\tau_p$ is a set of edges below $\alpha + 1$,
(iv) $\gamma_0$, $\gamma_1 \leq \alpha$ for every $\langle (N'_0, \gamma_0), (N'_1, \gamma_1) \rangle \in \tau_p \setminus N_0$,
(v) $p \restriction \alpha \in \mathbb{Q}_\alpha$,
(vi) $p \restriction N_0 \in \mathbb{Q}_{\alpha+1}$, and
(vii) if $\alpha \in \mathrm{dom}(f_p)$, then $p \restriction \alpha$ forces that $f_p(\alpha)$ is a partial specializing function for $\underset{\sim}{T}_\alpha$,

ordered by setting $p_1 \leq_{\mathbb{Q}^{N_0}_{\alpha+1}} p_0$ if

- $p_1 \restriction \alpha \leq_{\mathbb{Q}_\alpha} p_0 \restriction \alpha$ and
- $f_{p_0}(\alpha) \subseteq f_{p_1}(\alpha)$ in case $\alpha \in \mathrm{dom}(f_{p_0})$.

The extension relation on $\mathbb{Q}_\beta$ is defined in the following way:

Given $q_1, q_0 \in \mathbb{Q}_\beta$, $q_1 \leq_{\mathbb{Q}_\beta} q_0$ ($q_1$ is an extension of $q_0$) if and only if the following holds.

(1) $\mathrm{dom}(f_{q_0}) \subseteq \mathrm{dom}(f_{q_1})$
(2) For every $\alpha \in \mathrm{dom}(f_{q_0})$, $f_{q_0}(\alpha) \subseteq f_{q_1}(\alpha)$.
(3) For every $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_{q_0}$ there are some $\gamma'_0 \geq \gamma_0$ and $\gamma'_1 \geq \gamma_1$ such that $\langle (N_0, \gamma'_0), (N_1, \gamma'_1) \rangle \in \tau_{q_1}$.

**Remark 2.1.** *Given an ordinal $\alpha < \kappa^+$, the definition of $\mathbb{Q}_{\alpha+1}$ can be seen, because of clause (5) (b), as being by recursion on the supremum of the set of heights of models $N_0$ occurring in edges $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ in $\tau_q$.*

2.1. **Basic properties of $\langle \mathbb{Q}_\beta \mid \beta \leq \kappa^+ \rangle$.** Our first lemma follows from the choice of the predicates $\Phi_\alpha$.

**Lemma 2.2.** *For every $\alpha < \kappa^+$, $\mathbb{Q}_\alpha$ is definable over the structure*

$$(H(\kappa^+), \in, \Phi_\alpha)$$

*without parameters. Moreover, this definition can be taken to be uniform in $\alpha$.*

Lemma 2.3 follows immediately from the definition of condition.

**Lemma 2.3.** *Suppose $\alpha < \kappa^+$, $q \in \mathbb{Q}_{\alpha+1}$, $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q$, and $\gamma_0 = \alpha + 1$. Then $\mathbb{Q}^{N_0}_{\alpha+1}$ is a suborder of $\mathbb{Q}_{\alpha+1}$.*

The following standard lemma will be used in the proof of Lemma 2.12.

**Lemma 2.4.** *Suppose $\alpha < \kappa^+$, $q \in \mathbb{Q}_{\alpha+1}$, $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q$, and $\gamma_0 = \alpha + 1$. Then $\mathbb{Q}^{N_0}_{\alpha+1}$ has a dense subset consisting of conditions $p$ such that $p \restriction N_0$ forces, in $\mathbb{Q}_{\alpha+1} \cap N_0$, that every condition in $\dot{G}_{\mathbb{Q}_{\alpha+1} \cap N_0}$*

*is compatible, in* $\mathbb{Q}^{N_0}_{\alpha+1}$, *with* $p$. *In fact, for every* $p \in \mathbb{Q}^{N_0}_{\alpha+1}$ *there is an extension* $p' \in \mathbb{Q}^{N_0}_{\alpha+1}$ *of* $p$ *such that*

(1) $p' \restriction N_0$ *forces, in* $\mathbb{Q}_{\alpha+1} \cap N_0$, *that every condition in* $\dot{G}_{\mathbb{Q}_{\alpha+1}\cap N_0}$ *is compatible, in* $\mathbb{Q}^{N_0}_{\alpha+1}$, *with* $p'$,
(2) $\alpha \in \mathrm{dom}(f_{p'})$,
(3) $f_{p'}(\alpha) \restriction (\kappa \setminus \delta_{N_0}) \times \omega_1 = f_p(\alpha) \restriction (\kappa \setminus \delta_{N_0}) \times \omega_1$ *(where* $f_p(\alpha)$ *is set to be* $\emptyset$ *if* $\alpha \notin \mathrm{dom}(f_p)$*), and such that*
(4) $\langle (N_0', \gamma_0'), (N_1', \gamma_1') \rangle \in N_0$ *for every edge* $\langle (N_0', \gamma_0'), (N_1', \gamma_1') \rangle \in \tau_{p'}$ *such that* $\gamma_0' = \alpha + 1$ *or* $\gamma_1' = \alpha + 1$.

Our next lemma is obvious.

**Lemma 2.5.** $\mathbb{Q}_1$ *forces* $\kappa = \aleph_2$.

The following lemma is also an easy consequence of the definition of condition.

**Lemma 2.6.** *For every* $\beta \leq \kappa^+$,

(1) $\mathbb{Q}_\alpha \subseteq \mathbb{Q}_\beta$ *for all* $\alpha < \beta$, *and*
(2) *if* $\mathrm{cf}(\beta) \geq \kappa$, *then* $\mathbb{Q}_\beta = \bigcup_{\alpha<\beta} \mathbb{Q}_\alpha$.

The following lemma is also immediate and shows that the sequence $\langle \mathbb{Q}_\beta \mid \beta \leq \kappa^+ \rangle$ is a forcing iteration, in the sense that $\mathbb{Q}_\alpha \lessdot \mathbb{Q}_\beta$ for all $\alpha < \beta$. We will use this lemma very often throughout the remainder of the paper, most of the time without mentioning it.

**Lemma 2.7.** *For all* $\alpha < \beta \leq \kappa^+$, $q \in \mathbb{Q}_\beta$, *and* $r \in \mathbb{Q}_\alpha$, *if* $r \leq_\alpha q \restriction \alpha$, *then*

$$(f_r \cup f_q \restriction [\alpha,\, \beta), \tau_q \cup \tau_r)$$

*is a common extension of* $q$ *and* $r$ *in* $\mathbb{Q}_\beta$.

*Proof.* It suffices to consider clause (3)(b), as all other clauses can be checked easily. Thus suppose that $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q \cup \tau_r$ and let $\xi \in \mathrm{dom}(f_r \cup f_q) \cap N_0 \cap \gamma_0$ be such that $\Psi_{N_0,N_1}(\xi) < \gamma_1$.

If $\xi \geq \alpha$, then we must have $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q$ and $\xi \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_r)$, and the conclusion of (3)(b) follows from the fact that $q$ is a condition.

Thus suppose that $\xi < \alpha$. Then $\xi \in \mathrm{dom}(f_r)$ and

$$(f_r \cup f_q)(\xi) \restriction \delta_{N_0} \times \omega_1 = f_r(\xi) \restriction \delta_{N_0} \times \omega_1 \subseteq f_r(\Psi_{N_0,N_1}(\xi)).$$

Since $\Psi_{N_0,N_1}(\xi) \leq \xi < \alpha$, we have $f_r(\Psi_{N_0,N_1}(\xi)) = (f_r \cup f_q)(\Psi_{N_0,N_1}(\xi))$, and hence $(f_r \cup f_q)(\xi) \restriction \delta_{N_0} \times \omega_1 \subseteq (f_r \cup f_q)(\Psi_{N_0,N_1}(\xi))$, which gives the desired result. □

We say that a partial order $\mathcal{P}$ is $\sigma$-closed if every descending sequence $(p_n)_{n<\omega}$ of $\mathcal{P}$-conditions has a lower bound in $\mathcal{P}$.

**Lemma 2.8.** *$\mathbb{Q}_\beta$ is $\sigma$-closed for every $\beta \leq \kappa^+$. In fact, every decreasing $\omega$-sequence of $\mathbb{Q}_\beta$-conditions has a greatest lower bound in $\mathbb{Q}_\beta$. In particular, forcing with $\mathbb{Q}_\beta$ does not add new $\omega$-sequences of ordinals, and therefore it preserves both $\omega_1$ and CH.*

*Proof.* Given a decreasing sequence $(q_n)_{n<\omega}$ of $\mathbb{Q}_\beta$-conditions, it is immediate to check that $(f, \bigcup_{n<\omega} \tau_{q_n})$ is the greatest lower bound of $\{q_n \mid n < \omega\}$, where $\mathrm{dom}(f) = \bigcup_{n<\omega} \mathrm{dom}(f_{q_n})$ and, for each $n < \omega$ and $\alpha \in \mathrm{dom}(f_{q_n})$, $f(\alpha) = \bigcup\{f_{q_m}(\alpha),\, m \geq n\}$. □

Lemma 2.8, or rather its proof, will be used without mention in several places in which we run some construction, in $\omega$ steps, along which we build some decreasing sequence $(q_n)_{n<\omega}$ of conditions. At the end of such a construction we will have that the ordered pair $q = (f, \bigcup_{n<\omega} \tau_{q_n})$, where $f$ is given as in the above proof, is the greatest lower bound of $(q_n)_{n<\omega}$.

Given functions $f$ and $g$, let us momentarily denote by $f + g$ the function with $\mathrm{dom}(f + g) = \mathrm{dom}(f) \cup \mathrm{dom}(g)$ such that

- $(f + g)(x) = f(x)$ for $x \in \mathrm{dom}(f) \setminus \mathrm{dom}(g)$,
- $(f + g)(x) = g(x)$ for $x \in \mathrm{dom}(g) \setminus \mathrm{dom}(f)$, and
- $(f + g)(x) = f(x) \cup g(x)$ for $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$.

Given conditions $q_0$, $q_1 \in \mathbb{Q}_{\kappa^+}$, we denote by

$$q_0 \oplus q_1$$

the natural amalgamation of $q_0$ and $q_1$; i.e., $q_0 \oplus q_1$ is the ordered pair $(f, \tau)$ resulting from closing $q_0$ and $q_1$ under relevant isomorphisms $\Psi_{N_0,N_1}$ so that clause (3) in the definition of condition holds in the end. To be more specific, $f$ and $\tau$ are defined as $f = \bigcup_{n<\omega} f_n$ and $\tau = \bigcup_{n<\omega} \tau_n$, where $(f_n)_n$ and $(\tau_n)_n$ are the following sequences.

(1) $f_0 = f_{q_0} + f_{q_1}$
(2) $\tau_0 = \tau_{q_0} \cup \tau_{q_1}$
(3) For each $n < \omega$, $f_{n+1} = f_n + f'_n$, where $f'_n$ is the function with domain the set of ordinals of the form $\alpha' = \Psi_{N_0,N_1}(\alpha)$, for some edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_n$ and some $\alpha \in \mathrm{dom}(f_n) \cap N_0 \cap \gamma_0$ such that $\Psi_{N_0,N_1}(\alpha) < \gamma_1$, and such that for every $\alpha' \in \mathrm{dom}(f'_n)$, $f'_n(\alpha')$ is the union of all triples $((\rho, \zeta), i)$ for which there is some edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_n$ and some $\alpha \in \mathrm{dom}(f_n) \cap N_0 \cap \gamma_0$ such that $\Psi_{N_0,N_1}(\alpha) = \alpha' < \gamma_1$, $((\rho, \zeta), i) \in f_n(\alpha)$, and $\rho < \delta_{N_0}$.
(4) For each $n < \omega$, $\tau_{n+1} = \tau_n \cup \tau'_n$, where $\tau'_n$ is the set of edges of the form

$$\langle (\Psi_{N_0,N_1}(N'_0), \pi^{\gamma'_0,N'_0}_{N_0,\gamma_0,N_1,\gamma_1}), (\Psi_{N_0,N_1}(N'_1), \pi^{\gamma'_1,N'_1}_{N_0,\gamma_0,N_1,\gamma_1}) \rangle,$$

for edges $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle \in \tau_n$ and $\langle (N_0', \gamma_0'), (N_1', \gamma_1')\rangle \in N_0 \cap \tau_n$.

We finish the section with four easy lemmas that will be used in the next section.

**Lemma 2.9.** *Let $\beta \leq \kappa^+$, and suppose $q_0$, $q_1 \in \mathbb{Q}_\beta$ are such that for every $\alpha < \beta$, if*

$$(q_0 \restriction \alpha) \oplus (q_1 \restriction \alpha) \in \mathbb{Q}_\alpha,$$

*then*

$$(q_0 \restriction \alpha + 1) \oplus (q_1 \restriction \alpha + 1) \in \mathbb{Q}_{\alpha+1}$$

*Then $q_0 \oplus q_1 \in \mathbb{Q}_\beta$.*

*Proof.* The proof is by induction on $\beta$. We only need to argue for the conclusion in the case that $\beta$ is a nonzero limit ordinal. But in that case the conclusion follows easily from the induction hypothesis and the fact that for every $\alpha < \beta$, $(q_0 \oplus q_1) \restriction \alpha$ is the greatest lower bound of the set of $\mathbb{Q}_\alpha$-conditions given by $X = \{((q_0 \restriction \alpha') \oplus (q_1 \restriction \alpha')) \restriction \alpha \mid \alpha \leq \alpha' < \beta\}$. It is easy to see there is a decreasing sequence $(r_n)_{n<\omega}$ of $\mathbb{Q}_\alpha$-conditions in $X$ with the property that every $r \in X$ is such that $r_n \leq_{\mathbb{Q}_\alpha} r$ for some $n$, and thus this greatest lower bound exists. □

Given an ordinal $\alpha$ and a set $\tau$ of edges, we will call a finite sequence $(\alpha_i)_{i<n}$ of ordinals a *$\tau$-orbit of $\alpha$* if there is a sequence $\vec{\mathcal{E}} = (\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i)\rangle)_{i<n}$ of edges in $\tau$ and a sequence $(\epsilon_i)_{i<n}$ of ordinals in $\{0, 1\}$ such that

- $\alpha \in N_{\epsilon_0}^0 \cap \gamma_{\epsilon_0}^0$, and
- for each $i < n$,

$$\alpha_i = (\Psi_{N^i_{\epsilon_i}, N^i_{1-\epsilon_i}} \circ \Psi_{N^{i-1}_{\epsilon_{i-1}}, N^{i-1}_{1-\epsilon_{i-1}}} \circ \ldots \circ \Psi_{N^0_{\epsilon_0}, N^0_{1-\epsilon_0}})(\alpha)$$

  is such that $\alpha_i < \gamma^i_{1-\epsilon_i}$ and such that $\alpha_i \in N^{i+1}_{\epsilon_{i+1}} \cap \gamma^{i+1}_{\epsilon_{i+1}}$ if $i + 1 < n$.

We will call $(\alpha_i)_{i<n}$ a *descending orbit* if $\alpha_{i+1} \leq \alpha_i$ whenever $i+1 < n$.

In the above definition, we allow the empty orbit (with $n = 0$). This is for notational convenience in the proof of Lemma 2.10.[9]

Given two conditions $q_0$ and $q_1$ and an edge $e = \langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ in $\tau_{q_0 \oplus q_1}$, we define the *$(q_0, q_1)$-rank of $e$* as follows.

- $e$ has $(q_0, q_1)$-rank 0 if $e \in \tau_{q_0} \cup \tau_{q_1}$.

[9]Note that, with this definition, it makes sense to say that every ordinal is on an orbit of itself.

- for every $n < \omega$, $e$ has $(q_0, q_1)$-rank $n+1$ if there are edges $e^* = \langle (N_0^*, \gamma_0^*), (N_1^*, \gamma_1^*)\rangle$ and $e' = \langle (N_0', \gamma_0'), (N_1', \gamma_1')\rangle \in N_0$ in $\tau_{q_0 \oplus q_1}$ such that the maximum of the $(q_0, q_1)$-ranks of $e^*$ and $e'$ is $n$ and such that

$$e = \langle (\Psi_{N_0^*, N_1^*}(N_0'), \pi^{\gamma_0', N_0'}_{N_0^*, \gamma_0^*, N_1^*, \gamma_1^*}), (\Psi_{N_0^*, N_1^*}(N_1'), \pi^{\gamma_1', N_1'}_{N_0^*, \gamma_0^*, N_1^*, \gamma_1^*})\rangle$$

**Lemma 2.10.** *For all $\beta \leq \kappa^+$, $q_0$, $q_1 \in \mathbb{Q}_\beta$ and $\alpha \in \mathrm{dom}(f_{q_0 \oplus q_1})$ there is some $\alpha^* \in \mathrm{dom}(f_{q_0}) \cup \mathrm{dom}(f_{q_1})$ such that $\alpha$ is on some $\tau_{q_0} \cup \tau_{q_1}$–orbit of $\alpha^*$.*

*Proof.* It suffices to prove that for any $\tau_{q_0 \oplus q_1}$-orbit $(\alpha_i)_{i<n}$ of an ordinal $\alpha^*$ and any ordinal $\alpha$ on $(\alpha_i)_{i<n}$ there is in fact a (possibly longer) $\tau_{q_0} \cup \tau_{q_1}$-orbit of $\alpha^*$ on which $\alpha$ is. Let $(\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i)\rangle)_{i<n}$ be a sequence of edges in $\tau_{q_0 \oplus q_1}$ and $(\epsilon_i)_{i<n}$ a sequence of ordinals in $\{0, 1\}$ which together witness that $(\alpha_i)_{i<n}$ is a $\tau_{q_0 \oplus q_1}$-orbit. The conclusion can be easily proved by a double induction on, first, the maximum of the $(q_0, q_1)$-ranks of the edges $\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i)\rangle$ (for $i < n$) and, second, the length $n$.

Suppose there is some $i$ such that $\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i)\rangle$ is not an edge in $\tau_{q_0} \cup \tau_{q_1}$. Then $\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i)\rangle$ is of the form

$$\langle (\Psi_{N_0, N_1}(N_0'), \pi^{\gamma_0', N_0'}_{N_0, \gamma_0, N_1, \gamma_1}), (\Psi_{N_0, N_1}(N_1'), \pi^{\gamma_1', N_1'}_{N_0, \gamma_0, N_1, \gamma_1})\rangle$$

for edges $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ and $\langle (N_0', \gamma_0'), (N_1', \gamma_1')\rangle \in N_0$ in $\tau_{q_0 \oplus q_1}$ of $(q_0, q_1)$-rank less than $\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i)\rangle$. By induction hypothesis, $\alpha_i$ is on a $\tau_{q_0} \cup \tau_{q_1}$-orbit $o^*$ of $\alpha^*$ and $\alpha$ is on a $\tau_{q_0} \cup \tau_{q_1}$-orbit $o^{**}$ of $\alpha_{i+1}$. But $\alpha_{i+1}$ is also on the $q_0 \cup q_1$-orbit $o$ resulting from concatenating the following orbits $o_1$, $o_2$ and $o_3$:

(1) $o_1$ is a $\tau_{q_0} \cup \tau_{q_1}$-orbit joining $\alpha_i$ and $\Psi_{N_1, N_0}(\alpha_i))$, which exists by induction hypothesis since $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ has $(q_0, q_1)$-rank less than $\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i)\rangle$.
(2) $o_2$ is a $\tau_{q_0} \cup \tau_{q_1}$-orbit joining $\Psi_{N_1, N_0}(\alpha_i))$ and $\Psi_{N_1, N_0}(\alpha_{i+1})$ ($= \Psi_{N'_{\epsilon_i}, N'_{1-\epsilon_i}}(\Psi_{N_1, N_0}(\alpha_i)))$, which this times exists by induction hypothesis since $\langle (N_0', \gamma_0'), (N_1', \gamma_1')\rangle$ has $(q_0, q_1)$-rank less than $\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i)\rangle$.
(3) $o_3$ is a $\tau_{q_0} \cup \tau_{q_1}$-orbit joining $\Psi_{N_1, N_0}(\alpha_{i+1})$ and $\alpha_{i+1}$, which exists again by induction hypothesis since $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ has $(q_0, q_1)$-rank less than $\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i)\rangle$.

Now we obtain the desired $\tau_{q_0} \cup \tau_{q_1}$-orbit as the concatenation of these three orbits $o^*$, $o$ and $o^{**}$. $\square$

Given $\alpha < \kappa^+$ and given nodes $x$, $y \in \kappa \times \omega_1$, if $\mathbb{Q}_\alpha$ is $\kappa$-c.c., then we denote by $A^\alpha_{x,y}$ the first, in some well-order of $H(\kappa^+)$ canonically definable from $\Phi$, maximal antichain of $\mathbb{Q}_\alpha$ consisting of conditions deciding whether or not $x$ and $y$ are comparable in $\underset{\sim}{T}_\alpha$.[10] Given $q \in \mathbb{Q}_{\kappa^+}$, we will say that $q$ is *adequate* in case the following holds.

(1) For all nonzero $\alpha$, $\alpha'$ in $\mathrm{dom}(f_q)$,[11] if $x \in \mathrm{dom}(f_q(\alpha))$, $y \in \mathrm{dom}(f_q(\alpha'))$, and $\mathbb{Q}_\alpha$ is $\kappa$-c.c., then $q \restriction \alpha$ extends a condition in $A^\alpha_{x,y}$.
(2) For every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle \in \tau_q$ and every $\alpha \in \mathrm{dom}(f_q) \cap N_1 \cap \gamma_1$, if $\Psi_{N_1,N_0}(\alpha) < \gamma_0$, then $\Psi_{N_1,N_0}(\alpha) \in \mathrm{dom}(f_q)$ and

$$f_q(\Psi_{N_1,N_0}(\alpha)) \restriction \delta_{N_1} \times \omega_1 = f_q(\alpha) \restriction \delta_{N_1} \times \omega_1$$

Let us call a condition *weakly adequate* if it satisfies clause (1) in the above definition.

It is straightforward to see that the set of weakly adequate conditions is dense in $\mathbb{Q}_{\kappa^+}$.

**Lemma 2.11.** *Suppose $q$ is a weakly adequate $\mathbb{Q}_{\kappa^+}$-condition, $\alpha \in \mathrm{dom}(f_q)$, and $\alpha' < \kappa^+$ is on a descending $\tau_q$-orbit of $\alpha$ as witnessed by a sequence $\vec{\mathcal{E}} = (\langle (N^i_0, \gamma^i_0), (N^i_1, \gamma^i_1)\rangle)_{i \leq n}$ of edges in $\tau_q$. Suppose $x_0 = (\rho_0, \zeta_0)$ and $x_1 = (\rho_1, \zeta_1)$ are two nodes such that*

*(1) $\rho_0$, $\rho_1 < \min\{\delta_{N^i_0} \mid i \leq n\}$, and*
*(2) $q \restriction \alpha$ forces that $x$ and $y$ are incomparable in $\underset{\sim}{T}_\alpha$.*

*Then $q \restriction \alpha'$ forces that $x_0$ and $x_1$ are incomparable in $\underset{\sim}{T}_{\alpha'}$.*

*Proof.* By the definition of weak adequacy there is a condition $r \in A^\alpha_{x_0,x_1}$ extended by $q \restriction \alpha$ and forcing that $x_0$ and $x_1$ are incomparable in $\underset{\sim}{T}_\alpha$. For each $i \leq n$, let $\alpha_i = (\Psi_{N^i_0,N^i_1} \circ \ldots \circ \Psi_{N^0_0,N^0_1})(\alpha)$. For each such $i$ we have, by definability of $A^{\alpha_i}_{x_0,x_1}$ over $(H(\kappa^+), \in, \Phi_{\alpha_i})$, that $A^{\alpha_i}_{x_0,x_1} \in N^i_0$, and therefore also that $A^{\alpha_i}_{x_0,x_1} \subseteq N^i_0$ (since $|A^{\alpha_i}_{x_0,x_1}| < \kappa$ and hence $N^i_0$ contains a bijection between $A^{\alpha_i}_{x_0,x_1}$ and some cardinal $\mu < \kappa$, which of course is such that $\mu \subseteq N^i_0$). It follows from the above that $r \in N^0_0$ and, inductively, using the fact that $\Psi_{N^i_0,N^i_1}$ is an isomorphism between $(N^i_0, \in \Phi_{\alpha_{i-1}})$ and $(N^i_1, \in, \Phi_{\alpha_i})$ for every $i$ (where $\alpha_{-1} = \alpha$), that for every $i \leq n$, $(\Psi_{N^i_0,N^i_1} \circ \ldots \circ \Psi_{N^0_0,N^0_1})(r)$ is a condition in $\mathbb{Q}_{\alpha_i}$ extended by $q \restriction \alpha_i$ (by clause (3) in the definition of condition) and forcing, in $\mathbb{Q}_{\alpha_i}$, that $x_0$ and $x_1$ are also incomparable in $\mathbb{Q}_{\alpha_i}$. But now we reach the desired conclusion setting $i = n$. □

[10]Note that, if $\mathbb{Q}_\alpha$ is $\kappa$-c.c., then every maximal antichain of $\mathbb{Q}_\alpha$ is a member of $H(\kappa^+)$, and so this definition makes sense.

[11]$\alpha$ and $\alpha'$ may or may not be equal. Also, if $\alpha \neq \alpha'$, we may have $\alpha < \alpha'$ or $\alpha' < \alpha$.

**Lemma 2.12.** *For every $\beta \leq \kappa^+$, the set of adequate $\mathbb{Q}_\beta$-conditions is dense in $\mathbb{Q}_\beta$.*

*Proof.* The proof is by induction on $\beta$. First assume $\beta = \alpha + 1$ for some $\alpha$. Let $q$ be a condition in $\mathbb{Q}_\beta$, which we may assume is weakly adequate. We may also assume that there is a minimal $\delta_0$ for which there is an edge $\langle (N_0, \alpha+1), (N_1, \gamma_1)\rangle \in \tau_q$ such that $\delta_{N_0} = \delta_0$ and some $\alpha' \in \mathrm{dom}(f_q) \cap N_1 \cap \gamma_1$ such that $\Psi_{N_1,N_0}(\alpha') = \alpha$, since otherwise we are done. By Lemma 2.4 we may find a condition $q_0$ extending $q$ and such that

(1) $q_0 \restriction N_0$ forces, in $\mathbb{Q}_{\alpha+1} \cap N_0$, that every condition in $\dot{G}_{\mathbb{Q}_{\alpha+1}\cap N_0}$ is compatible, in $\mathbb{Q}^{N_0}_{\alpha+1}$, with $q_0$,
(2) $\alpha \in \mathrm{dom}(f_{q_0})$,
(3) $f_{q_0}(\alpha) \restriction (\kappa \setminus \delta_0) \times \omega_1 = f_q(\alpha) \restriction (\kappa \setminus \delta_0) \times \omega_1$ (where $f_q(\alpha)$ denotes $\emptyset$ if $\alpha \notin \mathrm{dom}(f_q)$), and such that
(4) $\langle (N_0', \gamma_0'), (N_1', \gamma_1')\rangle \in N_0$ for every edge $\langle (N_0', \gamma_0'), (N_1', \gamma_1')\rangle \in \tau_{q_0}$ such that $\gamma_0' = \alpha + 1$ or $\gamma_1' = \alpha + 1$.

By extending $q_0 \restriction \alpha$ in a construction in $\omega$ steps using the induction hypothesis and the fact that the set of weakly adequate conditions is dense, we may assume that $q_0$ is weakly adequate and that $q_0 \restriction \alpha$ is adequate. Let us consider the ordered pair $q_0'$ obtained from $q_0$ by simply adding to $f_{q_0}(\alpha)$ all missing pairs $(x, i)$ such that $x = (\rho, \zeta)$, for some $\rho < \delta_0$, and $(f_{q_0}(\alpha'))(x) = i$. By weak adequacy of $q_0$, $q_0' \restriction N_0$ is a condition in $\mathbb{Q}_\beta$, and by the choice of $q_0$ we may extend $q_0' \restriction \alpha$ $(= q_0 \restriction \alpha)$ to a condition $r \in \mathbb{Q}_\alpha$ which, for every two distinct $x_0$, $x_1 \in \mathrm{dom}(f_{q_0'}(\alpha))$ such that $(f_{q_0'}(\alpha))(x_0) = (f_{q'}(\alpha))(x_1)$, forces that $x_0$ and $x_1$ are incomparable in $\underset{\sim}{T}_\alpha$. Finally, we may naturally amalgamate $r$, $q_0'$ and $q_0$ into a $\mathbb{Q}_\beta$-condition $q_0^*$.

Assuming there is a least $\delta_1 > \delta_0$ with the property that there is an edge $\langle (N_0, \alpha + 1), (N_1, \gamma_1)\rangle \in \tau_q$ such that $\delta_{N_0} = \delta_1$ and some $\alpha' \in \mathrm{dom}(f_q) \cap N_1 \cap \gamma_1$ such that $\Psi_{N_1,N_0}(\alpha') = \alpha$, we run a construction as above with $q_0^*$ and $\delta_1$ instead of $q$ and $\delta_0$ and end up with a condition $q_1^*$. Proceeding in this way for countably many steps, taking greatest lower bounds at limit stages, we end up with an adequate extension of $q$.

Now suppose $\beta$ is a limit ordinal and let $q$ be a condition in $\mathbb{Q}_\beta$ which we may assume is weakly adequate. Let $\langle \alpha_i \mid i < \nu\rangle$, where $\nu < \omega_1$, be the increasing enumeration of the set of ordinals $\alpha$ for which there are some edge $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle \in \tau_q$ and some $\alpha' \in \mathrm{dom}(f_q) \cap N_1 \cap \gamma_1$ such that $\Psi_{N_1,N_0}(\alpha') < \gamma_0$ and $\alpha = \Psi_{N_1,N_0}(\alpha')$.

Using Lemmas 2.7 and 2.8, and by induction on $\alpha_i$, for $i < \nu$, we define a sequence $\langle q_i \mid i < \nu\rangle$ of conditions such that the following holds for all $i$.

- $q_i \in \mathbb{Q}_{\alpha_i+1}$ is adequate.
- $q_i \leq q \restriction \alpha_i + 1$
- For $i < j < \nu$, $q_j \restriction \alpha_i + 1 \leq q_i$.

Let $q^* \in \mathbb{Q}_\beta$ be the greatest lower bound of $\{q_i \mid i < \nu\} \cup \{q\}$. Then $q^* \leq q$ and $q^*$ is adequate. □

## 3. The chain condition

This section is devoted to proving Lemma 3.1.

**Lemma 3.1.** *For each $\beta \leq \kappa^+$, $\mathbb{Q}_\beta$ has the $\kappa$-chain condition.*

As we will see, the weak compactness of $\kappa$ is crucially used in order to prove Lemma 3.1. Let $\mathcal{F}$ be the weak compactness filter on $\kappa$, i.e., the filter on $\kappa$ generated by the sets $\{\lambda < \kappa \mid (V_\lambda, \in, B \cap V_\lambda) \models \psi\}$, where $B \subseteq V_\kappa$ and where $\psi$ is a $\Pi^1_1$ sentence for the structure $(V_\kappa, \in, B)$. $\mathcal{F}$ is a proper normal filter on $\kappa$. Let also $\mathcal{S}$ be the collection of $\mathcal{F}$-positive subsets of $\kappa$, i.e., $\mathcal{S} = \{X \subseteq \kappa \mid X \cap C \neq \emptyset \text{ for all } C \in \mathcal{F}\}$.

We will call a model $Q$ *suitable* if $Q$ is an elementary submodel of cardinality $\kappa$ of some high enough $H(\theta)$, closed under $<\kappa$-sequences, and such that $\langle \mathbb{Q}_\alpha \mid \alpha < \kappa^+\rangle \in Q$. Given a suitable model $Q$, a bijection $\varphi : \kappa \to Q$, and an ordinal $\lambda < \kappa$, we will denote $\varphi``\lambda$ by $M^\varphi_\lambda$.

Given $\beta \leq \kappa^+$, we will say that $\mathbb{Q}_\beta$ *has the strong $\kappa$-chain condition* if for every $X \in \mathcal{S}$, every suitable model $Q$ such that $\beta, X \in Q$, every bijection $\varphi : \kappa \to Q$, and every two sequences $(q^0_\lambda \mid \lambda \in X) \in Q$ and $(q^1_\lambda \mid \lambda \in X) \in Q$ of $\mathbb{Q}_\beta$-conditions, if $q^0_\lambda \restriction M^\varphi_\lambda = q^1_\lambda \restriction M^\varphi_\lambda$ for every $\lambda \in X$, then there is some $Y \in \mathcal{S}$, $Y \subseteq X$, together with sequences $(q^{00}_\lambda \mid \lambda \in Y)$ and $(q^{11}_\lambda \mid \lambda \in Y)$ of $\mathbb{Q}_\beta$-conditions with the following properties.

(1) $q^{00}_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ and $q^{11}_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$ for every $\lambda \in Y$.
(2) For all $\lambda < \lambda^*$ in $Y$, $q^{00}_\lambda \oplus q^{11}_{\lambda^*}$ is a common extension of $q^{00}_\lambda$ and $q^{11}_{\lambda^*}$.

The following lemma is immediate.

**Lemma 3.2.** *For every $\beta \leq \kappa^+$, if $\mathbb{Q}_\beta$ has the strong $\kappa$-chain condition, then $\mathbb{Q}_\beta$ has the $\kappa$-chain condition.*

Following [5], given $\beta \leq \kappa^+$, a suitable model $Q$ such that $\beta \in Q$, a bijection $\varphi : \kappa \to Q$, and a $\mathbb{Q}_\beta$-condition $q \in Q$, let us say that $q$ *is $\lambda$-compatible with respect to $\varphi$ and $\beta$* if, letting $\mathbb{Q}^*_\beta = \mathbb{Q}_\beta \cap Q$, we have that

- $\mathbb{Q}^*_\beta \cap M^\varphi_\lambda \lessdot \mathbb{Q}^*_\beta$,
- $q \restriction M^\varphi_\lambda \in \mathbb{Q}^*_\beta$, and
- $q \restriction M^\varphi_\lambda$ forces in $\mathbb{Q}^*_\beta \cap M^\varphi_\lambda$ that $q$ is in the quotient forcing $\mathbb{Q}^*_\beta/\dot{G}_{\mathbb{Q}^*_\beta \cap M^\varphi_\lambda}$; equivalently, for every $r \leq_{\mathbb{Q}^*_\beta \cap M^\varphi_\lambda} q \restriction M^\varphi_\lambda$, $r$ is compatible with $q$.[12]

Adopting the approach from [10], rather than proving Lemma 3.1 we will prove the following more informative lemma.

**Lemma 3.3.** *The following holds for every $\beta \leq \kappa^+$.*

$(1)_\beta$ *$\mathbb{Q}_\beta$ has the strong $\kappa$-chain condition.*

$(2)_\beta$ *Suppose $D \in \mathcal{F}$, $Q$ is a suitable model, $\beta$, $D \in Q$, $\varphi : \kappa \to Q$ is a bijection, and $(q^0_\lambda \mid \lambda \in D) \in Q$ and $(q^1_\lambda \mid \lambda \in D) \in Q$ are sequences of adequate $\mathbb{Q}_\beta$-conditions. Then there is some $D' \in \mathcal{F}$ such that $D' \subseteq D$ and such that for every $\lambda \in D'$, if $q^0_\lambda \restriction M^\varphi_\lambda = q^1_\lambda \restriction M^\varphi_\lambda$, then there are conditions $q'^0_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ and $q'^1_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$ such that*
   (a) *$q'^0_\lambda \restriction M^\varphi_\lambda = q'^1_\lambda \restriction M^\varphi_\lambda$ and*
   (b) *$q'^0_\lambda$ and $q'^1_\lambda$ are both $\lambda$-compatible with respect to $\varphi$ and $\beta$.*

The rest of the section is thus devoted to a proof of the above lemma.

*Proof.* (of Lemma 3.3) The proof is by induction on $\beta$. Let $\beta \leq \kappa^+$ and suppose $(1)_\alpha$ and $(2)_\alpha$ holds for all $\alpha < \beta$. We will show that $(1)_\beta$ and $(2)_\beta$ also hold.

There is nothing to prove for $\beta = 0$, and the case $\beta = 1$ is trivial, using the inaccessibility of $\kappa$.

Let us proceed to the case when $\beta > 1$. The proof in the case $\beta = \kappa^+$ follows immediately from Lemma 2.6 (2) together with the induction hypothesis.

Suppose next that $\beta < \kappa^+$. We start with the proof of $(1)_\beta$.

Let $X \in \mathcal{S}$ be given, together with a suitable model $Q$ such that $\beta, X \in Q$, a bijection $\varphi : \kappa \to Q$, and sequences $\sigma^0 = (q^0_\lambda \mid \lambda \in X) \in Q$ and $\sigma^1 = (q^1_\lambda \mid \lambda \in X) \in Q$ of $\mathbb{Q}_\beta$-conditions such that $q^0_\lambda \restriction M^\varphi_\lambda = q^1_\lambda \restriction M^\varphi_\lambda$ for every $\lambda \in X$. We need to prove that there is some $Y \in \mathcal{S}$, $Y \subseteq X$, together with sequences $(q^{00}_\lambda \mid \lambda \in Y)$ and $q^{11}_\lambda \mid \lambda \in Y)$ of $\mathbb{Q}_\beta$-conditions such that the following holds.

(1) $q^{00}_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ and $q^{11}_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$ for every $\lambda \in Y$.
(2) For all $\lambda < \lambda^*$ in $Y$, $q^{00}_\lambda \oplus q^{11}_{\lambda^*}$ is a common extension of $q^{00}_\lambda$ and $q^{11}_{\lambda^*}$.

[12] In [10], this situation is denoted by $*^\beta_\lambda(q_0, q_0 \restriction M^\varphi_\lambda)$.

In what follows, we will write $M_\lambda$ instead of $M^\varphi_\lambda$.

Let $\mathbb{Q}^*_\alpha = \mathbb{Q}_\alpha \cap Q$ for every $\alpha \in Q \cap (\beta+1)$. By the induction hypothesis, $\mathbb{Q}_\alpha$ has the $\kappa$-c.c. for every $\alpha \in Q \cap \beta$. Hence, since ${}^{<\kappa}Q \subseteq Q$, we have that $\mathbb{Q}^*_\alpha \lessdot \mathbb{Q}_\alpha$ for every such $\alpha$; in particular, we have that for every $\alpha \in Q \cap \beta$, $\mathbb{Q}^*_\alpha$ forces over $V$ that $\underset{\sim}{T}_\alpha$ does not have $\kappa$-branches.

Given a nonzero $\alpha \in \beta$, a node $x = (\rho, \zeta)$ and an ordinal $\bar\rho < \rho$, let $B^\alpha_{x,\bar\rho}$ denote the least, in some well–order of $H(\kappa^+)$ canonically defined from $\Phi$, maximal antichain of $\mathbb{Q}_\alpha$ consisting of conditions deciding some $\bar\zeta < \omega_1$ such that the node $\bar x = (\bar\rho, \bar\zeta)$ is below $x$ in $\underset{\sim}{T}_\alpha$. Also, given

- conditions $q^0$, $q^1$,
- $\alpha \in \mathrm{dom}(f_{q^0})$ and $\alpha' \in \mathrm{dom}(f_{q^1})$,
- nodes $x = (\rho_0, \zeta_0)$ and $y = (\rho_1, \zeta_1)$ such that $x \in \mathrm{dom}(f_{q^0}(\alpha))$ and $y \in \mathrm{dom}(f_{q^1}(\alpha'))$,[13] and
- $\lambda < \kappa$,

we will say that $x$ *and* $y$ *are separated below* $\lambda$ *at stages* $\alpha$ *and* $\alpha'$ *by* $q^1 \restriction \alpha$ *and* $q^0 \restriction \alpha'$ *(via* $\bar x$, $\bar y$*)* if there are $\bar\rho < \lambda$ and $\zeta \neq \zeta'$ in $\omega_1$ such that $\bar x = (\bar\rho, \zeta)$, $\bar y = (\bar\rho, \zeta')$, and such that

(1) $q^0 \restriction \alpha$ extends a condition in $B^\alpha_{x,\bar\rho}$ forcing $\bar x$ to be below $x$ in $\underset{\sim}{T}_\alpha$ and
(2) $q^1 \restriction \alpha'$ extends a condition in $B^{\alpha'}_{y,\bar\rho}$ forcing $\bar y$ to be below $y$ in $\underset{\sim}{T}_{\alpha'}$.

**Definition 3.4.** *Given* $Y \in \mathcal{S}$ *such that* $Y \subseteq X$ *and such that* $M_\lambda \prec Q$, $M_\lambda \cap \kappa = \lambda$, *and* ${}^{<\lambda}M_\lambda \subseteq M_\lambda$ *for all* $\lambda \in Y$, *and given two sequences* $\sigma^{00} = (q^{00}_\lambda \mid \lambda \in Y)$, $\sigma^{11} = (q^{11}_\lambda \mid \lambda \in Y)$ *of adequate* $\mathbb{Q}^*_\beta$*-conditions, we say that* $\sigma^{00}$, $\sigma^{11}$ is a separating pair for $\sigma^0$ and $\sigma^1$ *if the following holds.*

(1) $q^{00}_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ *and* $q^{11}_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$ *for all* $\lambda \in Y$.
(2) *For all* $\lambda \in Y$, *all nonzero* $\alpha \in \mathrm{dom}(f_{q^{00}_\lambda}) \cap M_\lambda$ *and* $\alpha' \in \mathrm{dom}(f_{q^{11}_\lambda})$ *such that* $\alpha' \leq \alpha$, *and all* $x \in \mathrm{dom}(f_{q^{00}_\lambda}(\alpha)) \setminus (\lambda \times \omega_1)$ *and* $y \in \mathrm{dom}(f_{q^{11}_\lambda}(\alpha')) \setminus (\lambda \times \omega_1)$, $x$ *and* $y$ *are separated below* $\lambda$ *at stages* $\alpha$ *and* $\alpha'$ *by* $q^{00}_\lambda \restriction \alpha$ *and* $q^{11}_\lambda \restriction \alpha'$ *via some pair* $\chi_0(x, y, \alpha, \alpha', \lambda)$, $\chi_1(x, y, \alpha, \alpha', \lambda)$.
(3) *The following holds for all* $\lambda_0 < \lambda_1$ *in* $Y$.
 (a) $q^{00}_{\lambda_0} \restriction M_{\lambda_0} = q^{11}_{\lambda_1} \restriction M_{\lambda_1}$
 (b) $\mathrm{dom}(f_{q^{00}_{\lambda_0}}) \cap M_{\lambda_0} = \mathrm{dom}(f_{q^{11}_{\lambda_1}}) \cap M_{\lambda_1}$
 (c) $q^{00}_{\lambda_0} \in M_{\lambda_1}$
 (d) *For some ordinal* $\varsigma$,

$$sup(R_{\lambda_0} \cup \Delta_{\lambda_0}) = sup(R_{\lambda_1} \cup \Delta_{\lambda_1}) = \varsigma$$

[13] $\alpha$ and $\alpha'$ may or may not be equal, and the same applies to $x$ and $y$.

*where, for every* $\epsilon \in \{0,1\}$,

$$R_\epsilon = \{\rho < \lambda_\epsilon \mid \alpha \in \mathrm{dom}(f_{q^{\epsilon\epsilon}_{\lambda_\epsilon}}),\ (\rho,\zeta) \in \mathrm{dom}(f_{q^{\epsilon\epsilon}_{\lambda_\epsilon}}(\alpha))\}$$

*and*

$$\Delta_\epsilon = \{\delta_{N_0} < \lambda_\epsilon \mid \langle (N_0,\gamma_0),(N_1,\gamma_1)\rangle \in \tau_{q^{\epsilon\epsilon}_{\lambda_\epsilon}}\}$$

(e) *For every edge* $\langle (N_0,\gamma_0),(N_1,\gamma_1)\rangle \in \tau_{q^{11}_{\lambda^*}}$, *if* $\delta_{N_0} < \lambda^*$, *then* $(N_0 \cup N_1) \cap (\mathrm{dom}(f_{q^{00}_\lambda}) \cup \tau_{q^{00}_\lambda}) \subseteq M_\lambda$.

(4) *For all* $\lambda_0 < \lambda_1$ *in* $X$, *all nonzero* $\alpha \in \mathrm{dom}(f_{q^{00}_{\lambda_0}}) \cap M_{\lambda_0}$ *and* $\alpha' \in \mathrm{dom}(f_{q^{11}_{\lambda_1}})$ *such that* $\alpha' \leq \alpha$, *and all nodes*

$$x \in \mathrm{dom}(f_{q^{00}_{\lambda_0}}(\alpha)) \setminus (\lambda_0 \times \omega_1)$$

*and*

$$y' \in \mathrm{dom}(f_{q^{11}_{\lambda_1}(\alpha')}) \setminus (\lambda_1 \times \omega_1)$$

*there are*

- *a node* $x' \in \mathrm{dom}(f_{q^{00}_{\lambda_1}(\alpha)}) \setminus (\lambda_1 \times \omega_1)$,
- *a stage* $\alpha^\dagger \in \mathrm{dom}(f_{q^{11}_{\lambda_0}})$ *such that* $\alpha^\dagger \leq \alpha$, *and*
- *a node* $y \in \mathrm{dom}(f_{q^{11}_{\lambda_0}(\alpha^\dagger)}) \setminus (\lambda_0 \times \omega_1)$

*such that*

$$\chi_0(x,y,\alpha,\alpha^\dagger,\lambda_0) = \chi_0(x',y',\alpha,\alpha',\lambda_1)$$

*and*

$$\chi_1(x,y,\alpha,\alpha^\dagger,\lambda_0) = \chi_1(x',y',\alpha,\alpha',\lambda_1)$$

Let us now prove the following.

**Claim 3.5.** *Let* $Y \in \mathcal{S}$ *be such that* $M_\lambda \cap \kappa = \lambda$ *for all* $\lambda \in Y$ *and suppose* $\sigma^{00} = (q^{00}_\lambda \mid \lambda \in Y)$, $\sigma^{11} = (q^{11}_\lambda \mid \lambda \in Y)$ *is a separating pair for* $\sigma^0$ *and* $\sigma^1$. *Then for all* $\lambda_0 < \lambda < \lambda_1$ *in* $Y$, $q^{00}_{\lambda_0} \oplus q^{11}_{\lambda_1}$ *is a common extension of* $q^{00}_{\lambda_0}$ *and* $q^{11}_{\lambda_1}$ *in* $\mathbb{Q}_\beta$.

*Proof.* Suppose, towards a contradiction, that there are $\lambda_0 < \lambda < \lambda_1$ in $Y$ such that $q^{00}_{\lambda_0} \oplus q^{11}_{\lambda_1}$ is not a common extension of $q^{00}_{\lambda_0}$ and $q^{11}_{\lambda_1}$. It then follows that $q^{00}_{\lambda_0} \oplus q^{11}_{\lambda_1}$ is not a condition. Hence, by Lemma 2.9, there is a least $\alpha < \beta$ such that $(q^{00}_{\lambda_0} \restriction \alpha) \oplus (q^{11}_{\lambda_1} \restriction \alpha)$ is a condition yet $(q^{00}_{\lambda_0} \restriction \alpha+1) \oplus (q^{11}_{\lambda_1} \restriction \alpha+1)$ is not. Assuming that we are in this situation, we will derive a contradiction. Set $q = (q^{00}_{\lambda_0} \restriction \alpha) \oplus (q^{11}_{\lambda_1} \restriction \alpha)$.

To start with, note that $\alpha > 0$. One possibility for the ordered pair $(q^{00}_{\lambda_0} \restriction \alpha+1) \oplus (q^{11}_{\lambda_1} \restriction \alpha+1)$ not to be a condition is that $\alpha \in \mathrm{dom}(f_{q^{00}_{\lambda_0}}) \cap \mathrm{dom}(f_{q^{11}_{\lambda_1}})$ and that there are $x = (\rho_0,\zeta_0) \in \mathrm{dom}(f_{q^{00}_{\lambda_0}}(\alpha))$ and $y' = (\rho'_1,\zeta'_1) \in \mathrm{dom}(f_{q^{11}_{\lambda_1}}(\alpha))$ such that the following holds.

- $(f_{q^{00}_{\lambda_0}}(\alpha))(x) = (f_{q^{11}_{\lambda_1}}(\alpha))(y')$,
- $q \restriction \alpha$ does not force $x$ and $y'$ to be incomparable in $\underset{\sim}{T}_\alpha$.

By clauses (3) (a) and (c) in Definition 3.4, we have of course that $\rho_0 \geq \lambda_0$ and $\rho_1' \geq \lambda_1$. The rest of the argument, in this case, is now essentially as in the corresponding proof in [10]. By an instance of clause (4) in Definition 3.4 with $\alpha' = \alpha$, we may pick $x' = (\rho_0', \zeta_0') \in \mathrm{dom}(f_{q^{00}_{\lambda_1}}(\alpha)) \setminus (\lambda_1 \times \omega_1)$, $\alpha^\dagger \in \mathrm{dom}(f_{q^{11}_{\lambda_0}})$ such that $\alpha^\dagger \leq \alpha$, and $y = (\rho_1, \zeta_1) \in \mathrm{dom}(f_{q^{11}_{\lambda_0}(\alpha^\dagger)}) \setminus (\lambda_0 \times \omega_1)$ such that

$$\chi_0(x, y, \alpha, \alpha^\dagger, \lambda_0) = \chi_0(x', y', \alpha, \alpha, \lambda_1)$$

and

$$\chi_1(x, y, \alpha, \alpha^\dagger, \lambda_0) = \chi_1(x', y', \alpha, \alpha, \lambda_1)$$

(where $\chi_0$ and $\chi_1$ are the projections in Definition 3.4). We have that $q \restriction \alpha$ forces $\chi_0(x, y, \alpha, \alpha^\dagger, \lambda_0)$ to be below $x$ in $\underset{\sim}{T}_\alpha$ (because this is true about $q^{00}_{\lambda_0} \restriction \alpha$). Also, $q^{11}_{\lambda_1} \restriction \alpha$ forces that $\chi_1(x', y', \alpha, \alpha, \lambda_1)$ is below $y'$ in $\underset{\sim}{T}_\alpha$, and therefore so does $q \restriction \alpha$. But this is a contradiction since there are $\bar{\rho} < \lambda_0$ and $\zeta_0 \neq \zeta_1$ in $\omega_1$ such that

$$\chi_0(x, y, \alpha, \alpha^\dagger, \lambda_0) = \chi_0(x', y', \alpha, \alpha, \lambda_1) = (\bar{\rho}, \zeta_0)$$

and

$$\chi_1(x, y, \alpha, \alpha^\dagger, \lambda_0) = \chi_1(x', y', \alpha, \alpha, \lambda_1) = (\bar{\rho}, \zeta_1),$$

and hence $q \restriction \alpha$ forces $x$ and $y'$ to be incomparable in $\underset{\sim}{T}_\alpha$. This contradiction rules out the above situation.

It will be convenient to isolate the following subclaim in order to block, in an efficient way, all other situations in which the ordered pair $(q^{00}_{\lambda_0} \restriction \alpha + 1) \oplus (q^{11}_{\lambda_1} \restriction \alpha + 1)$ would not be a condition.

**Subclaim 3.6.** *Suppose $\alpha_1 \leq \alpha_0$ are ordinals in $M_{\lambda_0}$ such that $\alpha_1$ is on a $\tau_{q^{11}_{\lambda_1} \restriction \alpha+1}$-orbit of $\alpha_0$, as witnessed by a sequence of edges $\vec{\mathcal{E}} = (\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i) \rangle)_{i \leq n}$ in $\tau_{q^{11}_{\lambda_1} \restriction \alpha+1}$ such that $\delta_{N_0^i} \geq \lambda_1$ for all $i \leq n$ and a sequence $(\epsilon_i)_{i \leq n}$ of ordinals in $\{0, 1\}$. Then $\alpha_1 = \alpha_0$.*

*Proof.* Suppose first that $\delta_{N_0^i} = \delta_{N_0^{i'}}$ for all $i$, $i'$. By correctness of the structure $(N^0_{\epsilon_0}, \in, \Phi \cap N^0_{\epsilon_0})$ within $(H(\kappa^+), \in, \Phi)$, we may pick some model $M \in N^0_{\epsilon_0}$ closed under $\vec{e}$ such that $\alpha_0 \in M$, $\delta_M = \lambda$, and $|M| = \lambda$ (since this is true for $M_\lambda$). Given that $\alpha_1 \leq \alpha_0$ are both in $M_\lambda$, $\delta_{M_\lambda} = \lambda$, and $\alpha_0 \in M$, by the first part of Fact 2.1 we then have that $\alpha_1 \in M \subseteq N^0_{\epsilon_0}$. But that means, by the second part of Fact 2.1, that $(\Psi_{N^n_{\epsilon_0}, N^n_{1-\epsilon_0}} \circ \ldots \circ \Psi_{N^0_{\epsilon_0}, N^0_{1-\epsilon_0}})(\alpha_0) = \alpha_1$ is in fact $\alpha_0$ since $\alpha_1 \in N^0_{\epsilon_0} \cap N^n_{1-\epsilon_n}$.

Suppose now that $\delta_{N_0^i} < \delta_{N_0^{i'}}$ for some $i \neq i'$ and let $i^*$ be such that $\delta_{N_0^{i^*}} = \min\{\delta_{N_0^i} \mid i \leq n\}$. The rest of the proof is like in the proof of Fact 2.1. Since each $N_\epsilon^i$ is closed under sequences of length less than $|N_\epsilon^i|$ and $|N_\epsilon^{i^{**}}| = |\delta_{N_0^{i^*}}|$ for every $\epsilon$ and every $i^{**}$ such that $\delta_{N_0^{i^{**}}} = \delta_{N_0^{i^*}}$, there is a sequence $(\langle \bar{N}_0^i, \bar{N}_1^i \rangle)_{i \leq n}$ of pairs of models such that the following holds.

- For all $i \leq n$, $(\bar{N}_0^i, \in, \Phi_1 \cap \bar{N}_0^i)$ and $(\bar{N}_1^i, \in, \Phi_1 \cap \bar{N}_1^i)$ are elementary submodels of $(H(\kappa^+), \in, \Phi_1)$ and
$$(\bar{N}_0^i, \in, \Phi_1 \cap \bar{N}_0^i) \cong (\bar{N}_1^i, \in, \Phi_1 \cap \bar{N}_1^i)$$
- $\delta_{\bar{N}_0^i} = \delta_{\bar{N}_0^{i'}} = \delta_{N_0^{i^*}}$ for all $i$, $i' \leq n$.
- For every $i \leq n$ and every $\epsilon \in \{0, 1\}$, $\bar{N}_\epsilon^i = N_\epsilon^i$ or $\bar{N}_\epsilon^i \in N_\epsilon^i$.
- $\alpha_0 \in \bar{N}_{\epsilon_0}^0$ and $\alpha_1 \in \bar{N}_{\epsilon_0}^0 \cap \bar{N}_{1-\epsilon_n}^n$.
- $\alpha_1 = (\Psi_{N_{\epsilon_n}^n, N_{1-\epsilon_n}^n} \circ \ldots \circ \Psi_{N_{\epsilon_0}^0, N_{1-\epsilon_0}^0})(\alpha_0) = (\Psi_{\bar{N}_{\epsilon_n}^n, \bar{N}_{1-\epsilon_n}^n} \circ \ldots \circ \Psi_{\bar{N}_{\epsilon_0}^0, \bar{N}_{1-\epsilon_0}^0})(\alpha_0)$

But now we are done by the previous case. □

By Lemmas 2.2 and 2.11, the only possibility left for the ordered pair
$$(q_{\lambda_0}^{00} \restriction \alpha + 1) \oplus (q_{\lambda_1}^{11} \restriction \alpha + 1)$$
not to be a condition is that there are ordinals $\bar{\alpha} \leq \alpha'$, $\alpha^* \leq \alpha$, and $\epsilon$, $\epsilon' \in \{0, 1\}$ such that $\alpha^* \in \mathrm{dom}(f_{q_{\lambda_\epsilon}^{\epsilon\epsilon}})$ and $\alpha' \in \mathrm{dom}(f_{q_{\lambda_{\epsilon'}}^{\epsilon'\epsilon'}})$, $\bar{\alpha}$ is on a descending $\tau_{(q_{\lambda_0}^{00} \restriction \alpha+1) \oplus (q_{\lambda_1}^{11} \restriction \alpha+1)}$-orbit of $\alpha^*$, as witnessed by a sequence $\vec{\mathcal{E}} = (\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i) \rangle)_{i \leq n}$ of edges in $\tau_{(q_{\lambda_0}^{00} \restriction \alpha+1) \oplus (q_{\lambda_1}^{11} \restriction \alpha+1)}$, together with $x = (\rho, \zeta) \in \mathrm{dom}(f_{q_{\lambda_\epsilon}^{\epsilon\epsilon}}(\alpha^*))$ and $y' = (\rho', \zeta') \in \mathrm{dom}(f_{q_{\lambda_{\epsilon'}}^{\epsilon'\epsilon'}}(\alpha'))$ such that $\rho < \delta_{N_0^i}$ for all $i \leq n$, such that one of the following holds,

(1) $\bar{\alpha} = \alpha'$
(2) $\bar{\alpha} < \alpha'$ and $\bar{\alpha}$ is on a descending $\tau_{(q_{\lambda_0}^{00} \restriction \alpha+1) \oplus (q_{\lambda_1}^{11} \restriction \alpha+1)}$-orbit of $\alpha'$, as witnessed by a sequence $\vec{\mathcal{E}'} = (\langle (N_0'^i, \gamma_0'^i), (N_1'^i, \gamma_1'^i) \rangle)_{i < n'}$ of edges belonging to $\tau_{(q_{\lambda_0}^{00} \restriction \alpha+1) \oplus (q_{\lambda_1}^{11} \restriction \alpha+1)}$ such that $\rho' < \delta_{N_0'^i}$ for all $i < n'$,

and such that

- $x = y'$ and $(f_{q_{\lambda_\epsilon}^{\epsilon\epsilon}}(\alpha^*))(x) \neq (f_{q_{\lambda_{\epsilon'}}^{\epsilon'\epsilon'}}(\alpha'))(y')$, or else
- $(f_{q_{\lambda_\epsilon}^{\epsilon\epsilon}}(\alpha^*))(x) = (f_{q_{\lambda_{\epsilon'}}^{\epsilon'\epsilon'}}(\alpha'))(y')$ and $q \restriction \bar{\alpha}$ does not force $x$ and $y'$ to be incomparable in $\underset{\sim}{T}_{\bar{\alpha}}$.

We may assume that $n + n'$ is the minimal integer for which the above can be realized (where we set $n' = 0$ in case (1)). We then have that

$\Psi_{N_0^0,N_1^0}(\alpha^*) < \alpha^*$, since otherwise we may as well get rid of the edge $\langle (N_0^0,\gamma_0^0),(N_1^0,\gamma_1^0)\rangle$, contradicting the minimality of $n+n'$. Also, we may assume that we are in situation (2) only if (1) does not obtain.

Let us first consider the case when $\alpha^* = \alpha$. In this case it is easy to see, using clauses (3) (a) and (c) in Definition 3.4 and the minimality of $n+n'$, that $\epsilon = 0$ and $x \in \mathrm{dom}(f_{q^{00}_{\lambda_0}}(\alpha))$, and that we may assume that $\langle (N_0^0, N_1^0)\rangle \in \tau_{q^{11}_{\lambda_1}}$. By Lemma 2.10, $\alpha'$ and $\bar\alpha$ are on $\tau_{q^{00}_{\lambda_0}\restriction\alpha+1} \cup \tau_{q^{11}_{\lambda_1}\restriction\alpha+1}$-orbits of $\alpha$. By adequacy of $q^{00}_{\lambda_0}$ and $q^{11}_{\lambda_1}$ together with Subclaim 3.6 and clause (3) in Definition 3.4, we then have that $\alpha' \in \mathrm{dom}(q^{11}_{\lambda_1})$, and that $\bar\alpha$ and $\alpha'$ are on $\tau_{q^{11}_{\lambda_1}\restriction\alpha+1}$-orbits of $\alpha$; in particular, all of $\alpha$, $\alpha'$ and $\bar\alpha$ are in $\mathrm{dom}(f_{q^{11}_{\lambda_1}})$ by adequacy of $q^{11}_{\lambda_1}$. Since then $f_{q^{00}_{\lambda_0}}(\alpha)\restriction \lambda_0\times\omega_1 = f_{q^{11}_{\lambda_1}}(\alpha')\restriction\lambda_1\times\omega_1$ by clause (3) in Definition 3.4 and the adequacy of $q^{11}_{\lambda_1}$, we may of course assume that $\rho \geq \lambda_0$, as otherwise we obtain a contradiction by Lemma 2.11. Also, since $\bar\alpha \in \mathrm{dom}(f_{q^{11}_{\lambda_1}})$, it follows that we are in fact in situation (1). We also may assume that $y' \in \mathrm{dom}(f_{q^{11}_{\lambda_1}}(\alpha'))$ (if $\alpha' \in \mathrm{dom}(f_{q^{00}_{\lambda_0}})$ and $y' \in \mathrm{dom}(f_{q^{00}_{\lambda_0}}(\alpha'))$ were such that $\rho' > \varsigma$, then $\bar\alpha = \alpha'$ would not be in $M_{\lambda_0}$ by Subclaim 3.6, which would give a contradiction). But then we must have that $\rho' \geq \lambda_1$—otherwise $(f_{q^{11}_{\lambda_1}}(\alpha'))(y') = (f_{q^{11}_{\lambda_1}}(\alpha))(y') = (f_{q^{00}_{\lambda_0}}(\alpha))(y')$, and we obtain a contradiction again by Lemma 2.11. However, the possibility that $\rho' \geq \lambda_1$ is ruled out by a separation argument as at the beginning, making use of an appropriate instance of clause (4) from Definition 3.4 (with the current choice of $\alpha$ and $\alpha' < \alpha$ and with the nodes $x$ and $y'$).

We are left with the case when $\alpha^* < \alpha$ (and of course also $\alpha' < \alpha$). In this case we may assume that

- $\langle N_0^0, N_1^0\rangle$ is of the form

  $$\langle (\Psi_{N_0,N_1}(N_0^*), \pi^{\gamma_0^*,N_0^*}_{N_0,\gamma_0,N_1,\gamma_1}), (\Psi_{N_0,N_1}(N_1^*), \pi^{\gamma_1^*,N_1^*}_{N_0,\gamma_0,N_1,\gamma_1})\rangle,$$

  for edges $\langle (N_0,\gamma_0),(N_1,\gamma_1)\rangle \in \tau_{q^{11}_{\lambda_1}\restriction\alpha+1}$ and $\langle (N_0^*,\gamma_0^*),(N_1^*,\gamma_1^*)\rangle \in \tau_{q^{00}_{\lambda_0}\restriction\alpha+1} \cap N_0$ such that $\Psi_{N_1,N_0}(\alpha^*) = \alpha$ and $\rho < \delta_{N_0}$,
- $\alpha^* \in \mathrm{dom}(f_{q^{00}_{\lambda_0}})$, and
- $x \in \mathrm{dom}(f_{q^{00}_{\lambda_0}}(\alpha^*))$.

Indeed, if $\alpha^* \in \mathrm{dom}(f_{q^{11}_{\lambda_1}}) \setminus \mathrm{dom}(f_{q^{00}_{\lambda_0}})$, then we would have by adequacy of $q^{11}_{\lambda_1}$ that $\alpha \in \mathrm{dom}(f^{11}_{\lambda_1})$, and therefore also $\alpha^{**} := \Psi_{N_0,N_1}(\alpha^*) \in \mathrm{dom}(f_{q^{11}_{\lambda_1}})$, and $(f_{q^{11}_{\lambda_1}}(\alpha^{**})))(x) = (f_{q^{11}_{\lambda_1}}(\alpha^*))(x)$. But then $\alpha^{**}$, $\alpha'$, $x$ and $y'$, together with the sequences $(\langle (N_0^i,\gamma_0^i),(N_1^i,\gamma_1^i)\rangle)_{0<i\leq n}$ and $\vec{\mathcal{E}}'$ of length, respectively, $n-1$ and $n'$ (where $\vec{\mathcal{E}}' = \emptyset$ if $n' = 0$), would

witness the fact that $(q^{00}_{\lambda_0} \restriction \alpha+1) \oplus (q^{11}_{\lambda_1} \restriction \alpha+1)$ is not a condition, contradicting the minimality of $n+n'$.

By clause (3) in Definition 3.4 the situation displayed in the above bullet points is only possible if $\delta_{N_0} < \lambda_0$. But in this case

- $\langle (N_0^*, \gamma_0^*), (N_1^*, \gamma_1^*) \rangle \in \tau_{q^{11}_{\lambda_1} \restriction \alpha+1}$,
- $\alpha^* \in \mathrm{dom}(f_{q^{11}_{\lambda_1}})$, and
- $f_{q^{11}_{\lambda_1}}(\alpha^*) \restriction \lambda_1 \times \omega_1 = f_{q^{00}_{\lambda_1}}(\alpha^*) \restriction \lambda_0 \times \omega_1$.

It follows that also $\langle (N_0^0, \gamma_0^0), (N_1^0, \gamma_1^0) \rangle \in \tau_{q^{11}_{\lambda_1} \restriction \alpha+1}$, $\alpha^{**} := \Psi_{N_0^0, N_1^0}(\alpha^*) \in \mathrm{dom}(f_{q^{11}_{\lambda_1}})$, and $f_{q^{11}_{\lambda_1}}(\alpha^{**}) \restriction \delta_{N_0^0} \times \omega_1 = f_{q^{11}_{\lambda_1}}(\alpha^*) \restriction \delta_{N_0^0} \times \omega_1$. But then we have, as before, that $\alpha^{**}$, $\alpha'$, $x$ and $y'$, together with the sequences $(\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i) \rangle)_{0<i\leq n}$ and $\vec{\mathcal{E}}'$, witness the fact that $(q^{00}_{\lambda_0} \restriction \alpha+1) \oplus (q^{11}_{\lambda_1} \restriction \alpha+1)$ is not a condition, and again this contradicts the minimality of $n+n'$.

We have thus proved that $(q^{00}_{\lambda_0} \restriction \alpha+1) \oplus (q^{11}_{\lambda_1} \restriction \alpha+1)$ is a condition after all, which is the contradiction that finishes the proof of the claim. □

The following technical fact appears essentially in [10].

**Claim 3.7.** *Suppose $Z \in \mathcal{S}$, $(p^0_\lambda \mid \lambda \in Z) \in Q$ and $(p^1_\lambda \mid \lambda \in Z) \in Q$ are sequences of conditions in $\mathbb{Q}^*_\beta$ such that $p^0_\lambda \restriction M_\lambda$ and $p^1_\lambda \restriction M_\lambda$ are compatible conditions in $\mathbb{Q}^*_\beta \cap M_\lambda$ for every $\lambda \in Z$, and suppose that for every $\lambda \in Z$,*

- *$p^0_\lambda$ and $p^1_\lambda$ are $\lambda$-compatible with respect to $\varphi$ and $\alpha$ for all $\alpha \in \beta \cap Q$,*
- *$\alpha_\lambda \in \mathrm{dom}(f_{p^0_\lambda}) \cap M_\lambda$,*
- *$\alpha'_\lambda \in \mathrm{dom}(f_{p^1_\lambda}) \cap M_\lambda$ is a nonzero ordinal such that $\alpha'_\lambda \leq \alpha_\lambda$, and*
- *$x_\lambda = (\rho^0_\lambda, \zeta^0_\lambda)$ and $y_\lambda = (\rho^1_\lambda, \zeta^1_\lambda)$ are nodes in $(\kappa \setminus \lambda) \times \omega_1$ such that $x_\lambda \in \mathrm{dom}(f_{p^0_\lambda}(\alpha_\lambda))$ and $y_\lambda \in \mathrm{dom}(f_{p^1_\lambda}(\alpha'_\lambda))$.*

*Then there is $D \in \mathcal{F}$, together with two sequences $(p^2_\lambda \mid \lambda \in Z \cap D)$, $(p^3_\lambda \mid \lambda \in Z \cap D)$ of conditions in $\mathbb{Q}^*_\beta$ such that*

1. *for each $\lambda \in Z \cap D$, $p^2_\lambda \leq q^0_\lambda$ and $p^3_\lambda \leq p^1_\lambda$,*
2. *for each $\lambda \in Z \cap D$, $p^2_\lambda \restriction M_\lambda$ and $p^3_\lambda \restriction M_\lambda$ are compatible in $\mathbb{Q}^*_\beta \cap M_\lambda$, and*
3. *for each $\lambda \in Z \cap D$, $x_\lambda$ and $y_\lambda$ are separated below $\lambda$ at stages $\alpha_\lambda$ and $\alpha'_\lambda$ by $p^2_\lambda \restriction \alpha_\lambda$ and $p^3_\lambda \restriction \alpha'_\lambda$.*

*Proof.* Let $B \subseteq V_\kappa$ code $\varphi$, $(\mathbb{Q}^*_\alpha)_{\alpha \in (\beta+1) \cap Q}$, the collection of maximal antichains of $\mathbb{Q}^*_\alpha$, for $\alpha \in \beta \cap Q$, and $(\underset{\sim}{T}_\alpha)_{\alpha \in \beta}$. By a reflection argument with a suitable $\Pi^1_1$ sentence over the structure $(V_\kappa, \in, B)$, together with the fact that $\mathbb{Q}^*_\alpha$ has the $\kappa$-c.c. for every $\alpha \in \beta \cap Q$, there is a set $D \in \mathcal{F}$ consisting of inaccessible cardinals $\lambda < \kappa$ for which $M_\lambda$ is a model such that $M_\lambda \cap \kappa = \lambda$, $M_\lambda$ is closed under $<\lambda$-sequences, and such that for every $\alpha \in M_\lambda \cap \beta$,

- $\mathbb{Q}^*_\alpha \cap M_\lambda$ forces, over $V$, that $\underset{\sim}{T}_\alpha \cap M_\lambda$ has no $\lambda$-branches,
- $\mathbb{Q}^*_\alpha \cap M_\lambda$ has the $\lambda$-c.c., and
- $\mathbb{Q}^*_\alpha \cap M_\lambda \lessdot \mathbb{Q}^*_\alpha$

Fix $\lambda \in Z \cap D$. Thanks to Lemma 2.7, it suffices to show that there are extensions $p^2_\lambda$ and $p^3_\lambda$ of $p^0_\lambda \restriction \alpha_\lambda$ and $p^1_\lambda \restriction \alpha_\lambda$, respectively, such that $p^2_\lambda \restriction M_\lambda$ and $p^3_\lambda \restriction M_\lambda$ are compatible in $\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda$, and such that $x_\lambda$ and $y_\lambda$ are separated below $\lambda$ at stages $\alpha_\lambda$ and $\alpha'_\lambda$ by $p^2_\lambda \restriction \alpha_\lambda$ and $p^3_\lambda \restriction \alpha'_\lambda$. By the above bullet points, we may view $\mathbb{Q}^*_{\alpha_\lambda}$ as a two-step forcing iteration $(\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda) * \underset{\sim}{\mathbb{S}}$. By $\lambda$-compatibility we may then identify $p^0_\lambda \restriction \alpha_\lambda$ and $p^1_\lambda \restriction \alpha_\lambda$ with, respectively, $\langle r^0, \underset{\sim}{s}^0\rangle$ and $\langle r^1, \underset{\sim}{s}^1\rangle$, both in $(\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda) * \underset{\sim}{\mathbb{S}}$.

Note that $r^0$ and $r^1$ are compatible in $\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda$. Working in an $(\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda)$-generic extension $V[G]$ of $V$ containing $r^0$ and $r^1$, we note that there have to be

- extensions $\langle r^{00}, \underset{\sim}{s}^{00}\rangle$ and $\langle r^{01}, \underset{\sim}{s}^{01}\rangle$ of $\langle r^0, \underset{\sim}{s}^0\rangle$ and
- an extension $\langle r^3, \underset{\sim}{s}^3\rangle$ of $\langle r^1, \underset{\sim}{s}^1\rangle$

such that $r^{00}$, $r^{01}$ and $r^3$ are all in $G$, together with some $\bar\rho < \lambda$ for which there are pairs $\zeta^{00} \neq \zeta^{01}$ of ordinals in $\omega_1$ and there is $\zeta^3 \in \omega_1$ such that, identifying $\underset{\sim}{T}_{\alpha_\lambda}$ and $\underset{\sim}{T}_{\alpha'_\lambda}$ with $(\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda) * \underset{\sim}{\mathbb{S}}$-names, we have the following.

- $\langle r^{00}, \underset{\sim}{s}^{00}\rangle$ forces that $(\bar\rho, \zeta^{00})$ is below $x_\lambda$ in $\underset{\sim}{T}_{\alpha_\lambda}$.
- $\langle r^{01}, \underset{\sim}{s}^{01}\rangle$ forces that $(\bar\rho, \zeta^{01})$ is below $x_\lambda$ in $\underset{\sim}{T}_{\alpha_\lambda}$.
- $\langle r^3, \underset{\sim}{s}^3\rangle \restriction \alpha'_\lambda$ forces that $(\bar\rho, \zeta^3)$ is below $y_\lambda$ in $\underset{\sim}{T}_{\alpha'_\lambda}$.

Indeed, any condition $\langle r, \underset{\sim}{s}\rangle$ in $(\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda) * \underset{\sim}{\mathbb{S}}$ such that $r \in G$ can be extended, for any $\bar\rho < \lambda$, to a condition $\langle r^+, \underset{\sim}{s}^+\rangle$ deciding some node $(\bar\rho, \zeta)$ to be below $x_\lambda$ in $\underset{\sim}{T}_{\alpha_\lambda}$ and such that $r^+ \in G$, and similarly with $y_\lambda$ and $\underset{\sim}{T}_{\alpha'_\lambda}$ in place of $x_\lambda$ and $\underset{\sim}{T}_{\alpha_\lambda}$. Hence, if the above were to fail, then there would be some $\rho^* < \lambda$ with the following property.

- For every $\bar\rho < \lambda$ above $\rho^*$ there is exactly one $\zeta < \omega_1$ such that some condition $\langle r, \underset{\sim}{s}\rangle$ in $(\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda) * \underset{\sim}{\mathbb{S}}$ with $r \in G$ forces that $(\bar\rho, \zeta)$ is below $x_\lambda$ in $\underset{\sim}{T}_{\alpha_\lambda}$.

It would then follow that $\underset{\sim}{T}_{\alpha_\lambda}$ has a $\lambda$-branch in $V[G]$, which is a contradiction.

Let $\zeta^3 < \omega_1$ be such that some condition $\langle r^3, \underset{\sim}{s}^3\rangle$ extending $\langle r^1, \underset{\sim}{s}^1\rangle$ is such that $\langle r^3, \underset{\sim}{s}^3\rangle \restriction \alpha'_\lambda$ forces $(\bar\rho, \zeta^3)$ to be below $y_\lambda$ in $\underset{\sim}{T}_{\alpha'_\lambda}$. But now, given conditions $\langle r^{0i}, \underset{\sim}{s}^{0i}\rangle$ as above (for $i \in \{0, 1\}$) there must be

$i \in \{0,1\}$ such that $\zeta^{0i} \neq \zeta^3$. We may then set $p^2_\lambda = \langle r^{0i}, \underset{\sim}{s}^{0i}\rangle$ and $p^3_\lambda = \langle r^3, \underset{\sim}{s}^3\rangle$. □

By Claim 3.5, in order to conclude the proof of the current instance of $(1)_\beta$, it suffices to prove the following.

**Claim 3.8.** *There is a separating pair for $\sigma^0$ and $\sigma^1$.*

*Proof.* This follows from first applying Lemma 2.12, Claim 3.7, and $(2)_\alpha$, for $\alpha < \beta$, countably many times, using the normality of $\mathcal{F}$, and then running a pressing–down argument again using the normality of $\mathcal{F}$.

To be more specific, we start by building sequences

$$\sigma^0_n = (q^0_{\lambda,n} \mid \lambda \in X \cap D_n)$$

and

$$\sigma^1_n = (q^1_{\lambda,n} \mid \lambda \in X \cap D_n),$$

for a $\subseteq$-decreasing sequence $(D_n)_{n<\omega}$ of sets in $\mathcal{F}$, such that $\sigma^0_0 = \sigma^0$ and $\sigma^1_0 = \sigma^1$, and such that for every $n < \omega$, $\sigma^0_{n+1}$ and $\sigma^1_{n+1}$ are obtained from $\sigma^0_n$ and $\sigma^1_n$ in the following way.

We first let

$$\sigma^0_{n,+} = (q^0_{\lambda,n,+} \mid \lambda \in X \cap D_n)$$

and

$$\sigma^1_{n,+} = (q^1_{\lambda,n,+} \mid \lambda \in X \cap D_n)$$

be sequences of adequate $\mathbb{Q}^*_\beta$-conditions, $\lambda$-compatible with respect to $\varphi$ and $\alpha$, for $\alpha \in \beta \cap Q$, such that $q^0_{\lambda,n,+} \leq_{\mathbb{Q}_\beta} q^0_{\lambda,n}$ and $q^0_{\lambda,n,+} \leq_{\mathbb{Q}_\beta} q^1_{\lambda,n}$ for all $\lambda \in X \cap D_n$. Given $\lambda$, $q^0_{\lambda,n,+}$ and $q^0_{\lambda,n,+}$ can be found by a simple construction in countably many steps, along which we

- apply Lemma 2.12, and
- apply $(2)_\alpha$, for some $\alpha \in \beta \cap Q$.

Now we find $D_{n+1}$ and $\sigma^0_{n+1}$, $\sigma^1_{n+1}$ by an application of Claim 3.7 to $\sigma^0_{n,+}$ and $\sigma^1_{n,+}$ with a suitable sequence $\alpha_\lambda$, $\alpha'_\lambda$, $x_\lambda$, $y_\lambda$ (for $\lambda \in X \cap D_n$).

By a standard book-keeping argument we can ensure that all relevant objects have been chosen in such a way that in the end, letting $q^{00}_\lambda$ and $q^{11}_\lambda$ be the greatest lower bound of, respectively, $(q^0_{\lambda,n})_{n<\omega}$ and $(q^1_{\lambda,n})_{n<\omega}$, for $\lambda \in X \cap \bigcap_n D_n$, $(q^{00}_\lambda \mid \lambda \in X \cap \bigcap_n D_n)$ and $(q^{11}_\lambda \mid \lambda \in X \cap \bigcap_n D_n)$ satisfy clause (2) in Definition 3.4.

Finally, by the normality of $\mathcal{F}$, we may find $Y \in \mathcal{S}$, $Y \subseteq X \cap \bigcap_n D_n$, such that $\sigma^{00} = (q^{00}_\lambda \mid \lambda \in Y)$ and $\sigma^{11} = (q^{11}_\lambda \mid \lambda \in Y)$ satisfy clauses (3) and (4) in Definition 3.4. □

We are left with proving $(2)_\beta$. This is established with the same argument as in the corresponding proof in [10]. The case when $\beta$ is a limit ordinal follows from the induction hypothesis, using the normality of $\mathcal{F}$ (cf. the proof in [10]). Hence, we may assume that $\beta$ is a successor ordinal, $\beta = \beta_0 + 1$. Suppose $D \in \mathcal{F}$, $Q$ is a suitable model such that $\beta, D \in Q$, $\varphi : \kappa \to Q$ is a bijection, and $(q^0_\lambda \mid \lambda \in D) \in Q$ and $(q^1_\lambda \mid \lambda \in D) \in Q$ are sequences of $\mathbb{Q}_\beta$-conditions, which we may assume are such that $q^0_\lambda \restriction M_\lambda = q^1_\lambda \restriction M_\lambda$ for each $\lambda \in D$. We want to show that there is some $D' \in \mathcal{F}$, $D' \subseteq D$, with the property that for every $\lambda \in D'$ there are conditions $q'^0_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ and $q'^1_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$ such that

(1) $q'^0_\lambda \restriction M^\varphi_\lambda = q'^1_\lambda \restriction M^\varphi_\lambda$ and
(2) $q'^0_\lambda$ and $q'^1_\lambda$ are both $\lambda$-compatible with respect to $\varphi$ and $\beta$.

Assuming the above fails, there is some $X \in \mathcal{S}$, $X \subseteq D$, with the property that, given $\lambda \in X$, there are no conditions $q'^0_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ and $q'^1_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$ such that (1) and (2) above holds.

Thanks to the induction hypothesis applied to $\beta_0$ we may assume, after shrinking $X$ to some $Y \in \mathcal{S}$ and extending the corresponding conditions if necessary, that for each $\lambda \in Y$,

- $q^0_\lambda \restriction \beta_0$ and $q^1_\lambda \restriction \beta_0$ are both $\lambda$-compatible with respect to $\varphi$ and $\beta_0$, and
- $q^0_\lambda \oplus q^1_{\lambda^*}$ is a condition for each $\lambda^* \in Y$, $\lambda^* > \lambda$.

Since the desired conclusion fails we may then assume, after shrinking $Y$ if necessary, that for each $\lambda \in Y$ there is a maximal antichain $A_\lambda$ of $\mathbb{Q}_\beta \cap M^\varphi_\lambda$ below $q^0_\lambda \restriction M^\varphi_\lambda$ consisting of conditions $r$ such that at least one of the following statements holds.

$\theta_{r,0,\lambda}$: $r$ is incompatible with $q^0_\lambda$.
$\theta_{r,1,\lambda}$: $r$ is incompatible with $q^1_\lambda$.

By the definition of $\mathcal{F}$ coupled with a suitable $\Pi^1_1$-reflection argument, we may further assume that each $A_\lambda$ is in fact a maximal antichain of $\mathbb{Q}_\beta$ below $q^0_\lambda \restriction M^\varphi_\lambda$ and that it has cardinality less than $\lambda$ (cf. the proof of Claim 3.7). Hence, after shrinking $Y$ yet another time using the normality of $\mathcal{F}$, we may assume, for all $\lambda < \lambda^*$ in $Y$, that

- $A_\lambda = A_{\lambda^*}$ and that
- for every $r \in A_\lambda$, $\theta_{r,0,\lambda}$ holds if and only if $\theta_{r,0,\lambda^*}$ does, and $\theta_{r,1,\lambda}$ holds if and only if $\theta_{r,1,\lambda^*}$ does.

Let us now fix any $\lambda < \lambda^*$ in $Y$. Since $A_\lambda$ is a maximal antichain of $\mathbb{Q}_\beta$ below $q^0_\lambda \restriction M^\varphi_\lambda$, we may find some $r \in A_\lambda$ compatible with $q^0_\lambda \oplus q^1_{\lambda^*}$. We have that $\theta_{r,0,\lambda}$ cannot hold since $q^0_\lambda \oplus q^1_{\lambda^*}$ extends $q^0_\lambda$. Therefore $\theta_{r,1,\lambda}$ holds, and hence also $\theta_{r,1,\lambda^*}$ does. But that is also a contradiction since $q^0_\lambda \oplus q^1_{\lambda^*}$ extends $q^1_{\lambda^*}$.

This contradiction concludes the proof of $(2)_\beta$, and hence the proof of the lemma. $\square$

## 4. Completing the proof of Theorem 1.2

In this final section we conclude the proof of Theorem 1.2. By Lemma 2.8, $\mathbb{Q}_{\kappa^+}$ does not add new $\omega$-sequences of ordinals and hence it preserves CH. We will start this section by proving that $\mathbb{Q}_{\kappa^+}$ also preserves $2^{\aleph_1} = \aleph_2$.

**Lemma 4.1.** $\Vdash_{\mathbb{Q}_{\kappa^+}} 2^{\aleph_1} = \kappa$

*Proof.* Suppose, towards a contradiction, that there is a condition $q \in \mathbb{Q}_{\kappa^+}$ and a sequence $(\underset{\sim}{r}_i)_{i<\kappa^+}$ of $\mathbb{Q}_{\kappa^+}$-names for subsets of $\omega_1$ such that

$$q \Vdash_{\mathbb{Q}_{\kappa^+}} \underset{\sim}{r}_i \neq \underset{\sim}{r}_{i'} \text{ for all } i < i' < \kappa^+$$

By Lemma 3.1 we may assume, for each $i$, that $\underset{\sim}{r}_i \in H(\kappa^+)$ and $\underset{\sim}{r}_i$ is a $\mathbb{Q}_{\beta_i}$-name for some $\beta_i < \kappa^+$.

Let $\theta$ be a large enough regular cardinal. For each $i < \kappa^+$ let $N_i^* \preceq H(\theta)$ be such that

(1) $|N_i^*| = |N_i^* \cap \kappa|$,
(2) $N_i^*$ is closed under sequences of length less than $|N_i^*|$,
(3) $q$, $\underset{\sim}{r}_i$, $\beta_i$, $(\Phi_\alpha)_{\alpha<\kappa^+}$, $(\mathbb{Q}_\alpha)_{\alpha<\kappa^+} \in N_i^*$, and
(4) $\mathbb{Q}_\alpha \cap N_i^* \lessdot \mathbb{Q}_\alpha$ for every $\alpha \in \kappa^+ \cap N_i^*$.

$N_i^*$ can be found by a $\Pi^1_1$-reflection argument, using the weak compactness of $\kappa$ and the $\kappa$-chain condition of each $\mathbb{Q}_\alpha$, as in the proof of Claim 3.7. Let $N_i = N_i^* \cap H(\kappa^+)$ for each $i$.

Let now $P$ be the satisfaction predicate for the structure

$$\langle H(\kappa^+), \in, \vec{\Phi}\rangle,$$

where $\vec{\Phi} \subseteq H(\kappa^+)$ codes $(\Phi_\alpha)_{\alpha<\kappa^+}$ in some canonical way, and let $M$ be an elementary submodel of $H(\theta)$ containing $q$, $\underset{\sim}{r}_i$, $(\beta_i)_{i<\kappa^+}$, $(\mathbb{Q}_\alpha)_{\alpha\leq\kappa^+}$, $(N_i^*)_{i<\kappa^+}$ and $P$, and such that $|M| = \kappa$ and ${}^{<\kappa}M \subseteq M$.

Let $i_0 \in \kappa^+ \setminus M$. By a standard reflection argument we may find $i_1 \in \kappa^+ \cap M$ for which there exists an isomorphism

$$\Psi : (N_{i_0}, \in, P, \underset{\sim}{r}_{i_0}, \beta_{i_0}, q) \cong (N_{i_1}, \in, P, \underset{\sim}{r}_{i_1}, \beta_{i_1}, q),$$

such that $\Psi(\xi) \leq \xi$ for every ordinal in $N_{i_0}$. Indeed, the existence of such an $i_1$ follows from the correctness of $M$ in $H(\theta)$ about a suitable statement with parameters $(N_i)_{i<\kappa^+}$, $q$, $P$, $(\beta_i)_{i<\kappa^+}$, $(\underset{\sim}{r}_i)_{i<\kappa^+}$, and $N_{i_0} \cap M$, all of which are in $M$. Let $\bar{q} = (f_q, \tau_{\bar{q}})$, where

$$\tau_{\bar{q}} = \tau_q \cup \{\langle (N_{i_0}, \beta_{i_0} + 1), (N_{i_1}, \beta_{i_1} + 1)\rangle\}$$

Thanks the choice of $N^*_{i_0}$ and $N^*_{i_1}$, together with Lemma 2.3, it is then easy to see that $\bar{q} \in \mathbb{Q}_{\kappa^+}$. We show that $\bar{q} \Vdash_{\mathbb{Q}_{\kappa^+}} \underaccent{\sim}{r}_{i_0} = \underaccent{\sim}{r}_{i_1}$.

Suppose not, and we will derive a contradiction. Thus we can find $\nu < \omega_1$ and $q' \leq_{\kappa^+} \bar{q}$ such that

$$q' \Vdash_{\mathbb{Q}_{\kappa^+}} \text{“}\nu \in \underaccent{\sim}{r}_{i_0} \iff \nu \notin \underaccent{\sim}{r}_{i_1}\text{”}.$$

Let us assume, for concreteness, that $q' \Vdash_{\mathbb{Q}_{\kappa^+}}$ “$\nu \in \underaccent{\sim}{r}_{i_0}$ and $\nu \notin \underaccent{\sim}{r}_{i_1}$” (the proof in the case that $q' \Vdash_{\mathbb{Q}_{\kappa^+}}$ “$\nu \in \underaccent{\sim}{r}_{i_1}$ and $\nu \notin \underaccent{\sim}{r}_{i_0}$” is exactly the same). By correctness of $N^*_{i_0}$ we have that this model contains a maximal antichain $A$ of conditions in $\mathbb{Q}_{\beta_{i_0}}$ deciding the statement “$\nu \in \underaccent{\sim}{r}_{i_0}$”. By Lemma 3.1 we know that $|A| < \kappa$ and hence, since $N^*_{i_0} \cap \kappa \in \kappa$, $A \subseteq N^*_{i_0} \cap H(\kappa^+) = N_{i_0}$ (cf. the proof of Lemma 2.11). Hence, we may find a common extension $q''$ of $q'$ and some $r \in N_{i_0} \cap A$ such that $r \Vdash_{\mathbb{Q}_{\kappa^+}}$ “$\nu \in \underaccent{\sim}{r}_{i_0}$”.

Also, note that, since $\Psi$ is an isomorphism between the structures $(N_{i_0}, \in, P, \underaccent{\sim}{r}_{i_0}, \beta_{i_0}, q)$ and $(N_{i_1}, \in, P, \underaccent{\sim}{r}_{i_1}, \beta_{i_1}, q)$, and by the choice of $P$, we have that

$$\Psi(r) \Vdash_{\mathbb{Q}_{\beta_{i_1}}} \text{“}\nu \in \Psi(\underaccent{\sim}{r}_{i_0}) = \underaccent{\sim}{r}_{i_1}\text{”}$$

But then, by clause (3) in the definition of condition, together with the fact that $(N_{i_1}, \beta_{i_1+1})$ is a projection of $(N_{i_0}, \beta_{i_0+1})$, we have that $q'' \leq \Psi(r)$. We thus obtain that $q'' \Vdash_{\mathbb{Q}_{\kappa^+}}$ “$\nu \in \underaccent{\sim}{r}_{i_1}$”, which is impossible as $q' \Vdash_{\mathbb{Q}_{\kappa^+}}$ “$\nu \notin \underaccent{\sim}{r}_{i_1}$” and $q'' \leq q'$.

We get a contradiction and the lemma follows.[14] □

**Corollary 4.2.** $\mathbb{Q}_{\kappa^+}$ *forces GCH.*

Lemma 4.3, which completes the proof of Theorem 1.2, follows immediately from earlier lemmas, together with a standard density argument.

**Lemma 4.3.** $\mathbb{Q}_{\kappa^+}$ *forces* $\mathrm{SATP}_{\aleph_2}$.

*Proof.* Let $G$ be $\mathbb{Q}_{\kappa^+}$-generic over $V$. Since CH holds in $V[G]$, there are $\aleph_2$-Aronszajn trees there. Hence, it suffices to prove that, in $V[G]$, every $\aleph_2$-Aronszajn tree is special.

Let $T \in V[G]$ be an $\aleph_2$-Aronszajn tree. Note that $\aleph_2 = \kappa$ in $V[G]$ by Lemmas 2.5 and 3.1. We need to prove that $T$ is special in $V[G]$. Let us go down to $V$ and let us note there that, by the $\kappa$-chain condition of $\mathbb{Q}_{\kappa^+}$ together with the choice of $\Phi$, we may find some $\alpha < \kappa^+$ such that $\Phi(\alpha)$ is a $\mathbb{Q}_\alpha$-name for an $\aleph_2$-Aronszajn tree such that $\Phi(\alpha)_G = T$. We then have that $\underaccent{\sim}{T}_\alpha = \Phi(\alpha)$.

[14] Note the resemblance of this proof with the proof of Lemma 2.11.

Let

$$f = \bigcup \{ f_q(\alpha) \mid q \in G, \, \alpha \in \mathrm{dom}(f_q) \}$$

By the definition of the forcing, we have that $f$ is a (partial) specializing function for $T$. Also, given any adequate condition $q \in \mathbb{Q}_{\kappa^+}$, which we may assume is such that $\alpha \in \mathrm{dom}(f_q)$, and any node $x \in \kappa \times \omega_1$ such that $x \notin \mathrm{dom}(f_q(\alpha))$, it is easy to see that we may extend $q$ to a condition $q^*$ such that $x \in \mathrm{dom}(f_{q^*}(\alpha))$; indeed, it suffices for this to pick any $i^* < \omega_1$ such that $i^* \notin \bigcup_{\alpha' \in \mathrm{dom}(f_q)} \mathrm{range}(f_q(\alpha'))$, which of course is possible since $\bigcup_{\alpha' \in \mathrm{dom}(f_q)} \mathrm{range}(f_q(\alpha'))$ is countable, extend $f_q$ to a function $f$ such that $x \in \mathrm{dom}(f(\alpha))$ and $(f(\alpha))(x) = i^*$, and close under the relevant isomorphisms $\Psi_{N_0,N_1}$.[15] The above density argument shows that $F$ is defined everywhere on $\kappa \times \omega_1$. It follows that $T$ is special in $V[G]$, which concludes the proof. □

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