STRONGLY COMPCT DIAGONAL PRIKRY FORCING

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ABSTRACT. We define a version of Gitik-Sharon diagonal Prikry forcing using a strongly compact cardinal, and prove its basic properties.

1. Introduction

In [3], Gitik and Sharon introduced a new forcing notion, diagonal (supercompact) Prikry forcing, to answer some questions of Cummings, Foreman, Magidor and Woodin. So starting from a supercompact cardinal κ, they introduced a generic extension in which the following hold:

(1) κ is a singular limit cardinal of cofinality ω and $2^\kappa > \kappa^+$,

(2) There exists a very good scale at κ,

(3) There is a bad scale at κ.

In this paper we define a strongly compact version of Gitik-Sharon forcing that we call *strongly compact diagonal Prikry forcing*, prove its basic properties and show that it shares all properties of diagonal Prikry forcing.

2. Strongly compact diagonal Prikry forcing

In this section we define our *strongly compact diagonal Prikry forcing.* Assume κ is a strongly compact cardinal, and let

$$\kappa = \kappa_0 < \kappa_1 < \cdots < \kappa_n < \dots$$

be an increasing sequence of regular cardinals with limit κ_ω. Let U be a fine measure on $P_\kappa(\kappa_\omega^+)$, and for each $n < \omega$ let U_n be its projection to $P_\kappa(\kappa_n)$:

$$X \in U_n \Leftrightarrow X \subseteq P_\kappa(\kappa_n) \wedge \{P \in P_\kappa(\kappa_\omega^+) : P \cap \kappa_n \in X\} \in U.$$

Let

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$$K_n = \{P \in P_\kappa(\kappa_n) : P \cap \kappa \text{ is inaccessible } \}.$$

Then $K_n \in U_n$. Corresponding to the sequences $\bar{\kappa} = \langle \kappa_0, \dots, \kappa_n, \dots \rangle$ and $\bar{U} = \langle U_0, \dots, U_n, \dots \rangle$ we define the forcing notion $\mathbb{P} = \mathbb{P}_{\bar{\kappa}, \bar{U}}$ as follows.

Definition 2.1. *A condition in $\mathbb{P}$ is a finite sequence*

$$p = \langle P_0, \dots, P_{n-1}, T \rangle$$

where:

(1) *For $i < n, P_i \in K_i$,*

(2) *$P_0 \prec P_1 \prec \cdots \prec P_{n-1}$, where*

$$P \prec Q \Leftrightarrow otp(P) = \lambda_P < \kappa_Q = Q \cap \kappa,$$

(3) *T is a $\bar{U}$-tree with trunk $\langle P_0, \dots, P_{n-1} \rangle$, which means:*

  (a) *T is a tree, whose nodes are finite sequences $\langle Q_0, \dots, Q_{m-1} \rangle$, such that each $Q_i \in K_i$ and $Q_0 \prec Q_1 \prec \cdots \prec Q_{m-1}$, ordered by end extension,*

  (b) *The trunk of T is $t = \langle P_0, \dots, P_{n-1} \rangle$, which means $t \in T$ and for any $s \in T, s \trianglelefteq t$ or $t \trianglelefteq s$,*

  (c) *If $s = \langle Q_0, \dots, Q_{m-1} \rangle \trianglerighteq t$, then*

$$\mathrm{Suc}_T(s) = \{Q \in K_m : s^\frown \langle Q \rangle \in T\} \in U_m.$$

Given a condition $p \in \mathbb{P}$, we denote it by

$$p = \langle P^p_0, \dots, P^p_{\mathrm{lh}(p)-1}, T^p \rangle$$

and call $\mathrm{lh}(p)$ the length of p. We allow $\mathrm{lh}(p) = 0$, which just means p has no P's in its definition. We also call $\langle P^p_0, \dots, P^p_{\mathrm{lh}(p)-1} \rangle$ the lower part of p.

Definition 2.2. *Let T be a tree as above and $s \in T$. Then*

$$T_s = \{u \in T : u \trianglelefteq s \text{ or } s \trianglelefteq u\}.$$

Definition 2.3. *Let $p, q \in \mathbb{P}$. Then $p \leq q$ iff*

(1) $\mathrm{lh}(p) \geq \mathrm{lh}(q)$,

(2) *For all $i < \mathrm{lh}(q), P^p_i = P^q_i$,*

(3) *For all* $\mathrm{lh}(q) \leq i < \mathrm{lh}(p), P^p_i \in \mathrm{Suc}_{T^q}(\langle P^p_0, \dots P^p_{i-1}\rangle)$,

(4) $T^p \subseteq T^q_{\langle P^p_0,\dots,P^p_{\mathrm{lh}(p)-1}\rangle}$.

Definition 2.4. *Let* $p, q \in \mathbb{P}$. *We say* p *is a Prikry or a direct extension of* $q, p \leq^* q$, *iff* $p \leq q$ *and* $\mathrm{lh}(p) = \mathrm{lh}(q)$.

Before we continue, let us introduce a notation that will become useful later.

Notation 2.5. *Let* Ξ *be the tree of possible lower parts:*

$$\Xi = \{\langle P_0, \dots, P_{n-1}\rangle : n < \omega, P_i \in K_i, P_0 \prec \dots P_{n-1}\}.$$

Also we denote each $t \in \Xi$ *as* $t = \langle P^t_0, \dots P^t_{\mathrm{lh}(t)-1}\rangle$.

We now study the basic properties of the forcing notion $(\mathbb{P}, \leq, \leq^*)$.

Lemma 2.6. $(\mathbb{P}, \leq)$ *satisfies the* κ^+_ω*-c.c.*

Proof. This follows easily using the fact that if p and q have the same lower part, then they are compatible, and that

$$|\{\langle P^p_0, \dots, P^p_{\mathrm{lh}(p)-1}\rangle : p \in \mathbb{P}\}| \leq \kappa_\omega.$$

□

Lemma 2.7. $(\mathbb{P}, \leq^*)$ *is* κ*-closed.*

Proof. By the κ-completeness of U_n's. □

We now show that $(\mathbb{P}, \leq, \leq^*)$ is a Prikry type forcing notion.

Lemma 2.8. $(\mathbb{P}, \leq, \leq^*)$ *satisfies the Prikry property.*

Proof. Let $p \in \mathbb{P}$ and let σ be a statement of the forcing language $(\mathbb{P}, \leq)$. We find $q \leq^* p$ which decides σ. Assume this is not true.

Call a lower part $t = \langle P_0, \dots, P_{n-1}\rangle$ indecisive if there is no tree T with trunk t such that $p = \langle P_0, \dots, P_{n-1}, T\rangle \in \mathbb{P}$ and p decides σ. Otherwise t is called decisive. Note that by our assumption the lower part of p is indecisive.

Claim 2.9. *If $t = \langle P_0, \dots, P_{n-1}\rangle$ is indecisive, then*

$$\{P \in K_n : t^\frown\langle P\rangle \text{ is indecisive}\} \in U_n.$$

Proof. Assume otherwise, so

$$X = \{P \in K_n : t^\frown\langle P\rangle \text{ is decisive}\} \in U_n.$$

For $P \in X$ pick a tree T_P and $i < 2$ such that $q_P = \langle t^\frown\langle P\rangle, T_P\rangle \in \mathbb{P}$ and $q_P \Vdash {}^i\sigma$ (where ${}^0\sigma = \sigma$ and ${}^1\sigma = \neg\sigma$). Let $i < 2$ be such that

$$Y = \{P \in X : q_P \Vdash {}^i\sigma\} \in U_n.$$

Let T be a tree with trunk t, so that $\mathrm{Suc}_T(s) = Y$, and for each $P \in Y, T_{\langle t^\frown\langle P\rangle\rangle} = T_P$. Let $p = \langle t, T\rangle$. Then $p \in \mathbb{P}$, and any extension of p extends some $q_P, P \in Y$. It follows that $p \Vdash^i \sigma$, hence t is decisive, a contradiction. □

By the above claim and by induction, we can find a tree T with trunk $\langle P^p_0, \dots, P^p_{\mathrm{lh}(p)-1}\rangle$ such that all nodes $t \in T, t \trianglerighteq \langle P^p_0, \dots, P^p_{\mathrm{lh}(p)-1}\rangle$ are indecisive. Let $q = \langle P^p_0, \dots, P^p_{\mathrm{lh}(p)-1}, T\rangle$. Let $r \leq q$ and r decides σ. Then $\langle P^r_0, \dots, P^r_{\mathrm{lh}(r)-1}\rangle \in T$ and it is decisive, a contradiction. The lemma follows. □

Let G be $\mathbb{P}$-generic over V, and let $\langle P_i : i < \omega\rangle$ be the Prikry sequence added by G, where $P_i = P^p_i$, for some (and hence all) $p \in G$ wit $\mathrm{lh}(p) > i$. Then

$$P_0 \prec P_1 \prec \cdots \prec P_i \prec \dots.$$

Lemma 2.10. *For any $n \leq \omega$,*

$$\kappa_n = \bigcup\{P_i \cap \kappa_n : i < \omega\},$$

in particular all cardinals in (κ, κ_ω) are collapsed into κ.

Let us summarize the properties of forcing notion $\mathbb{P}$.

Theorem 2.11. *Let G be $\mathbb{P}$-generic over V. Then*

(*a*) $cf^{V[G]}(\kappa) = \omega$,

(*b*) $\kappa^{+V[G]} = \kappa^+_\omega$,

(*c*) *No bounded subsets of κ are added, in particular all cardinals $\leq \kappa$ are preserved.*

3. More on strongly compact diagonal Prikry forcing

In this section we prove some more properties of the forcing notion $\mathbb{P}$ introduced in the previous section. Let G be $\mathbb{P}$-generic over V, and let $\langle P_i : i < \omega \rangle$ be the corresponding Prikry generic sequence. It is easily seen that

$$G = \{p \in \mathbb{P} : \langle P^p_0, \dots, P^p_{\mathrm{lh}(p)-1} = \langle P_0, \dots, P_{\mathrm{lh}(p)-1} \rangle \text{ and } \forall i \geq \mathrm{lh}(p), P_i \in \mathrm{Suc}_{T^p}(\langle P_0, \dots, P_{i-1} \rangle)\},$$

hence $V[G] = V[\langle P_i : i < \omega \rangle]$.

Lemma 3.1. *(Diagonal intersection lemma) For each $t \in \Xi$, let T^t be a $\bar{U}$-tree with trunk t such that $\langle t, T^t \rangle \in \mathbb{P}$. Then there is a $\bar{U}$-tree S with trunk $\langle \rangle$, so that for each $t \in S$, $\langle t, S_t \rangle \leq \langle t, T^t \rangle$.*

Proof. Define the tree S by induction on levels so that for each $t \in S$,

$$\mathrm{Suc}_S(t) = \bigcap_{i \leq \mathrm{lh}(t)} \mathrm{Suc}_{T^{t \restriction i}}(t) \in K_{\mathrm{lh}(t)}.$$

We show that S is as required. Thus let $t \in S$. We need to show that $\langle t, S_t \rangle \leq \langle t, T^t \rangle$, i.e., $S_t \subseteq T^t$. Thus assume $t \trianglelefteq s \in S$. Then

$$s \in \mathrm{Suc}_S(s \restriction \mathrm{lh}(s) - 1) \subseteq \mathrm{Suc}_{T^t}(s \restriction \mathrm{lh}(s) - 1),$$

so $s \in T^t$. □

Lemma 3.2. *Assume $A \in V[G]$ is a set of ordinals of order type β, where $\omega < \beta = cf^V(\beta) < \kappa$. Then there exists an unbounded $B \subseteq A$ with $B \in V$.*

Proof. For each $p \in G$ set $A_p = \{\alpha : p \Vdash \alpha \in \dot{A}\}$. Then $A = \bigcup_{p \in G} A_p$. Note that in $V[G]$, $cf(\beta) = \beta > \omega$, so for some $n < \omega$, the set $A' = \bigcup_{p \in G, \mathrm{lh}(p) = n} A_p$ is an unbounded subset of A.

Let $f \in V[G], f : \beta \to A'$ enumerate A'. For each $\alpha < \beta$ let $p_\alpha = \langle P_0, \dots, P_{n-1}, T^\alpha \rangle \in \mathbb{P}$ be such that p_α decides $\dot{f}(\alpha)$, where $\langle P_0, \dots, P_i, \dots \rangle$ is the generic Prikry sequence. Let p be such that the lower part of p is $\langle P_0, \dots, P_{n-1} \rangle$ and for each $\langle P_0, \dots, P_{n-1} \rangle \trianglelefteq t \in T^p$, $\mathrm{Suc}_{T^p}(t) = \bigcap_{\alpha < \beta} \mathrm{Suc}_{T^\alpha}(t)$.

Then $p \in \mathbb{P}$ and p decides $\dot{f}$. The result follows immediately. □

Lemma 3.3. *(Bounding lemma) Assume* $\forall n < \omega, \kappa_n = \kappa^{+n}$ *(recall* $\langle \kappa_n : n < \omega \rangle$ *is the sequence we fixed at the beginning). Let* $\eta : \omega \to \kappa$ *be such that* $\eta(n) > n$ *is a successor ordinal. Let* $\langle P_i : i < \omega \rangle$ *be the Prikry generic sequence, and let* $h \in V[\langle P_i : i < \omega \rangle]$ *with* $h \in \prod_{i<\omega} \kappa_{P_i}^{+\eta(i)}$*. Then there exists* $\langle H_i : i < \omega \rangle \in V$*, so that:*

(1) *For each* i, $\mathrm{dom}(H_i) = K_i$,

(2) *For all* $Q \in \mathrm{dom}(H_i)$, $H_i(Q) < \kappa_Q^{+\eta(i)}$,

(3) *For all large* i, $h(i) < H_i(P_i)$.

Proof. Assume for simplicity that the trivial condition forces $\dot{h}$ is as in the statement of the lemma. For any $t \in \Xi$, by the Prikry property, let $q_t = \langle t, H^t \rangle \in \mathbb{P}$ be such that q_t decides $\dot{h}(\mathrm{lh}(t) - 1)$, say $q_t \Vdash \dot{h}(\mathrm{lh}(t) - 1) = g(t) < \kappa_{P^t_{\mathrm{lh}(t)-1}}^{+\eta(\mathrm{lh}(t)-1)}$.

By diagonal intersection lemma, we can find a tree S so that for each $t \in S, \langle t, S_t \rangle \leq q_t$. Let $p = \langle \langle\rangle, S \rangle$. Then for any $i < \omega$,

$$p \Vdash \dot{h}(i) = g(\langle P_0, \dots, P_i \rangle).$$

For any $i < \omega$ let $\mathrm{dom}(H_i) = K_i$, and for $Q \in K_i$ set

$$H_i(Q) = \sup\{g(t) : t \in \Xi, \mathrm{lh}(t) = i + 1, P^t_i = Q\} + 1.$$

By a simple counting argument, $H_i(Q) \leq \kappa_Q^{+i} < \kappa_Q^{+\eta(i)}$.

□

References

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