SPECIALIZING TREES AND ANSWER TO A QUESTION OF WILLIAMS

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Abstract. We show that if $cf(2^{\aleph_0}) = \aleph_1$, then any non-trivial $\aleph_1$-closed forcing notion of size $\leq 2^{\aleph_0}$ is forcing equivalent to $\mathrm{Add}(\aleph_1, 1)$, the Cohen forcing for adding a new Cohen subset of ω_1. We also produce, relative to the existence of some large cardinals, a model of ZFC in which $2^{\aleph_0} = \aleph_2$ and all $\aleph_1$-closed forcing notion of size $\leq 2^{\aleph_0}$ collapse $\aleph_2$, and hence are forcing equivalent to $\mathrm{Add}(\aleph_1, 1)$. Our results answer a question of Scott Williams from 1978. We also extend a result of Todorcevic and Foreman-Magidor-Shelah by showing that it is consistent that every partial order which adds a new subset of $\aleph_2$, collapses $\aleph_2$ or $\aleph_3$.

1. introduction

For an infinite cardinal κ let $\mathrm{Add}(\kappa, 1)$ denote the Cohen forcing for adding a new Cohen subset of κ; thus conditions are partial functions $p : \kappa \to \{0, 1\}$ of size less than κ ordered by reverse inclusion. The forcing is $cf(\kappa)$-closed and satisfies the $(2^{<\kappa})^+$-c.c., in particular if κ is regular and $2^{<\kappa} = \kappa$, then it preserves all cardinals.

It is well-known that if the continuum hypothesis holds, then any $\aleph_1$-closed forcing notion of size continuum is forcing equivalent to $\mathrm{Add}(\aleph_1, 1)$. In [16] (see also [17]), Scott Williams asked if the converse is also true, i.e., if CH follows from the assumption "any $\aleph_1$-closed forcing notion of size continuum is forcing equivalent to the Cohen forcing $\mathrm{Add}(\aleph_1, 1)$". We give a negative answer to his question; in fact we will prove the following stronger result.

Theorem 1.1. *Assume $cf(2^{\aleph_0}) = \aleph_1$. Then any non-trivial $\aleph_1$-closed forcing notion of size $\leq 2^{\aleph_0}$ is forcing equivalent to* $\mathrm{Add}(\aleph_1, 1)$.

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Remark 1.2. (1) *We can replace* $\aleph_1, 2^{\aleph_0}$ *by* $\kappa = \mu^+, 2^\mu$ *resp., with* $cf(2^\mu) = \kappa$; *or by* $\kappa, 2^\mu$ *resp., if* κ *is weakly inaccessible,* $\mu < \kappa$, $2^\mu = 2^{<\kappa}$ *and* $cf(2^\mu) = \kappa$.

(2) *If* $2^{\aleph_0} = 2^{\aleph_1}$, *then* $\mathrm{Add}(\aleph_2, 1)$ *is* $\aleph_1$*-closed of size continuum, but it is not forcing equivalent to* $\mathrm{Add}(\aleph_1, 1)$.

On the other hand, it is not difficult to prove the consistency of "$2^{\aleph_0} = \aleph_2$ and there exists a non-trivial $\aleph_1$-closed (but not $\aleph_2$-closed) forcing notion of size $\aleph_2$ which preserves all cardinals" (see [7] for this and generalizations).

So it is natural to ask if we can have the same result as in Theorem 1.1 with $2^{\aleph_0}$ being regular. We show that this is indeed the case, if we assume the existence of some large cardinals.

Theorem 1.3. *Assume* κ *is weakly compact and* $\lambda > \kappa$ *is a* 2*-Mahlo cardinal. Then there is a generic extension of the universe in which the following hold:*

(*a*) $2^{\aleph_0} = \kappa = \aleph_2$,

(*b*) $2^{\aleph_1} = \lambda = \aleph_3$,

(*c*) *Any* $\aleph_1$*-closed forcing notion of size* $\leq \aleph_2$ *collapses* $\aleph_2$ *into* $\aleph_1$, *in particular it is forcing equivalent to* $\mathrm{Add}(\aleph_1, 1)$.

Following [4], let *Todorcevic's maximality principle* be the assertion: "every partial order which adds a new subset of $\aleph_1$, collapses $\aleph_1$ or $\aleph_2$", where by a new subset of $\aleph_1$ we mean a subset of $\aleph_1$ which is not in the ground model but all of its proper initial segments are in the ground model.

In [15], Todorcevic showed that if $2^{\aleph_0} = \aleph_2$ and every $\aleph_1$-tree of size $\aleph_1$ is special, then Todorcevic's maximality principle holds.

By results of Baumgartner [2] and Todorcevic [14], "$2^{\aleph_0} = \aleph_2 +$ every $\aleph_1$-tree of size $\aleph_1$ is special" is consistent, and hence Todorcevic's maximality principle is consistent as well. On the other hand, Foreman-Magidor-Shelah [10] proved that PFA implies the same conclusion. In [3], Cox and Krueger introduced the new principles GMP (guessing model principle) and IGMP (indestructible guessing model principle) and showed that PFA implies both of them and that the IGMP implies Todorcevic's maximality principle. On the other hand, in [4], they showed that Todorcevic's maximality principle does not follow from GMP.

We show that a simple modification of the proof of Theorem 1.3, yields the following, which extends the above result of Todorcevic [15] to higher cardinals.

Theorem 1.4. *Assume κ is weakly compact and $\lambda > \kappa$ is a 2-Mahlo cardinal. Then there is a generic extension of the universe in which the following hold:*

(*a*) $2^{\aleph_0} = \aleph_1$,

(*b*) $\kappa = \aleph_2$,

(*c*) $2^{\aleph_1} = \lambda = \aleph_3$,

(*d*) *Every partial order which adds a new subset of* $\aleph_2$, *collapses* $\aleph_2$ *or* $\aleph_3$.

Remark 1.5. *In theorems 1.3 and 1.4, we can replace the cardinals $\aleph_0, \aleph_1$ and $\aleph_2$ by the cardinals η, η^+ and η^{++} respectively, where η is a regular cardinal less than κ.*

The above result is also connected to Foreman's maximality principle [9], which asserts that any non-trivial forcing notion either adds a new real or collapses some cardinals. See [7] for more on this.

The structure of the paper is as follows. In Section 2 we prove Theorem 1.1. In sections 3 and 4 we present some results that will be used in section 5 for the proof of Theorem 1.3. Finally in section 6 we complete the proof of Theorem 1.4.

To avoid trivialities, by a forcing notion we always mean a non-trivial separative forcing notion. We use $\simeq$ for the equivalence of forcing notions, so

$$\mathbb{P} \simeq \mathbb{Q} \ \Leftrightarrow \ RO(\mathbb{P}) \text{ is isomorphic to } RO(\mathbb{Q}),$$

where $RO(\mathbb{P})$ denotes the Boolean completion of $\mathbb{P}$. Also $\mathbb{P} \lessdot \mathbb{Q}$ means that $\mathbb{P}$ is a regular sub-forcing of $\mathbb{Q}$.

2. A negative answer to Williams question when the continuum is singular

In this section we prove Theorem 1.1. In [7] it is shown that if $\mathbb{Q}$ is any $\aleph_1$-closed forcing notion [1] of size $\leq 2^{\aleph_0}$ and if λ is the least cardinal such that forcing with $\mathbb{Q}$ adds a new λ-sequence of ordinals, then forcing with $\mathbb{Q}$ collapses $2^{\aleph_0}$ into λ; if in addition $\lambda = \aleph_1$, then $\mathbb{Q} \simeq \mathrm{Add}(\aleph_1, 1)$. Thus to prove Theorem 1.1, it is sufficient to show that if $cf(2^{\aleph_0}) = \aleph_1$,

[1] In fact being $\omega + 1$-strategically closed is sufficient

then forcing with any $\aleph_1$-closed forcing notion $\mathbb{Q}$ adds a new set of ordinals of size $\aleph_1$. The proof presented here avoids the use of the results from [7] and is more direct.

If $2^{\aleph_0} = \aleph_1$, then the result is known, so we assume that $\aleph_1 < 2^{\aleph_0}$ and $cf(2^{\aleph_0}) = \aleph_1$. Let $\mathbb{Q}$ be a non-trivial $\aleph_1$-closed forcing notion of size $\leq 2^{\aleph_0}$. We are going to show that $\mathbb{Q}$ is forcing equivalent to $\mathrm{Add}(\aleph_1, 1)$.

Notation 2.1. *For a forcing notion $\mathbb{P}$ and a condition $p \in \mathbb{P}$, let $\mathbb{P} \downarrow p$ denote the set of all conditions in $\mathbb{P}$ which extend p; so $\mathbb{P} \downarrow p = \{q \in \mathbb{P} : q \leq_{\mathbb{P}} p\}$.*

As $cf(2^{\aleph_0}) = \aleph_1$, we can find a sequence $\langle \mathbb{Q}_i : i < \omega_1 \rangle$ of subsets of $\mathbb{Q}$ with union $\mathbb{Q}$ which is $\subseteq$-increasing and continuous, so that $\mathbb{Q}_0 = \emptyset$ and for all $i < \omega_1, |\mathbb{Q}_i| < 2^{\aleph_0}$.

Lemma 2.2. *For every $i < \omega_1$ and every $p \in \mathbb{Q}$, there exists $q \leq_{\mathbb{Q}} p$ such that there is no $r \in \mathbb{Q}_i$ with $r \leq q$. Moreover for every $r \in \mathbb{Q}_i, r \nVdash$ "$q \in \dot{G}_{\mathbb{Q}}$".*

Proof. Let A be a maximal antichain in $\mathbb{Q}$ below p of size $2^{\aleph_0}$, which exists as $\mathbb{Q}$ is non-trivial and $\aleph_1$-closed. As $|\mathbb{Q}_i| < 2^{\aleph_0}$, we can find $q \in A$ such that $\mathbb{Q} \downarrow q \cap \mathbb{Q}_i = \emptyset$. Then q is easily seen to be as required. □

We now define by induction on $i < \omega_1$ a sequence $\bar{p}_i$ such that:

(1) $\bar{p}_i = \langle p_\eta : \eta \in^{i+1} (2^{\aleph_0}) \rangle$ is a maximal antichain in $\mathbb{Q}$,
(2) If $j < i$ and $\eta \in^{i+1} (2^{\aleph_0})$, then $p_\eta \leq_{\mathbb{Q}} p_{\eta \restriction (j+1)}$,
(3) If $\eta \in^{i+1} (2^{\aleph_0})$, then there is no member of $\mathbb{Q}_i$ which is below p_η. Moreover for each $r \in \mathbb{Q}_i, r \nVdash$"$p_\eta \in \dot{G}_{\mathbb{Q}}$".

$\underline{i = 0}$: Then take $\bar{p}_0 = \langle p_\eta : \eta \in (2^{\aleph_0}) \rangle$ be any maximal antichain in $\mathbb{Q}$. Note that clauses (2) and (3) above are vacuous as $\mathbb{Q}_0$ is empty.

$\underline{i > 0}$: For every $\eta \in^i (2^{\aleph_0})$ let $\bar{p}^1_{i,\eta} = \langle p_{\eta \restriction (j+1)} : j < i \rangle$. Note that $\bar{p}^1_{i,\eta}$ is a countable decreasing sequence of conditions in $\mathbb{Q}$, and so if we set

$$\mathbb{P}^2_{i,\eta} = \{q \in \mathbb{Q} : j < i \Rightarrow q \leq p_{\eta \restriction (j+1)}\},$$

then $\mathbb{P}^2_{i,\eta}$ is non-empty. Let

$$\mathbb{P}^3_{i,\eta} = \{q \in \mathbb{P}^2_{i,\eta} : \forall z \in \mathbb{Q}_i [\neg z \leq_{\mathbb{Q}} q \text{ and moreover } z \nVdash \text{"} q \in \dot{G}_{\mathbb{Q}} \text{"}]\}.$$

$\mathbb{P}^3_{i,\eta}$ is easily seen to be a dense subset of $\mathbb{P}^2_{i,\eta}$, hence we can find a maximal antichain, say $\bar{\mathbb{P}}_{i,\eta} = \{p_{i,\eta}(\alpha) : \alpha < 2^{\aleph_0}\}$, in it. For $\eta \in^{i+1}(2^{\aleph_0})$ set $p_\eta = p_{i,\eta\restriction i}(\eta(i))$. Then it is easily seen that $\bar{p}_i = \langle p_\eta : \eta \in^{i+1}(2^{\aleph_0})\rangle$ is as required.

Let $\underset{\sim}{v} \in V^{\mathbb{Q}}$ be a $\mathbb{Q}$-name such that $1_{\mathbb{Q}} \Vdash$"$\underset{\sim}{v} \in^{\omega_1}(2^{\aleph_0})$", and for every $\eta \in^{i+1}(2^{\aleph_0}), p_\eta \Vdash$"$\eta = \underset{\sim}{v} \restriction i+1$". We now define a $\mathbb{Q}$-name $\underset{\sim}{\tau} \in^{\omega_1}2$ as follows: let $\langle \rho_\alpha : \alpha < 2^{\aleph_0}\rangle$ be an enumeration of ${}^{\omega}2$ with no repetitions. Then let

$$\Vdash_{\mathbb{Q}} \underset{\sim}{\tau}(\omega \cdot i + n) = \rho_{\underset{\sim}{v}(i)}(n).$$

Lemma 2.3. $\Vdash_{\mathbb{Q}}$ *"$\underset{\sim}{\tau} \in {}^{\omega_1}2$ and $\underset{\sim}{\tau} \notin \check{V}$".*

Proof. Let $q_1 \in \mathbb{Q}$. Then for some $i < \omega_1, q_1 \in \mathbb{Q}_i$. Now $\langle p_\eta : \eta \in^{i+1}(2^{\aleph_0})\rangle$ is a maximal antichain, hence we can find $\eta \in^{i+1}(2^{\aleph_0})$ such that q_1 is compatible with p_η. But $q_1 \nVdash$"$p_\eta \in \dot{G}_{\mathbb{Q}}$", so there is $q_2 \leq q_1$ such that q_2 is incompatible with p_η. But again as $\langle p_\eta : \eta \in^{i+1}(2^{\aleph_0})\rangle$ is a maximal antichain, there exists $\rho \in^{i+1}(2^{\aleph_0})$ such that q_2 and p_ρ are compatible. Let $q_3 \leq q_2, p_\rho$, and let $j \leq i$ be maximal such that $\eta \restriction j = \rho \restriction j$ and $\eta \restriction (j+1) \neq \rho \restriction (j+1)$. Then q_3, p_η are compatible with q_1, but they force contradictory information about $\underset{\sim}{\tau} \restriction [\omega \cdot j, \omega \cdot j + \omega)$. The result follows immediately. □

Lemma 2.4. *There is a dense subset $\mathbb{Q}'$ of $\mathbb{Q}$ which is the union of $\aleph_1$-maximal antichains $\langle I^*_i : i < \omega_1\rangle$ of $\mathbb{Q}$.*

Proof. For any $p \in \mathbb{Q}$, by the previous lemma, p does not force any value for $\underset{\sim}{\tau}$, hence there are ordinal $i < \omega_1$ and conditions $p_0, p_1 \leq p$ such that $p_l \Vdash$"$\underset{\sim}{\tau}(i) = l$", $l = 0, 1$. Hence we can define by recursion a sequence

$$\langle\langle q_{p,\eta}, i_{p,\eta}\rangle : \eta \in {}^{<\omega}2\rangle$$

such that

(4) $q_{p,\langle\rangle} = p$,

(5) $\nu \lhd \eta \Rightarrow q_{p,\eta} \leq q_{p,\nu}$,

(6) $i_{p,\eta}$ is the least ordinal i less than ω_1 such that $q_{p,\eta}$ does not decide $\underset{\sim}{\tau}(i)$,

(7) $q_{p,\eta} \Vdash$"$\forall j < i_{p,\eta}, \underset{\sim}{\tau}(i_{p,\eta\restriction j}) = \sigma_{p,\eta}(j)$".

It is evident that if $\nu \lhd \eta$, then $i_{p,\nu} < i_{p,\eta}$.

Claim 2.5. *For any $p \in \mathbb{Q}$, there exists a perfect subtree T_p of ${}^{\omega}2$ such that for some limit ordinal δ_p, we have*

$$\forall \rho \in Lim(T_p)(\bigcup_n i_{p,\rho\restriction n}) = \delta_p,$$

where $Lim(T_p)$ is the set of all branches through T.

Proof. For any $\eta \in {}^{<\omega}2$ set

$$\delta_{p,\eta} = \sup\{i_{p,\nu} : \eta \lhd \nu \in {}^{<\omega}2\}.$$

For some η_*, the ordinal δ_{p,η_*} is minimal. δ_{p,η_*} is a limit ordinal of cofinality $\aleph_0$, so let $\langle \eta_{p,m} : m < \omega \rangle$ be an increasing sequence with limit δ_{p,η_*} such that $\eta_{p,0} = \mathrm{lh}(\eta_*)$. We define $h_m : {}^{m}2 \to {}^{\eta_{p,m}}2$, by induction on $m < \omega$, such that

(1) h_m is 1-1,

(2) $h_0(\langle\rangle) = \eta_*$,

(3) If $n < m$ and $\eta \in {}^{m}2$, then $h_n(\eta \restriction n) \lhd h_m(\eta)$,

(4) If $\eta \in {}^{m}2$, then $i_{p,h_m(\eta)} > \eta_{p,m}$.

Then $T_p = \{h_m(\eta) : m < \omega \text{ and } \eta \in {}^{m}2\}$ and $\delta_p = \delta_{p,\eta_*}$ are as required. □

For each limit ordinal $\delta < \omega_1$ set

$$I^1_\delta = \{p \in \mathbb{Q} : \delta_p = \delta\}.$$

Then clearly $\mathbb{Q} = \bigcup\{I^1_\delta : \delta \text{ is a limit ordinal less than } \omega_1\}$.

Claim 2.6. *Let δ be a countable limit ordinal. Then there exists an antichain $\bar{q}^\delta = \langle q^\delta_p : p \in I^1_\delta \rangle$ such that for each $p \in I^1_\delta$, $q^\delta_p \leq p$.*

Proof. Let $\langle p_\alpha : \alpha < \alpha_\delta \leq 2^{\aleph_0} \rangle$ enumerate I^1_δ. We choose, by induction on α, a pair $\langle r_\alpha, v_\alpha \rangle$ such that

(8) $r_\alpha \leq p_\alpha$, and $v_\alpha \in {}^{\delta}2$,

(9) $r_\alpha \Vdash$ "$\underset{\sim}{\tau} \restriction \delta_{p_\alpha} = v_\alpha$",

(10) $\alpha \neq \beta \Rightarrow v_\alpha \neq v_\beta$.

Suppose $\alpha < \alpha_\delta$ and we have defined $\langle r_\beta, v_\beta \rangle$ for all $\beta < \alpha$ as above. We define $\langle r_\alpha, v_\alpha \rangle$.

For every $\rho \in Lim(T_{p_\alpha})$, the sequence $\langle q_{p,\rho\restriction n} : n < \omega \rangle$ is a decreasing chain of conditions in $\mathbb{Q}$, and hence there is a condition $q^*_{\rho,\alpha}$ which extends all of them. We may further suppose that it forces a value $v_{\rho,\alpha}$ for $\underset{\sim}{\tau} \restriction \delta$, where $\delta = \delta_{p_\alpha}$. Also note that by the choice of $\langle q_{\rho,\alpha} : \rho \in {}^{<\omega}2\rangle$, for $\rho_1 \neq \rho_2$ in $Lim(T_{p_\alpha})$, we have $v_{\rho_1,\alpha} \neq v_{\rho_2,\alpha}$. Now $\{v_\beta : \beta < \alpha\} \subseteq {}^{\delta}2$, hence for some $\rho = \rho_\alpha \in Lim(p_\alpha)$ we have that $v_{\rho,\alpha} \notin \{v_\beta : \beta < \alpha\}$. Let $r_\alpha = q_{\rho_\alpha,\alpha}$ and $v_\alpha = v_{\rho_\alpha,\alpha}$. □

Now for each limit ordinal $\delta < \omega_1$ let I_δ be a maximal antichain of $\mathbb{Q}$, such that $I_\delta \supseteq \{q_{p,\delta} : p \in I^1_\delta\}$, and let $\mathbb{Q}' = \bigcup\{I_\delta : \delta$ is a countable limit ordinal $\}$. Clearly $\mathbb{Q}'$ is as required and lemma 2.4 follows. □

As each I^*_i is a maximal antichain in $\mathbb{Q}'$ and hence also in $\mathbb{Q}$, it can easily seen that there are $\bar{p}^*_i, i < \omega_1$, such that

(11) $\bar{p}^*_i = \langle p^*_\eta : \eta \in {}^{i+1}(2^{\aleph_0})\rangle$ is a maximal antichin of $\mathbb{Q}'$ (and hence of $\mathbb{Q}$),

(12) If $j < i$ and $\eta \in {}^{i+1}(2^{\aleph_0})$, then $p^*_\eta \leq p^*_{\eta\restriction(j+1)}$,

(13) If $i = j+1$ and $\eta \in {}^{i+1}(2^{\aleph_0})$, then p^*_η is stronger than some condition in I^*_i.

Let

$$\mathbb{Q}'' = \{p^*_\eta : \exists i < \omega_1, \eta \in {}^{i+1}(2^{\aleph_0})\}.$$

Lemma 2.7. *$\mathbb{Q}''$ is a dense subset of $\mathbb{Q}$.*

Proof. Let $p \in \mathbb{Q}$. By Lemma 2.4, we can find some $i < \omega_1$ and some $p_1 \in I^*_i \subseteq \mathbb{Q}'$ such that $p_1 \leq p$. By (13), each $p^*_\eta, \eta \in {}^{i+1}(2^{\aleph_0})$, is stronger than some condition in I^*_i. If there is no η with $p^*_\eta \leq p_1$, then we contradict with (11). The result follows immediately. □

Finally note that the map

$$\eta \mapsto p^*_\eta$$

defines an isomorphism between a dense subset of $Col(\aleph_1, 2^{\aleph_0})$ and $\mathbb{Q}''$. It follows that

$$\mathbb{Q} \simeq \mathbb{Q}'' \simeq Col(\aleph_1, 2^{\aleph_0}) \simeq \mathrm{Add}(\aleph_1, 1).$$

The theorem follows. □

3. A note on $\aleph_1$-closed forcing notions of size continuum

In this section we present a result about $\aleph_1$-closed forcing notions of size continuum which will be used in section 5 for the proof of Theorem 1.3.

Assume that $2^{\aleph_0} = \aleph_2$ and that $\mathbb{R}$ is an $\aleph_1$-closed forcing notions of size continuum which does not collapse $\aleph_2$. It follows from [7] that the forcing notion $\mathbb{R}$ does not add a new sequence of ordinals of size $\aleph_1$, hence is is $\aleph_2$-distributive. The following result is proved in [1] Theorem 2.1.

Lemma 3.1. *There exists a sequence $\langle T_\alpha : \alpha < \aleph_2 \rangle$ of subsets of $\mathbb{R}$ such that:*

1. *Each T_α is a maximal antichain in $\mathbb{R}$,*
2. *If $T = \bigcup\{T_\alpha : \alpha < \aleph_2\}$, then $(T, \geq_{\mathbb{R}})$ is a tree of height $\aleph_2$, where T_α is the α-th level of T,*
3. *Each $t \in T$ has $\aleph_2$-many immediate successors,*
4. *T is dense in $\mathbb{R}$.*

We denote the above tree T by $T(\mathbb{R})$, and call it a base tree of $\mathbb{R}$. Note that by clause (4), $\mathbb{R} \simeq T(\mathbb{R})$.

4. Specializing $\aleph_2$-trees which have few branches

In this section we consider trees of size and height $\aleph_2$ which have $\leq \aleph_2$-many branches, and define a suitable forcing notion for specializing them. As we allow our trees to have branches, so we need a different definition of the concept of a special tree than the usual ones.

Definition 4.1. *Let $\kappa = \lambda^+$, where λ is a regular cardinal.*

1. *A κ-tree is a tree of height and size κ (so we allow the levels of the tree to have size κ).*
2. *([2]) Let T be a κ-tree. T is called special if there exists a function $F : T \to \lambda$ such that for all $x, y, z \in T$ if $x \leq_T y, z$ and $f(x) = f(y) = f(z)$, then either $y \leq_T z$ or $z \leq_T y$.*

As in [2] Theorem 8.1, we can show that a κ-special tree has at most κ-many cofinal branches. For the rest of this section we concentrate on $\kappa = \aleph_2$.

Suppose that T is an $\aleph_2$-tree with at most $\aleph_2$-many cofinal branches. Further suppose that for all $t \in T, Suc_T(t)$, the set of successors of t in T, has size $\aleph_2$. We introduce a forcing notion for specializing T. Let $\langle b_\alpha : \alpha < \aleph_2 \rangle$ be an enumeration of the cofinal branches through T, and for each α set

$$s_\alpha = \text{the } \leq_T\text{-least element of } b_\alpha \setminus \bigcup_{\beta<\alpha} b_\beta.$$

Also let

$$T^* = \{t \in T : \exists \alpha,\ s_\alpha <_T t \in b_\alpha\}.$$

Finally set $S_T = T \setminus T^*$.

Lemma 4.2. *(a) S_T has no cofinal branches.*

(b) $(S_T, \geq_T)$ is dense in $(T, \geq_T)$ (when considered as forcing notions), in particular $(S_T, \geq_T) \simeq (T, \geq_T)$.

Proof. (a) Assume not, and let b be a branch through S_T. then for some $\alpha, b \subseteq b_\alpha$, and then clearly $b \cap T^* \neq \emptyset$, which is a contradiction.

(b) Let $t \in T$. If $t \notin T^*$, then $t \in S_T$ and we are done; so assume that $t \in T^*$. Then for some $\alpha < \aleph_2, s_\alpha <_T t \in b_\alpha$. Let $t' \in Suc_T(t) \setminus \bigcup_{\beta \leq \alpha} b_\beta$. Then $t' \in S_T$ and $t' \geq_T t$. □

Lemma 4.3. *Assume there exists $F : S_T \to \omega_1$ such that if $F(x) = F(y)$, then x and y are incomparable in T. Then there exists $F' : T \to \omega_1$ such that $F' \supseteq F$ and F' specializes T.*

Proof. Define $G : T^* \to \omega_1$ as follows: Let $t \in T^*$. Then for some $\alpha < \aleph_2$, $s_\alpha <_T t \in b_\alpha$. Set $G(t) = F(s_\alpha)$. It is now easily seen that $F' = F \cup G$ is as required. □

So it suffices to define a forcing notion which adds a function $F : S_T \to \omega_1$ as above.

Definition 4.4. *$\mathbb{Q}(S_T)$ is defined as follows:*

(a) *A condition in $\mathbb{Q}(S_T)$ is a partial function $f : S_T \to \omega_1$ such that:*

  (1) $\mathrm{dom}(f)$ *is countable,*

  (2) *If $x <_T y$ and $x, y \in \mathrm{dom}(f)$ then $f(x) \neq f(y)$.*

(b) $f \leq_{\mathbb{Q}(S_T)} g$ *iff $f \supseteq g$.*

It is clear that the forcing notion $\mathbb{Q}(S_T)$ is $\aleph_1$-closed. But in general, there is no guarantee that the forcing $\mathbb{Q}(S_T)$ satisfies the $\aleph_2$-c.c., or preserves all cardinals, even if we assume GCH (see [5] and [13]).

Let G be $\mathbb{Q}(S_T)$-generic over V and let $V_1 = V[G]$. Also let $F = \bigcup\{f : f \in G\} \in V_1$. Then $F : S_T \to \omega_1$ and for all $x <_T y$ in S_T we have $F(x) \neq F(y)$.

Lemma 4.5. *With the same notation as above, forcing with $\mathbb{Q}(S_T) * T$ collapses $\aleph_2$ into $\aleph_1$.*

Proof. By Lemma 4.2(b), $\mathbb{Q}(S_T) * T \simeq \mathbb{Q}(S_T) * S_T$. Let H be S_T-generic over V_1 and $V_2 = V_1[H]$. Also let $F : S_T \to \omega_1$ be defined as above. There exists $b \in V_2$, such that b is a branch of S_T. But then $F \restriction b : b \to \omega_1$ is an injection, which implies $\aleph_2$ is collapsed into $\aleph_1$, and the result follows. □

Given an infinite cardinal κ, let $\mathrm{Add}(\aleph_0, \kappa)$ denote the Cohen forcing for adding κ-many new Cohen reals; thus conditions are finite partial functions $p : \kappa \times \omega \to \{0, 1\}$ ordered by reverse inclusion. The forcing is c.c.c., and hence it preserves all cardinals and cofinalities.

For our purpose in the next section, we will work with $\mathrm{Add}(\aleph_0, \aleph_2)$-names of trees as above, and now we modify the above presentation to cover this case. Thus assume that $\underset{\sim}{T}$ is an $\mathrm{Add}(\aleph_0, \aleph_2)$-name for a subtree of ${}^{<\aleph_2}\aleph_2$ which is forced to have $\leq \aleph_2$-many cofinal branches. Let's assume without loss of generality that it is forced by $\mathrm{Add}(\aleph_0, \aleph_2)$ that "the set of nodes of $\underset{\sim}{T}$ is $\aleph_2 \times \aleph_2$ and for each $\alpha < \aleph_2$, the α-th level of $\underset{\sim}{T}$ is $\{\alpha\} \times \aleph_2$". Let $\underset{\sim}{S_T}$ be an $\mathrm{Add}(\aleph_0, \aleph_2)$-name such that it is forced by $\mathrm{Add}(\aleph_0, \aleph_2)$ that "$\underset{\sim}{S_T}$ is a subtree of $\underset{\sim}{T}$ which has no cofinal branches and is dense it $\underset{\sim}{T}$". We now define $\mathbb{Q}(\underset{\sim}{S_T}) \in V$ as follows:

Definition 4.6. (a) *A condition in $\mathbb{Q}(\underset{\sim}{S_T})$ is a partial function $f : \aleph_2 \to \omega_1$ such that:*

(1) $\mathrm{dom}(f)$ *is a countable subset of* $\aleph_2 \times \aleph_2$.

(2) *If $x, y \in \mathrm{dom}(f)$ and $f(x) = f(y)$, then $\Vdash_{\mathrm{Add}(\aleph_0,\aleph_2)}$ "x and y are incompatible in the tree ordering, $x \perp y$". .*

(b) *$f \leq_{\mathbb{Q}(\underset{\sim}{S_T})} g$ iff $f \supseteq g$.*

We may note that we defined the forcing notion $\mathbb{Q}(\underset{\sim}{S_T})$ in V and not in $V^{\mathrm{Add}(\aleph_0,\aleph_2)}$. The forcing notion $\mathbb{Q}(\underset{\sim}{S_T})$ is $\aleph_1$-closed. The following can be proved as in Lemma 4.5.

Lemma 4.7. *Let* $\underset{\sim}{T}$ *be an* $\mathrm{Add}(\aleph_0, \aleph_2)$*-name for a subtree of* ${}^{<\aleph_2}\aleph_2$ *which has* $\leq \aleph_2$*-many cofinal branches. Then*

$$\Vdash_{\mathbb{Q}(\underset{\sim}{S_T}) * \underset{\sim}{\mathrm{Add}}(\aleph_0, \aleph_2)} \text{“ forcing with } \underset{\sim}{T} \text{ collapses } \aleph_2\text{”.}$$

Proof. We have

$$(\mathbb{Q}(\underset{\sim}{S_T}) * \underset{\sim}{\mathrm{Add}}(\aleph_0, \aleph_2)) * \underset{\sim}{T} \simeq (\mathbb{Q}(\underset{\sim}{S_T}) \times \mathrm{Add}(\aleph_0, \aleph_2)) * \underset{\sim}{S_T},$$

so it suffices to show that

$$\Vdash_{\mathbb{Q}(\underset{\sim}{S_T}) \times \mathrm{Add}(\aleph_0, \aleph_2)} \text{“ forcing with } \underset{\sim}{S_T} \text{ collapses } \aleph_2\text{”.}$$

Let $(G_1 \times G_2) * H$ be $(\mathbb{Q}(\underset{\sim}{S_T}) \times \mathrm{Add}(\aleph_0, \aleph_2)) * \underset{\sim}{S_T}$-generic over V. Let

$$F = \bigcup \{f : f \in G_1\}.$$

Then by simple density arguments, and as before, $F : \underset{\sim}{S_T} \to \omega_1$, and

$$\Vdash_{\mathrm{Add}(\aleph_0, \aleph_2)} \text{“} x <_{\underset{\sim}{T}} y \text{”} \Rightarrow F(x) \neq F(y).$$

Let $b \in V[(G_1 \times G_2) * H]$ be a branch of $S_T = \underset{\sim}{S_T}[G_2]$. But then $F \restriction b : b \to \omega_1$ is an injection, which implies $\aleph_2$ is collapsed into $\aleph_1$, and the result follows. □

5. A negative answer to Williams question when the continuum is regular

In this section we prove Theorem 1.3.. In subsection 5.1 we define the main forcing construction $\mathbb{P}$ and prove some of its basic properties. In subsection 5.2 it is shown that forcing with $\mathbb{P}$ preserves κ. Then in subsection 5.3 more properties of the forcing notion $\mathbb{P}$ are proved and finally in subsection 5.4 we complete the proof of Theorem 1.3.

5.1. **The main forcing construction and its basic properties.** Assume that GCH holds, and $\lambda > \kappa$ are such that κ is weakly compact and λ is a 2-Mahlo cardinal. In this subsection we define the main forcing notion that will be used in the proof of Theorem 1.3.

Definition 5.1. *Let*

$$\langle \langle \mathbb{P}_\alpha : \alpha \leq \lambda \rangle, \langle \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle \rangle$$

be an iteration such that:

(1) *Any* $p \in \mathbb{P}_\alpha$ *has domain* α *with support of size less than* κ *such that* $\{\beta \in \mathrm{supp}(p) : \beta \equiv 0(mod\ 3) \text{ or } \beta \equiv 2(mod\ 3)\}$ *has cardinality less than* $\aleph_1$,

(2) *If* $\beta < \kappa$ *and* $\beta \equiv 0(mod\ 3)$ *or* $\beta \equiv 2(mod\ 3)$, *then* $\Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathrm{Col}}(\aleph_1, \aleph_2 + |\beta|)$",

(3) *If* $\beta \geq \kappa$, $\beta \equiv 0(mod\ 3)$ *and* β *is inaccessible, then* $\Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathrm{Add}}(\aleph_1, \kappa)$",

(4) *If* $\beta \geq \kappa$, $\beta \equiv 1(mod\ 3)$ *and* $\beta - 1$ *is inaccessible, then* $\Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathrm{Col}}(\kappa, 2^{|\mathbb{P}_\beta|}) = \underset{\sim}{\mathrm{Col}}(\kappa, 2^\beta)$" *(as* $|\mathbb{P}_\beta| = \beta$*)*,

(5) *If* $\beta \geq \kappa$, $\beta \equiv 2(mod\ 3)$ *and* $\beta - 2$ *is inaccessible, then* $\Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta = \mathbb{Q}(S_{\underset{\sim}{T'_\beta}})$", *where* $\underset{\sim}{T}'_\beta$ *is a* $\mathbb{P}_\beta * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$*-name for a subtree of* ${}^{<\kappa}\kappa$ *which has* $\leq \kappa$*-many cofinal branches,*

(6) *Otherwise,* $\Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta$ *is the trivial forcing notion*",

(7) *If* $\underset{\sim}{\mathbb{R}}$ *is a* $\mathbb{P}_\lambda * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$*-name for an* $\aleph_1$*-closed forcing notion of size* $\leq \kappa$ *that preserves* κ *and if* $\underset{\sim}{T} = T(\underset{\sim}{\mathbb{R}})$ *(see section 3), then there is a Mahlo cardinal* $\beta \in (\kappa, \lambda)$ *such that* $\underset{\sim}{T}$ *is a* $\mathbb{P}_\beta * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$*-name. Further* $\underset{\sim}{T}$ *is isomorphic to some* $\underset{\sim}{T}'$ *which is a* $\mathbb{P}_{\beta+2} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$*-name for a subtree of* ${}^{<\kappa}\kappa$ *with* $\leq \kappa$*-many cofinal branches and* $\underset{\sim}{T}' = \underset{\sim}{T}'_{\beta+2}$ *(see Remark 5.2).*

Finally set $\mathbb{P} = \mathbb{P}_\lambda$.

Remark 5.2. *As it is shown in Lemma 5.3, the forcing* $\mathbb{P}$ *satisfies the* λ*-c.c, so does* $\mathbb{P}_\lambda * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$*. It follows that there are only* λ*-many nice names* $\underset{\sim}{T}$ *as above; hence by a book-keeping argument, and using Lemma 5.15, part* (7) *of the above definition makes sense.*

We now prove basic properties of the forcing notion $\mathbb{P}$.

Lemma 5.3. (a) $\mathbb{P}$ *is* $\aleph_1$*-closed, and hence it preserves* CH.

(b) *If* $\mu \in (\kappa, \lambda]$ *is Mahlo, then* $\mathbb{P}_\mu$ *satisfies the* μ*-c.c.*

(c) $\mathbb{P}_\lambda$ *collapses all cardinals in* $(\aleph_1, \kappa)$ *into* $\aleph_1$*, so if* κ *is not collapsed, then* $\Vdash_{\mathbb{P}}$ "$\kappa = \aleph_2$".

(d) *In* $V^{\mathbb{P}}, \lambda$ *is preserved, but all* $\mu \in (\kappa, \lambda)$ *are collapsed, so if* κ *is not collapsed, then* $\Vdash_{\mathbb{P}}$ "$\lambda = \kappa^+ = \aleph_3$".

(e) $\Vdash_{\mathbb{P}}$ "$2^{\aleph_1} = \lambda$".

Proof. (a) is clear as all forcing notions considered in the iteration are $\aleph_1$-closed and the support of the iteration is at least countable.

(b) Assume $A \subseteq \mathbb{P}_\mu$ is a maximal antichain of size μ and let $\langle p^\xi : \xi < \mu \rangle$ be an enumeration of A. Define $F : \mu \to \mu$ by $F(\xi) =$ the least η such that $supp(p^\xi) \restriction \xi \subseteq \eta$. F is a regressive function on $X = \{\xi < \mu : \xi$ is inaccessible $\}$, so as μ is a Mahlo cardinal, F is constant on some stationary subset Y of X. Let η be the resulting fixed value. So for all $\xi \in Y, supp(p^\xi) \restriction \xi \subseteq \eta$. We may further suppose that if $\xi_1 < \xi_2$ are in Y, then $supp(p^{\xi_1}) \subseteq \xi_2$.

As $\mathbb{P}_\eta$ has size less than μ, there are $\xi_1 < \xi_2$ in Y such that $p^{\xi_1} \restriction \eta$ is compatible with $p^{\xi_2} \restriction \eta$. But then in fact p^{ξ_1} is compatible with p^{ξ_2} and we get a contradiction.

(c), (e) and the fact that forcing with $\mathbb{P}_\lambda$ collapses all cardinals in $(\aleph_1, \kappa)$ into $\aleph_1$ are clear and the rest of (d) follows from (b). The lemma follows. □

5.2. **Preservation of κ.** In this subsection, we prove the following

$$(*) \qquad \text{Forcing with } \mathbb{P} \text{ preserves } \kappa.$$

To prove $(*)$, first we define two forcing notions $\mathbb{P}^C$ and $\mathbb{P}^U$ which can be considered as sub-forcings of $\mathbb{P}$ and prove some basic facts about them. Then we show that

(1) $\mathbb{P}^U$ is κ-closed,

(2) $\mathbb{P}^C$ is κ-c.c.,

(3) There is a projection $\pi : \mathbb{P}^C \times \mathbb{P}^U \to \mathbb{P}$.

Using the above results, $(*)$ follows immediately: if $\mathbb{P}$ collapses κ, then by clause (3), the forcing notion $\mathbb{P}^C \times \mathbb{P}^U$ also collapses κ. On the other hand, by clauses (1), (2) and by Easton's lemma, the forcing notion $\mathbb{P}^C \times \mathbb{P}^U$ preserves κ and we get a contradiction.

Forcing notions $\mathbb{P}^C$ and $\mathbb{P}^U$. We define the forcing notions $\mathbb{P}^C$ and $\mathbb{P}^U$. Assume $\alpha \leq \lambda$ and $p \in \mathbb{P}_\alpha$. Then we set

$$\mathrm{supp}_C(p) = \mathrm{supp}(p) \cap \{\beta < \alpha : \beta \equiv 0 (\mathrm{mod}\ 3) \text{ or } \beta \equiv 2 (\mathrm{mod}\ 3)\},$$

$$\mathrm{supp}_U(p) = \mathrm{supp}(p) \cap \{\beta < \alpha : \beta \equiv 1 (\mathrm{mod}\ 3)\}.$$

Note that $\mathrm{supp}(p) = \mathrm{supp}_C(p) \cup \mathrm{supp}_U(p)$, $|\mathrm{supp}_C(p)| < \aleph_1$ and $|\mathrm{supp}_U(p)| < \kappa$.

Definition 5.4. *Assume $\alpha \leq \lambda$. Then*

(1) $\mathbb{P}^C_\alpha = \{p \in \mathbb{P}_\alpha : \forall \beta \in \mathrm{supp}_U(p), p \restriction \beta \Vdash_\beta \text{“} p(\beta) = 1_\beta \text{”}\}$.

(2) $\mathbb{P}^U_\alpha = \{p \in \mathbb{P}_\alpha : \forall \beta \in \mathrm{supp}_C(p), p \restriction \beta \Vdash_\beta \text{“} p(\beta) = 1_\beta \text{”}\}$.

We also set $\mathbb{P}^C = \mathbb{P}^C_\lambda$ and $\mathbb{P}^U = \mathbb{P}^U_\lambda$. Note that

$$p \in \mathbb{P}^C_\alpha \Rightarrow \mathrm{supp}(p) = \mathrm{supp}_C(p)$$

and

$$p \in \mathbb{P}^U_\alpha \Rightarrow \mathrm{supp}(p) = \mathrm{supp}_U(p).$$

The following can be proved easily.

Lemma 5.5. *Let $\alpha \leq \lambda$. Then*

(1) $\mathbb{P}^C_\alpha$ *is $\aleph_1$-closed.*

(2) $\mathbb{P}^U_\alpha$ *is κ-closed.*

The next lemma is the key step towards proving $(*)$.

Lemma 5.6. *For any $\alpha \leq \lambda$, the forcing notion $\mathbb{P}^C_\alpha$ is κ-c.c.*

Remark 5.7. *As κ is a weakly compact cardinal, it is easily seen that for a forcing notion $\mathbb{P}$, the notions of "$\mathbb{P}$ is κ-c.c." and "$\mathbb{P}$ is κ-Knaster" are equivalent. We will use this fact in the proof of the above Lemma, without mentioning it explicitly.*

Proof. We prove the lemma by induction on $\alpha \leq \lambda$. Let $\mathcal{F}$ be the weakly compact filter on κ. By a positive set, we mean a set in $\mathcal{F}$.

Case 1. $\alpha < \kappa$: This is trivial as $|\mathbb{P}^C_\alpha| < \kappa$.

Case 2. $\alpha = \kappa$: We show that $\mathbb{P}^C_\kappa$ is κ-c.c., and hence it preserves κ. Assume not, and let $\langle p_i : i < \kappa \rangle$ be an antichain in $\mathbb{P}^C_\kappa$. As κ is inaccessible and for each $i < \kappa, |\mathrm{supp}(p_i)| < \kappa$, so by Δ-system lemma, we can assume that $\{\mathrm{supp}_C(p_i) : i < \kappa\}$ forms a Δ-system with root, say, Δ. Pick $\alpha < \kappa$ such that $\Delta \subseteq \alpha$. By induction hypothesis, there are $i < j < \kappa$ such that $p_i \restriction \alpha$ is compatible with $p_j \restriction \alpha$. But then p_i and p_j are compatible and we get a contradiction.

Case 3. $\alpha = \bar{\alpha} + 1 > \kappa$ is a successor ordinal and $\alpha \notin \{\beta + 1, \beta + 2, \beta + 3 : \beta \text{ is Mahlo}\}$: Then

$$\Vdash_{\mathbb{P}^C_{\bar{\alpha}}} \text{"}\underset{\sim}{\mathbb{Q}}_{\bar{\alpha}} \text{ is the trivial forcing"},$$

so $\mathbb{P}^C_\alpha \simeq \mathbb{P}^C_{\bar{\alpha}}$, and the result follows from the induction hypothesis.

Case 4. $\alpha = \beta + 1 > \kappa$ is a successor ordinal and β is Mahlo: Then we have $\mathbb{P}^C_\alpha \simeq \mathbb{P}^C_\beta * \underset{\sim}{\mathrm{Add}}(\aleph_1, \kappa)$. By our assumption, forcing with $\mathbb{P}^C_\beta$ is κ-c.c. On the other hand

$$\Vdash_{\mathbb{P}^C_\beta} \text{"}\underset{\sim}{\mathrm{Add}}(\aleph_1, \kappa) \text{ is } \kappa\text{-c.c."}.$$

Hence $\mathbb{P}^C_\alpha$ is κ-c.c. as well.

Case 5. $\alpha=\beta+2>\kappa$ is a successor ordinal and β is Mahlo: In this case, $\mathbb{P}^C_\alpha \simeq \mathbb{P}^C_{\beta+1}$, and the result follows from the induction assumption.

Case 6. $\alpha=\beta+3>\kappa$ is a successor ordinal and β is Mahlo: Then $\mathbb{P}^C_\alpha \simeq \mathbb{P}^C_{\beta+2} * \mathbb{Q}(\underset{\sim}{S_{T'_\beta}})$, where $\underset{\sim}{T'_\beta}$ is a $\mathbb{P}^C_\beta * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)$-name for a subtree of ${}^{<\kappa}\kappa$ which has $\leq \kappa$-many cofinal branches. By our assumption, forcing with $\mathbb{P}^C_{\beta+2}$ is κ-c.c. We show that

$$\Vdash_{\mathbb{P}^C_{\beta+2}} \text{“}\mathbb{Q}(\underset{\sim}{S_{T'_\beta}}) \text{ is } \kappa\text{-c.c.”},$$

from which the result follows. The proof follows ideas of Laver-Shelah [12], but here we have one more difficulty, as we work with names for trees and not trees themselves. Similar arguments are given in [8]. The main technical tool is the following.

Claim 5.8. *(Separation claim) There exists $A\in\mathcal{F}$ such that if $\eta\in A,\theta,\tau$ are elements in $S_{T'_{\beta+2}}$ above η and $p\in\mathbb{P}^C_{\beta+2}\cap V_\eta$, then for every q',q'' with $q'\cap V_\eta=q''\cap V_\eta=p$, there are $p',p''\in\mathbb{P}^C_{\beta+2}$ and a sequence $\langle(b_n,\theta_n,\tau_n):n<\omega\rangle$ such that:*

1. *$p'\leq q'$, $p''\leq q''$ and $p'\cap V_\eta=p''\cap V_\eta$.*
2. *$b_n\in \mathrm{Add}(\aleph_0,\kappa)\cap V_\eta(=\mathrm{Add}(\aleph_0,\eta))$.*
3. *$\theta_n,\tau_n\in\eta\times\omega_2$ are at the same level and $\theta_n\neq\tau_n$.*
4. *$(p',b_n)\Vdash$ “ $\tau_n \leq_{\underset{\sim}{S_{T'_{\beta+2}}}} \tau$ ” and $(p'',b_n)\Vdash$ “ $\theta_n \leq_{\underset{\sim}{S_{T'_{\beta+2}}}} \theta$ ”.*
5. *$\{b_n \mid n<\omega\}$ is a maximal antichain in $\mathrm{Add}(\aleph_0,\kappa)$.*

Proof. By the induction hypothesis, $\mathbb{P}^C_{\beta+2}$ has the κ-c.c. and therefore by standard Π^1_1-reflection arguments, the set A consisting of inaccessible cardinals $\eta<\kappa$ such that

- $\mathbb{P}^C_{\beta+2}\cap V_\eta$ is a regular sub-forcing of $\mathbb{P}^C_{\beta+2}$.
- $\mathbb{P}^C_{\beta+2}\cap V_\eta$ forces $\underset{\sim}{S_{T'_\beta}}\cap V_\eta$ is $\mathrm{Add}(\aleph_0,\kappa)\cap V_\eta$-name for an η-Aronszajn tree, i.e., a λ-tree with no cofinal branches.

is in $\mathcal{F}$. We show that A is as required. Thus suppose that $\eta\in A$, θ,τ are elements in $S_{T'_{\beta+2}}$ above η and $p\in\mathbb{P}^C_{\beta+2}\cap V_\eta$.

The branches in $S_{T'_{\beta+2}}\cap V_\eta$ below θ and below τ are both new (relative to the forcing $\mathbb{P}^C_{\beta+2}\cap V_\eta$. Hence it is forced that there are dense many pairs of conditions (p'_0,p''_0) in $\mathbb{P}^C_{\beta+2}/(\mathbb{P}^C_{\beta+2}\cap V_\eta)$ such that p'_0,p''_0 force incompatible values for the branches below θ and

τ. Pick $b_0 \in \mathrm{Add}(\aleph_0, \kappa) \cap V_\eta$ such that b_0 decides the values of these two conditions and the elements $\theta_0, \tau_0 \in \eta \times \omega_2$ at the same level of the tree which witness the incompatibility.

If there is a condition in $\mathrm{Add}(\aleph_0, \kappa)$ that forces those two conditions do not witness the incompatibility by θ_0, τ_0, then it is incompatible with b_0, and we extend it to a condition b_1 that forces a further extension of p'_1 of p'_0 and p''_1 of p''_0 witness the incompatibility of of branches below θ, τ using $\theta_1, \tau_1 \in \eta \times \omega_2$.

As $\mathrm{Add}(\aleph_0, \kappa)$ is c.c.c., and $\mathbb{P}^C_{\beta+2}$ is $\aleph_1$-closed, we can continue this process, which terminates after at most countably many steps. At the end of the process, we get a countable ordinal ϑ, sequences $\langle p'_n : n < \vartheta \rangle$ and $\langle p''_n : n < \vartheta \rangle$ of conditions in $\mathbb{P}^C_{\beta+2}/(\mathbb{P}^C_{\beta+2} \cap V_\eta)$, a sequence $\{b_n : n < \vartheta\}$ of conditions in $\mathrm{Add}(\aleph_0, \kappa)$ and a sequence $\langle (\theta_n, \tau_n) : n < \vartheta \rangle$ such that

- The sequences $\langle p'_n : n < \vartheta \rangle$ and $\langle p''_n : n < \vartheta \rangle$ are decreasing and $p'_n \cap V_\eta = p''_n \cap V_\eta$.
- $\{b_n : n < \vartheta\}$ is a maximal antichain in $\mathrm{Add}(\aleph_0, \kappa)$.
- For all $n < \vartheta$, $\theta_n, \tau_n \in \eta \times \omega_2$ and $\theta_n \neq \tau_n$ are at the same level.
- For all $n < \vartheta, (b_n, p'_n) \Vdash$" $\tau_n \leq_{S_{\underset{\sim}{T}'_{\beta+2}}} \tau$" and $(b_n, p''_n) \Vdash$" $\theta_n \leq_{S_{\underset{\sim}{T}'_{\beta+2}}} \theta$".

Let p' extend all $p'_n, n < \vartheta$ and p'' extends all $p''_n, n < \vartheta$. Then p', p'' together with $\langle (b_n, \theta_n, \tau_n : n < \vartheta) \rangle$ are as required. $\square$

Let us call the sequence $\langle (b_n, \theta_n, \tau_n \mid n < \omega) \rangle$ a separating witness for θ, τ relative to p', p''. By repeated use of the claim, for every condition p, we may find an $\mathcal{F}$-positive set A such that for all $\eta \in A$, there exists a pair of conditions (p', p'') extending p with $p' \cap V_\eta = p'' \cap V_\eta$ such that every pair of elements above η in $\mathrm{dom}\, p'(\beta+2) \times \mathrm{dom}\, p''(\beta+2)$ has a separating witness relative to $p' \restriction \beta+2, p'' \restriction \beta+2$.

Let $\langle p_\eta : \eta < \kappa \rangle$ be a sequence of conditions in $\mathbb{P}^C_\alpha$. By the induction hypothesis we can assume that for all $\eta < \eta', p_\eta \restriction \beta+2$ is compatible with $p_{\eta'} \restriction \beta+2$.

By the normality of the filter $\mathcal{F}$, we may find $A \in \mathcal{F}$ such that for each $p_\eta, \eta \in A$, there is a pair (p'_η, p''_η), of extensions of p_η such that $p'_\eta \cap V_\eta = p''_\eta \cap V_\eta$ and every pair of elements above η in $\mathrm{dom}\, p'_\eta(\beta+2) \times \mathrm{dom}\, p''_\eta(\beta+2)$ has a separating witness relative to $p'_\eta \restriction \beta+2, p''_\eta \restriction \beta+2$. We may also assume that the separating witnesses are the same for all such η's; call it $\langle (b_n, \theta_n, \tau_n) : n < \omega) \rangle$.

We claim that for any $\eta < \eta'$ in A, p_η is compatible with $p_{\eta'}$. Let q be such that $q \restriction \beta+2 = (p_\eta \restriction \beta+2) \wedge (p_{\eta'} \restriction \beta+2)$ is the least common extension of $p_\eta \restriction \beta+2$ and $p_{\eta'} \restriction \beta+2$ (which exists by our assumption) and $q(\beta+2) = p'_\eta(\beta+2) \cup p''_{\eta'}(\beta+2)$.

It is enough to show that q is a condition, i.e., $q \in \mathbb{P}^C_\alpha$. Thus suppose $t, t' \in \mathrm{dom}(q(\beta+2))$ and $q(\beta+2)(t) = q(\beta+2)(t')$. We show that $q \restriction \beta+2 \Vdash_{\beta+2} \Vdash_{\mathrm{Add}(\aleph_0,\kappa)} \check{t} \perp_{S_{\underset{\sim}{T'_{\beta+2}}}} \check{t}'$. We may suppose that both t and t' are above η, as otherwise, we may use the fact $p'_\eta \cap V_\eta = p''_\eta \cap V_\eta$ to conclude the result. By the choice of (p'_η, p''_η), $p'_\eta \restriction \beta+2 \Vdash_{\beta+2} b_n \Vdash_{\mathrm{Add}(\aleph_0,\kappa)} \tau_n \leq t$ and $p''{}_\eta \restriction \beta+2 \Vdash_{\beta+2} b_n \Vdash_{\mathrm{Add}(\aleph_0,\kappa)} \theta_n \leq t'$, for some $\tau_n \neq \theta_n$. But $q \restriction \beta+2$ forces that b_n forces that t, t' are incompatible in the tree relation. Now if $q \restriction \beta+2 \not\Vdash_{\beta+2}$"$\Vdash_{\mathrm{Add}(\aleph_0,\kappa)} t \perp t'$", then there is a condition $q' \leq q \restriction \beta+2$ and $b \in \mathrm{Add}(\aleph_0, \kappa)$ such that $(q', b) \Vdash t \leq_{S_{\underset{\sim}{T'_{\beta+2}}}} t'$. But b is compatible with b_n (for some $n < \omega$) and q' is stronger than q and thus also force that b_n separates t and t'; a contradiction.

Case 7. $\alpha > \kappa$ is a limit ordinal. The proof is very similar to the proof of Case 6. By repeated use of separation claim, for every condition p, we may find a positive set and extend it to a pair of conditions p', p'' such that $p' \cap V_\eta = p'' \cap V_\eta$ and for every Mahlo $\beta < \alpha$ and any pair of elements above η in $\mathrm{dom}\, p'(\beta+2) \times \mathrm{dom}\, p''(\beta+2)$ has a separating witness relative to $p' \restriction \beta+2, p'' \restriction \beta+2$. Note that all of the separating witnesses are in V_η. We call this pair (p', p'') a separating pair.

Let $\langle p_\eta \mid \eta < \kappa \rangle$ be a sequence of conditions in $\mathbb{P}^C_\alpha$. For a positive set of $\eta < \kappa$, we may extend each p_η to a separating pair (p'_η, p''_η), above η. We can also assume that on a positive set A, $p'_\eta \cap V_\eta = p''_\eta \cap V_\eta$ and the separating witnesses are the same for all such η's; call it $\langle (b_n, \theta_n, \tau_n) \mid n < \omega) \rangle$. Narrowing A down, we may assume that for every $\beta < \alpha$, $p_\eta(\beta)$ is a Δ-system. Then as in Case 6, for any $\eta < \eta'$ in A, p_η is compatible with $p_{\eta'}$, and moreover it is witnessed by the condition q, which is defined by $q(\beta) = p'_\eta(\beta) \cup p''_{\eta'}(\beta)$ for every $\beta < \alpha$. □

We now show that there are projections from $\mathbb{P}$ onto both of $\mathbb{P}^C$ and $\mathbb{P}^U$. First, let us recall the definition of a projection between two forcing notions.

Definition 5.9. *Let $\mathbb{P}, \mathbb{Q}$ be two forcing notions. π is a projection from $\mathbb{P}$ onto $\mathbb{Q}$ if $\pi : \mathbb{P} \to \mathbb{Q}$, and it satisfies the following conditions:*

(1) $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$,

(2) π *is order preserving; i.e.,* $p \leq_{\mathbb{P}} q \Rightarrow \pi(p) \leq_{\mathbb{Q}} \pi(q)$,

(3) *If* $p \in \mathbb{P}, q \in \mathbb{Q}$ *and* $q \leq_{\mathbb{Q}} \pi(p)$, *then there exists* $p^* \leq_{\mathbb{P}} p$ *such that* $\pi(p^*) \leq_{\mathbb{Q}} q$.

If $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection, then clearly $\pi[\mathbb{P}]$ is dense in $\mathbb{Q}$. The next lemma shows that if $\mathbb{P}$ projects into $\mathbb{Q}$, then a generic filter for $\mathbb{P}$ yields a generic filter for $\mathbb{Q}$.

Lemma 5.10. *Let* $\pi : \mathbb{P} \to \mathbb{Q}$ *be a projection from* $\mathbb{P}$ *into* $\mathbb{Q}$, *let* G *be* $\mathbb{P}$*-generic over* V, *and let* $H \subseteq \mathbb{Q}$ *be the filter generated by* $\pi[G]$. *Then* H *is* $\mathbb{Q}$*-generic over* V *and* $V[H] \subseteq V[G]$. *Furthermore,* $V[G]$ *is a generic extension of* $V[H]$ *using the forcing notion* $\mathbb{P}/H = \{p \in \mathbb{P} : \pi(p) \in H\}$.

Lemma 5.11. *Let* $\alpha \leq \lambda$, *and define*

$$\pi_{C,\alpha} : \mathbb{P}_\alpha \to \mathbb{P}^C_\alpha$$

by $\mathrm{dom}(\pi_{C,\alpha}(p)) = \alpha$, *and for all* $\beta < \alpha$

$$\Vdash_\beta \text{“}\pi_{C,\alpha}(p)(\beta) = \begin{cases} p(\beta)\text{”} & \text{if } \beta \in \mathrm{supp}_C(p), \\ 1_\beta\text{”} & \text{if } \beta \notin \mathrm{supp}_C(p). \end{cases}$$

Then $\pi_{C,\alpha}$ *is a projection from* $\mathbb{P}_\alpha$ *onto* $\mathbb{P}^C_\alpha$.

Proof. The proof is by induction on α. It is clear that $\pi_{C,\alpha}(1_\alpha) = 1_{\mathbb{P}^C_\alpha}$ and that $\pi_{C,\alpha}$ is order preserving. Let $p \in \mathbb{P}_\alpha, q \in \mathbb{P}^C_\alpha$ and suppose that $q \leq \pi_{C,\alpha}(p)$. We find $p^* \leq p$ such that $\pi_{C,\alpha}(p^*) \leq q$. By the induction hypothesis, suppose that for all $\beta < \alpha$, we have defined $p^* \restriction \beta$ such that the following conditions are satisfied:

(1) $p^* \restriction \beta \leq p \restriction \beta$.

(2) $\pi_{C,\alpha}(p^*) \restriction \beta \leq q \restriction \beta$.

(3) $q \restriction \beta \leq p^* \restriction \beta$.

Now there are three cases to be considered:

Case 1. $\alpha = \beta + 1$ and $\beta \in \mathrm{supp}_C(p)$. Let $p^*(\beta)$ be a $\mathbb{P}_\beta$-name such that $q \restriction \beta \Vdash$“$p^*(\beta) = q(\beta)$” and for all $q' \leq p^* \restriction \beta$ with $q' \perp q \restriction \beta, q' \Vdash$“$p^*(\beta) = p(\beta)$”. It is easily seen that $p^* \restriction \alpha \leq p \restriction \alpha$, $\pi_{C,\alpha}(p^*) \restriction \alpha \leq q \restriction \alpha$ and $q \restriction \alpha \leq p^* \restriction \alpha$.

Case 2. $\alpha = \beta + 1$ and $\beta \notin \mathrm{supp}_C(p)$. Let $p^*(\beta)$ be a $\mathbb{P}_\beta$-name such that $p^* \restriction \beta \Vdash$“$p^*(\beta) =$

1". Again, one can easily verify that $p^* \restriction \alpha \leq p \restriction \alpha$, $\pi_{C,\alpha}(p^*) \restriction \alpha \leq q \restriction \alpha$ and $q \restriction \alpha \leq p^* \restriction \alpha$.

Case 3. α is a limit ordinal. Then set $p^* \restriction \alpha = \bigcup_{\beta<\alpha} p^* \restriction \beta$. As $\mathrm{supp}(p^*) \subseteq \mathrm{supp}(p) \cup \mathrm{supp}(q)$, we have $p^* \in \mathbb{P}_\alpha$ and clearly we have $p^* \restriction \alpha \leq p \restriction \alpha$, $\pi_{C,\alpha}(p^*) \restriction \alpha \leq q \restriction \alpha$ and $q \restriction \alpha \leq p^* \restriction \alpha$. □

Similarly we have the following.

Lemma 5.12. *Let $\alpha \leq \lambda$, and define*

$$\pi_{U,\alpha} : \mathbb{P}_\alpha \to \mathbb{P}^U_\alpha$$

by $\pi_{U,\alpha}(p) = p^U$, where $\mathrm{dom}(p^U) = \alpha$, and for all $\beta < \alpha$

$$\Vdash_\beta \text{“}p^U(\beta) = \begin{cases} p(\beta)\text{”} & \text{if } \beta \in \mathrm{supp}_U(p), \\ 1_\beta\text{”} & \text{if } \beta \notin \mathrm{supp}_U(p). \end{cases}$$

Then $\pi_{U,\alpha}$ is a projection from $\mathbb{P}_\alpha$ onto $\mathbb{P}^U_\alpha$.

We may note that the above definitions of $\pi_{C,\alpha}$ and $\pi_{C,\alpha}$ do not depend on the choice α, so from now on we remove the subscript α, and just use π_C and π_U to denote $\pi_{C,\alpha}$ and $\pi_{U,\alpha}$ respectively.

We also have the following lemma.

Lemma 5.13. *Let $\alpha \leq \lambda$, and define*

$$\pi_\alpha : \mathbb{P}^C_\alpha \times \mathbb{P}^U_\alpha \to \mathbb{P}_\alpha$$

by $\mathrm{dom}(\pi_\alpha(p_C, p_U)) = \alpha$, and for all $\beta < \alpha$

$$\Vdash_\beta \text{“}\pi_\alpha(p_C, p_U)(\beta) = \begin{cases} p_C(\beta)\text{”} & \text{if } \beta \equiv 0 (mod\ 3) \text{ or } \beta \equiv 2 (mod\ 3), \\ p_U(\beta)\text{”} & \text{if } \beta \equiv 1 (mod\ 3). \end{cases}$$

Then π_α is a projection from $\mathbb{P}^C_\alpha \times \mathbb{P}^U_\alpha$ onto $\mathbb{P}_\alpha$.

Proof. We prove the lemma by induction on α. As the definition of π_α does not depend on the choice of α, we remove the subscript α from it. It is clear that $\pi(1_{\mathbb{P}^C_\alpha}, 1_{\mathbb{P}^U_\alpha}) = 1_{\mathbb{P}_\alpha}$ and that π is order preserving.

Suppose that $(p_C, p_U) \in \mathbb{P}^C_\alpha \times \mathbb{P}^U_\alpha$, $q \in \mathbb{P}_\alpha$ and $q \leq \pi(p_C, p_U)$. We show there are $p^*_C \in \mathbb{P}^C_\alpha$ and $p^*_U \in \mathbb{P}^U_\alpha$ such that $(p^*_C, p^*_U) \leq (p_C, p_U)$ and $\pi(p^*_C, p^*_U) \leq q$.

By the induction hypothesis, suppose that for all $\beta < \alpha$, we have defined $p^*_C \restriction \beta \in \mathbb{P}^C_\beta$ and $p^*_U \restriction \beta \in \mathbb{P}^U_\beta$ such that the following conditions are satisfied:

(1) $(p^*_C \restriction \beta, p^*_U \restriction \beta) \leq (p_C \restriction \beta, p_U \restriction \beta)$.

(2) $\pi(p^*_C \restriction \beta, p^*_U \restriction \beta) \leq q \restriction \beta$.

(3) $q \restriction \beta \leq p^*_C \restriction \beta, p^*_U \restriction \beta$.

Case 1. $\alpha = \beta + 1$ is a successor ordinal and $\beta \equiv 0 (\text{mod } 3)$ or $\beta \equiv 2 (\text{mod } 3)$: Let $p^*_C(\beta)$ be a $\mathbb{P}_\beta$-name such that $q \restriction \beta \Vdash$"$p^*_C(\beta) = q(\beta)$ and for all $q' \leq p^*_C \restriction \beta$ with $q' \perp q \restriction \beta, q' \Vdash$"$p^*_C(\beta) = p_C(\beta)$". Also let $p^*_U(\beta)$ be a $\mathbb{P}_\beta$-name such that $p^*_U \restriction \beta \Vdash$"$p^*_U(\beta) = 1$. It is easily seen that $(p^*_C \restriction \alpha, p^*_U \restriction \alpha) \leq (p_C \restriction \alpha, p_U \restriction \alpha)$, $\pi(p^*_C \restriction \alpha, p^*_U \restriction \alpha) \leq q \restriction \alpha$ and that $q \restriction \alpha \leq p^*_C \restriction \alpha, p^*_U \restriction \alpha$.

Case 2. $\alpha = \beta + 1$ is a successor ordinal and $\beta \equiv 1 (\text{mod } 3)$: Let $p^*_U(\beta)$ be a $\mathbb{P}_\beta$-name such that $q \restriction \beta \Vdash$"$p^*_U(\beta) = q(\beta)$ and for all $q' \leq p^*_U \restriction \beta$ with $q' \perp q \restriction \beta, q' \Vdash$"$p^*_U(\beta) = p_U(\beta)$". Also let $p^*_C(\beta)$ be a $\mathbb{P}_\beta$-name such that $p^*_C \restriction \beta \Vdash$"$p^*_C(\beta) = 1$. One can easily verify that $(p^*_C \restriction \alpha, p^*_U \restriction \alpha) \leq (p_C \restriction \alpha, p_U \restriction \alpha)$, $\pi(p^*_C \restriction \alpha, p^*_U \restriction \alpha) \leq q \restriction \alpha$ and $q \restriction \alpha \leq p^*_C \restriction \alpha, p^*_U \restriction \alpha$.

Case 3. α is a limit ordinal: Then set $p^*_C \restriction \alpha = \bigcup_{\beta<\alpha} p^*_C \restriction \beta$ and $p^*_U \restriction \alpha = \bigcup_{\beta<\alpha} p^*_U \restriction \beta$. Then $p^*_C \restriction \alpha \in \mathbb{P}_\alpha$ as $\text{supp}(p^*_C) \subseteq \text{supp}(q) \cup \text{supp}(p_C)$. Similarly $p^*_U \restriction \alpha \in \mathbb{P}_\alpha$ as $\text{supp}(p^*_U) \subseteq \text{supp}(q) \cup \text{supp}(p_U)$. It is also evident that $(p^*_C \restriction \alpha, p^*_U \restriction \alpha) \leq (p_C \restriction \alpha, p_U \restriction \alpha)$, $\pi(p^*_C \restriction \alpha, p^*_U \restriction \alpha) \leq q \restriction \alpha$ and $q \restriction \alpha \leq p^*_C \restriction \alpha, p^*_U \restriction \alpha$. □

It follows from Easton's lemma, Lemma 5.5(2) and Lemma 5.10 that forcing with $\mathbb{P}^C_\alpha \times \mathbb{P}^U_\alpha$ preserves κ, and hence by Lemma 5.12, forcing with $\mathbb{P}_\alpha$ also preserves κ. The $(*)$ follows (by taking $\alpha = \lambda$), and we are done.

5.3. **More on the forcing notion $\mathbb{P}$.** It follows that

$$V^{\mathbb{P}} \models \text{“ } CH + \kappa = \aleph_2 + \lambda = \aleph_3 = 2^{\aleph_1}\text{”}.$$

We now prove the following:

$$(**) \qquad V^{\mathbb{P} * \underset{\sim}{\text{Add}}(\aleph_0, \kappa)} \models \text{“ Any } \aleph_1 - \text{closed forcing of size } \leq \kappa \text{ collapses } \kappa\text{”},$$

which completes the proof of Theorem 1.3; as then in $V^{\mathbb{P} * \underset{\sim}{\text{Add}}(\aleph_0, \kappa)}$, any $\aleph_1$-closed forcing notion of size $\leq \kappa = \aleph_2$ is forcing isomorphic to $\text{Col}(\aleph_1, \kappa) \simeq \text{Add}(\aleph_1, 1)$.

Lemma 5.14. *Assume that $\mu \in (\kappa, \lambda)$ is Mahlo. Let $\underset{\sim}{T}$ be a $\mathbb{P}_\mu * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$-name of a subtree of ${}^{<\kappa}\kappa$, all of whose levels have size $\leq \kappa$. Then*

$$\Vdash_{\mathbb{P}_{\mu+2} * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)} \text{“} \underset{\sim}{T} \textit{ has } \leq \kappa - \textit{many } \kappa - \textit{branches.}$$

Proof. We work in $V_1 = V^{\mathbb{P}_\mu}$, so that we can assume that $\underset{\sim}{T}$ is an $\mathrm{Add}(\aleph_0, \kappa)$-name. Note that in V_1, $\kappa = \aleph_2$, $\mu = \aleph_3$ and $2^{\aleph_1} = \aleph_3$. Further we have

$$\mathbb{P}_{\mu+2}/\mathbb{P}_\mu \simeq \mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1} = \mathrm{Add}(\aleph_1, \kappa) * \underset{\sim}{\mathrm{Col}}(\kappa, 2^\mu),$$

and since $(2^\kappa)^{V_1} \leq (2^\mu)^{V^{\mathbb{P}_{\mu+1}}}$, we have

$$\Vdash_{\mathbb{P}_{\mu+2} * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)} \text{“ } |\{b \in V_1 : b \text{ is a branch of } \underset{\sim}{T}\}| \leq \kappa\text{”}.$$

So it suffices to show that forcing with $\mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$ adds no new branches. Assume by contradiction that $\underset{\sim}{\eta}$ is a $\mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$-name which is forced to be a new κ-branch of $\underset{\sim}{T}$. The next claim follows easily from the assumption $\underset{\sim}{\eta}$ is forced to be a new branch.

Claim 5.15. *For every $\langle p^0, p^1, p^2\rangle \in \mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$ we can find conditions $\langle q_i^0, q_i^1, q_i^2\rangle$, for $i = 0, 1$, $\delta < \kappa$ and x_0, x_1 such that*

(a) $\langle q_0^0, q_0^1, q_0^2\rangle, \langle q_1^0, q_1^1, q_1^2\rangle \leq \langle p^0, p^1, p^2\rangle$,

(b) $x_0 \neq x_1$,

(c) $\Vdash$ “$x_0, x_1 \in \underset{\sim}{T}_\delta$, *the δ-th level of* $\underset{\sim}{T}$”,

(d) $\langle q_i^0, q_i^1, q_i^2\rangle \Vdash$ “$x_i \in \underset{\sim}{\eta}$” *(i=0,1).* □

In fact, as the forcing notions $\mathrm{Add}(\aleph_1, \kappa)$ and $\mathrm{Add}(\aleph_0, \kappa)$ are κ-c.c. and $\mathrm{Col}(\kappa, 2^\mu)$ is forced to be κ-closed, we can show that the conditions $\langle q_0^0, q_0^1, q_0^2\rangle$ and $\langle q_1^0, q_1^1, q_1^2\rangle$ in the claim can be chosen so that $q_0^0 = q_1^0 = p^0$ and $q_0^2 = q_1^2 = p^2$ (see [11] for similar arguments).

Now assume for simplicity that the empty condition forces $\underset{\sim}{\eta}$ is a new branch. So using the above claim we can build a sequence $\langle \underset{\sim}{q}{}^1_\nu : \nu \in {}^{<\omega_1}2\rangle$ of $\mathbb{Q}_\mu$-names of elements of $\underset{\sim}{\mathbb{Q}}_{\mu+1}$, an increasing continuous sequence $\langle \delta_i : i < \omega_1\rangle$ of ordinals less than κ and a sequence $\langle x_\nu : \nu \in {}^{<\omega_1}2\rangle$ such that:

(1) $\nu_1 \trianglelefteq \nu_2 \Rightarrow \Vdash_{\mathbb{Q}_\mu}$ “$\underset{\sim}{q}{}^1_{\nu_2} \leq \underset{\sim}{q}{}^1_{\nu_1}$”,

(2) $\langle \emptyset, \underset{\sim}{q}{}^1_\nu, \emptyset\rangle \Vdash$ “$x_\nu \in \underset{\sim}{T}_{\delta_i}$” where $i = \mathrm{lh}(\nu)$,

(3) $x_{\nu^\frown\langle 0\rangle} \neq x_{\nu^\frown\langle 1\rangle}$,

(4) $\langle \emptyset, \underset{\sim}{q}^1_\nu, \emptyset\rangle \Vdash$"$x_\nu \in \underset{\sim}{\eta}$",

(5) $\nu_1 \trianglelefteq \nu_2 \Rightarrow \langle \emptyset, \underset{\sim}{q}^1_{\nu_2}, \emptyset\rangle \Vdash$"$x_{\nu_1} <_{\underset{\sim}{T}} x_{\nu_2}$".

For some $\xi < \kappa$, $\langle \underset{\sim}{q}^1_\nu : \nu \in {}^{<\omega_1}2\rangle$ is in fact an $\mathrm{Add}(\aleph_1, \xi)$-name. Now we have

$$\mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa) \simeq \mathrm{Add}(\aleph_1, \xi) * \underset{\sim}{\mathrm{Add}}(\aleph_1, [\xi, \kappa)) * \underset{\sim}{\mathbb{Q}}_{\mu+1} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa),$$

and in the generic extension $V^{\mathbb{P}_\mu * \underset{\sim}{\mathrm{Add}}(\aleph_1, \xi)}$, we have an interpretation q^1_ν of the name $\underset{\sim}{q}^1_\nu$, where $\nu \in {}^{<\omega_1}2$.

Work in $V^{\mathbb{P}_\mu * \mathbb{Q}_\mu}$. For each $\tau \in {}^{\omega_1}2$, let $q^1_\tau \leq q^1_{\tau\restriction i}, i < \omega_1$ and let $\delta = \sup\{\delta_i : i < \omega_1\} < \kappa$. By extending q^1_τ if necessary, we can assume that for some x_τ,

$$\langle \emptyset, q^1_\tau, \emptyset\rangle \Vdash \text{" } x_\tau \in \underset{\sim}{T}_\delta \cap \underset{\sim}{\eta} \text{ ".}$$

But then for all $\tau_1 \neq \tau_2$ in ${}^{\omega_1}2$ we have $x_{\tau_1} \neq x_{\tau_2}$, and so

$$\Vdash_{\mathbb{P}_\mu * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)} \text{" the } \delta\text{-th level of the tree has at least } 2^{\aleph_1} = \mu = \aleph_3\text{-many nodes".}$$

But $\Vdash_{\mathbb{P}_\mu * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)} \models$" $|\underset{\sim}{T}_\delta| \leq \kappa < \mu$", and we get a contradiction. □

The next lemma follows from Lemma 5.13 and the fact that $\mathbb{P}_\mu * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa) \lessdot \mathbb{P}_{\mu+2} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$

Lemma 5.16. *With the same hypotheses as in Lemma 5.13, we have the following: In $V^{\mathbb{P}_{\mu+2}}$, $\underset{\sim}{T}$ is isomorphic to some $\underset{\sim}{T}'$, which is an $\mathrm{Add}(\aleph_0, \kappa)$-name of a subtree of ${}^{<\kappa}\kappa$ of height κ with $\leq \kappa$-many cofinal branches.*

5.4. **Completing the proof of Theorem 1.3.** We are now ready to give the proof of (∗∗) and hence of Theorem 1.3. First note that $\mathbb{P}_\lambda * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa) \simeq \mathbb{P}_\lambda \times \mathrm{Add}(\aleph_0, \kappa)$. Let $G \times H$ be $\mathbb{P}_\lambda \times \mathrm{Add}(\aleph_0, \kappa)$-generic over V. Also let $\mathbb{R} \in V[G \times H]$ be an $\aleph_1$-closed forcing notion of size $\leq \kappa = \aleph_2$.

Assume towards contradiction that forcing with $\mathbb{R}$ over $V[G \times H]$ does not collapse $\aleph_2$ and adds no new set of ordinals of size $\aleph_1$. Let $T = T(\mathbb{R})$. By Lemma 3.1, T is a dense subset of $\mathbb{R}$, and $(T, \geq_{\mathbb{R}})$ is a tree of height κ all of whose levels have size $\leq \kappa$. Note that T is isomorphic to a subtree of ${}^{<\kappa}\kappa$ of height κ, and hence we consider T as a subtree of ${}^{<\kappa}\kappa$.

Let $\underset{\sim}{T}$ be a $\mathbb{P}_\lambda \times \mathrm{Add}(\aleph_0, \kappa)$-name for T. By Lemma 5.15 and clause (7) of Definition 5.1, there exists some Mahlo cardinal $\beta \in (\kappa, \lambda)$ such that $\underset{\sim}{T}$ is a $\mathbb{P}_\beta * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$-name and such that $\underset{\sim}{T}$ is isomorphic to some $\underset{\sim}{T}'$ which is a $\mathbb{P}_{\beta+2} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$-name for a subtree of ${}^{<\kappa}\kappa$ and $\underset{\sim}{T}' = \underset{\sim}{T}_{\beta+2}$. Then $\mathbb{P}_{\beta+3} \simeq \mathbb{P}_{\beta+2} * \mathbb{Q}(S_{\underset{\sim}{T}'})$, and by Lemma 4.7.

$$\Vdash_{\mathbb{P}_{\beta+3} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)} \text{“ Forcing with } \underset{\sim}{T}' \text{ collapses } \kappa \text{ into } \aleph_1\text{”}.$$

As $\mathbb{P}_{\beta+3} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa) \lessdot \mathbb{P}_\lambda * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$, so

$$\Vdash_{\mathbb{P}_\lambda * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)} \text{“ Forcing with } \underset{\sim}{T}' \text{ collapses } \kappa \text{ into } \aleph_1\text{”}.$$

This means

$$\Vdash_{\mathbb{P} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)} \text{“ Forcing with } \underset{\sim}{\mathbb{R}} \text{ collapses } \kappa \text{ into } \aleph_1\text{”},$$

so

$$\Vdash_{\mathbb{P} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)} \text{“ } \underset{\sim}{\mathbb{R}} \simeq \underset{\sim}{\mathrm{Col}}(\aleph_1, \kappa) \simeq \underset{\sim}{\mathrm{Add}}(\aleph_1, 1)\text{”},$$

and the theorem follows. □

6. Every forcing which adds a new subset of $\aleph_2$ can collapse a cardinal

In this section we give a proof of Theorem 1.4. Thus assume that GCH holds and $\lambda > \kappa$ are such that κ is weakly compact and λ is a 2-Mahlo cardinal. The forcing notion we define is very similar to the one of section 5.

Definition 6.1. *Let*

$$\langle \langle \mathbb{P}_\alpha : \alpha \leq \lambda \rangle, \langle \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle \rangle$$

be an iteration such that:

(1) *Any $p \in \mathbb{P}_\alpha$ has domain α with support of size less than κ such that $\{\beta \in \mathrm{supp}(p) : \beta \equiv 0 (mod\ 3)$ or $\beta \equiv 2 (mod\ 3)\}$ has cardinality less than $\aleph_1$,*

(2) *If $\beta < \kappa$ and $\beta \equiv 0 (mod\ 3)$ or $\beta \equiv 2 (mod\ 3)$, then $\Vdash_\beta$ “ $\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathrm{Col}}(\aleph_1, \aleph_2 + |\beta|)$”,*

(3) *If $\beta \geq \kappa$, $\beta \equiv 0 (mod\ 3)$ and β is inaccessible, then $\Vdash_\beta$ “ $\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathrm{Add}}(\aleph_1, \kappa)$”,*

(4) *If $\beta \geq \kappa$, $\beta \equiv 1 (mod\ 3)$ and $\beta - 1$ is inaccessible, then $\Vdash_\beta$ “ $\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathrm{Col}}(\kappa, 2^{|\mathbb{P}_\beta|}) = \underset{\sim}{\mathrm{Col}}(\kappa, 2^\beta)$” (as $|\mathbb{P}_\beta| = \beta$),*

(5) *If $\beta \geq \kappa$, $\beta \equiv 2 (mod\ 3)$ and $\beta - 2$ is inaccessible, then $\Vdash_\beta$ “ $\underset{\sim}{\mathbb{Q}}_\beta = \mathbb{Q}(S_{\underset{\sim}{T}'_\beta})$”, where $\underset{\sim}{T}'_\beta$ is a $\mathbb{P}_\beta$-name for a subtree of ${}^{<\kappa}\kappa$ which has $\leq \kappa$-many cofinal branches,*

(6) *Otherwise, $\Vdash_\beta$ “ $\underset{\sim}{\mathbb{Q}}_\beta$ is the trivial forcing notion”,*

(7) *If* $\underset{\sim}{T}$ *is a* $\mathbb{P}_\lambda$*-name for a tree of size and height* κ*, then there is a Mahlo cardinal* $\beta \in (\kappa, \mu)$ *such that* $\underset{\sim}{T}$ *is a* $\mathbb{P}_\beta$*-name. Further* $\underset{\sim}{T}$ *is isomorphic to some* $\underset{\sim}{T}'$ *which is a* $\mathbb{P}_{\beta+2}$*-name for a subtree of* ${}^{<\kappa}\kappa$ *with* $\leq \kappa$*-many cofinal branches and* $\underset{\sim}{T}' = \underset{\sim}{T}'_{\beta+2}$.

Finally set $\mathbb{P} = \mathbb{P}_\lambda$.

The next lemma can be proved as in section 5.

Lemma 6.2. *Let* G *be* $\mathbb{P}$*-generic over* V*. Then the following hold in* $V[G]$*:*

(a) $2^{\aleph_0} = \aleph_1 < \kappa = \aleph_2 < 2^{\aleph_1} = \lambda = \aleph_3$,

(b) *Every Tree of size and height* $\aleph_2$ *is specialized.*

Thus (a)-(c) of Theorem 1.4 are satisfied. Let's prove Theorem 1.4.(d). The proof is similar to Todorcevic's proof in [15], and we present it here for completeness.

Work in $V[G]$. Let $\mathbb{P}$ be any forcing notion, and suppose forcing with $\mathbb{P}$ adds a new subset of $\aleph_2$ without collapsing it. We show that forcing with $\mathbb{P}$ collapses $\aleph_3$. Let $\mathbb{B} = RO(\mathbb{P})$. Also let $\underset{\sim}{\tau}$ be a name for a new subset of $\aleph_2$, so that

$$\|(\underset{\sim}{\tau} \subseteq \aleph_2) \wedge (\underset{\sim}{\tau} \notin V) \wedge (\forall \alpha < \aleph_2, \underset{\sim}{\tau} \cap \alpha \in V)\|_{\mathbb{B}} = 1$$

For $\alpha < \aleph_2$, set $a_{\alpha,0} = \|\alpha \in \underset{\sim}{\tau}\|_{\mathbb{B}}$ and $a_{\alpha,1} = \|\alpha \notin \underset{\sim}{\tau}\|_{\mathbb{B}}$. Let $T_0 = \{1_{\mathbb{B}}\}$, and for $0 < \alpha < \aleph_2$ set

$$T_\alpha = \{\bigwedge\{a_{\beta,f(\beta)} : \beta < \alpha\} : f \in {}^{\alpha}2, \bigwedge\{a_{\beta,f(\beta)} : \beta < \alpha\} \neq 0_{\mathbb{B}}\}.$$

By the assumption on $\underset{\sim}{\tau}$, each T_α is a partition of $1_{\mathbb{B}}$, for $\beta < \alpha, T_\alpha$ refines T_β and so $T = \bigcup\{T_\alpha : \alpha < \aleph_2\}$ is a tree of height $\aleph_2$, whose α-th level is T_α. Also clearly $|T| = 2^{\aleph_1} = \aleph_3$.

Claim 6.3. *For every* $0_{\mathbb{B}} \neq b \in \mathbb{B}$*, there exists* $\alpha < \aleph_2$ *such that*

$$|\{a \in T_\alpha : a \wedge b \neq 0_{\mathbb{B}}\}| > \aleph_2.$$

Proof. Suppose not. So we can find $0_{\mathbb{B}} \neq b \in \mathbb{B}$ such that for each $\alpha < \aleph_2$, $|\{a \in T_\alpha : a \wedge b \neq 0_{\mathbb{B}}\}| \leq \aleph_2$. Define a new tree $T^* = \bigcup\{T^*_\alpha : \alpha < \aleph_2\}$, where for each α,

$$T^*_\alpha = \{a \wedge b : a \in T_\alpha, a \wedge b > 0_{\mathbb{B}}\}.$$

Then T^* is an $\aleph_2$-tree of size $\aleph_2$, so it is specialized. But then

$$\|\dot{G}_{\mathbb{B}} \cap T^* \text{ is a new cofinal branch of } T^*\|_{\mathbb{B}} \geq b,$$

where $\dot{G}_{\mathbb{B}}$ is the canonical name for a generic ultrafilter over $\mathbb{B}$. This is impossible as T^* is specialized and forcing with $\mathbb{B}$ preserves $\aleph_2$. □

For each $\alpha < \aleph_2$, let $\langle a_\alpha(\xi) : \xi < \lambda_\alpha \leq \aleph_3 \rangle$ be an enumeration of T_α, and let $\underset{\sim}{f}$ be a name for a function from $\aleph_2$ into $\aleph_3$ defined by

$$\|\underset{\sim}{f}(\alpha) = \xi\|_{\mathbb{B}} = \begin{cases} a_\alpha(\xi) & \text{if } \xi < \lambda_\alpha, \\ 0_{\mathbb{B}} & \text{Otherwise.} \end{cases}$$

Claim 6.4. $\|$*range*$(\underset{\sim}{f})$ *is unbounded in* $\aleph_3\|_{\mathbb{B}} = 1_{\mathbb{B}}$.

Proof. Assume not. Then for some $\delta < \aleph_3$, $b = \|\text{range}(\underset{\sim}{f}) \subseteq \delta\|_{\mathbb{B}} > 0_{\mathbb{B}}$. By Claim 6.2, we can find $\alpha < \aleph_2$ such that $|\{a \in T_\alpha : a \wedge b \neq 0_{\mathbb{B}}\}| = \aleph_3$, so $\lambda_\alpha = \aleph_3$. Pick some $\xi > \delta$ so that $a_\alpha(\xi) \wedge b \neq 0_{\mathbb{B}}$. This implies

$$\|\underset{\sim}{f}(\alpha) = \xi\|_{\mathbb{B}} \wedge \|\text{range}(\underset{\sim}{f}) \subseteq \delta\|_{\mathbb{B}} \neq 0_{\mathbb{B}},$$

and we get a contradiction (as $\xi > \delta$). □

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