STRONGLY COMPACT DIAGONAL PRIKRY FORCING

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Abstract. We define a version of Gitik-Sharon diagonal Prikry forcing using a strongly compact cardinal, and prove its basic properties.

1. Introduction

In [3], Gitik and Sharon introduced a new forcing notion, diagonal (supercompact) Prikry forcing, to answer some questions of Cummings, Foreman, Magidor and Woodin. So starting from a supercompact cardinal $\kappa$, they introduced a generic extension in which the following hold:

1. $\kappa$ is a singular limit cardinal of cofinality $\omega$ and $2^\kappa > \kappa^+$,
2. There exists a very good scale at $\kappa$,
3. There is a bad scale at $\kappa$.

In this paper we define a strongly compact version of Gitik-Sharon forcing that we call strongly compact diagonal Prikry forcing, prove its basic properties and show that it shares all properties of diagonal Prikry forcing.

2. Strongly compact diagonal Prikry forcing

In this section we define our strongly compact diagonal Prikry forcing. Assume $\kappa$ is a strongly compact cardinal, and let

$$\kappa = \kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$$

be an increasing sequence of regular cardinals with limit $\kappa_\omega$. Let $U$ be a fine measure on $P_\kappa(\kappa_\omega^+)$, and for each $n < \omega$ let $U_n$ be its projection to $P_\kappa(\kappa_n)$:

$$X \in U_n \iff X \subseteq P_\kappa(\kappa_n) \land \{ P \in P_\kappa(\kappa_\omega^+) : P \cap \kappa_n \in X \} \in U.$$

Let

The first author’s research was in part supported by a grant from IPM (No. 98030417).
\[ K_n = \{ P \in P_\kappa(\kappa_n) : P \cap \kappa \text{ is inaccessible} \} \).

Then \( K_n \in U_n \). Corresponding to the sequences \( \bar{\kappa} = \langle \kappa_0, \ldots, \kappa_n, \ldots \rangle \) and \( \bar{U} = \langle U_0, \ldots, U_n, \ldots \rangle \)
we define the forcing notion \( P = P_{\bar{\kappa}, \bar{U}} \) as follows.

**Definition 2.1.** A condition in \( P \) is a finite sequence
\[ p = \langle P_0, \ldots, P_{n-1}, T \rangle \]
where:

1. For \( i < n \), \( P_i \in K_i \),
2. \( P_0 \prec P_1 \prec \cdots \prec P_{n-1} \), where \( P \prec Q \iff \text{otp}(P) = \lambda_P < \kappa_Q = Q \cap \kappa \),
3. \( T \) is a \( \bar{U} \)-tree with trunk \( \langle P_0, \ldots, P_{n-1} \rangle \), which means:
   a. \( T \) is a tree, whose nodes are finite sequences \( \langle Q_0, \ldots, Q_{m-1} \rangle \), such that each \( Q_i \in K_i \) and \( Q_0 \prec Q_1 \prec \cdots \prec Q_{m-1} \), ordered by end extension,
   b. The trunk of \( T \) is \( t = \langle P_0, \ldots, P_{n-1} \rangle \), which means \( t \in T \) and for any \( s \in T, s \leq t \) or \( t \leq s \),
   c. If \( s = \langle Q_0, \ldots, Q_{m-1} \rangle \triangleright t \), then \( \text{Suc}_T(s) = \{ Q \in K_m : s \triangleright (Q) \in T \} \in U_m \).

Given a condition \( p \in P \), we denote it by
\[ p = \langle P_0^p, \ldots, P_{\text{lh}(p)-1}^p, T^p \rangle \]
and call \( \text{lh}(p) \) the length of \( p \). We allow \( \text{lh}(p) = 0 \), which just means \( p \) has no \( P \)'s in its definition. We also call \( \langle P_0^p, \ldots, P_{\text{lh}(p)-1}^p \rangle \) the lower part of \( p \).

**Definition 2.2.** Let \( T \) be a tree as above and \( s \in T \). Then
\[ T_s = \{ u \in T : u \leq s \text{ or } s \leq u \} \).

**Definition 2.3.** Let \( p, q \in P \). Then \( p \leq q \) iff

1. \( \text{lh}(p) \geq \text{lh}(q) \),
2. For all \( i < \text{lh}(q) \), \( P_i^p = P_i^q \).
(3) For all \( \text{lh}(q) \leq i < \text{lh}(p) \), \( P_i^p \in \text{Suc}_{T_q}(\langle P_0^p, \ldots, P_i^p \rangle) \),

(4) \( T_p \subseteq T_{(P_0^p, \ldots, P_{\text{lh}(p)-1}^p)} \).

**Definition 2.4.** Let \( p, q \in \mathbb{P} \). We say \( p \) is a Prikry or a direct extension of \( q \), \( p \preceq^* q \), iff \( p \leq q \) and \( \text{lh}(p) = \text{lh}(q) \).

Before we continue, let us introduce a notation that will become useful later.

**Notation 2.5.** Let \( \Xi \) be the tree of possible lower parts:

\[ \Xi = \{ \langle P_0, \ldots, P_{n-1} \rangle : n < \omega, P_i \in K_i, P_0 \prec \ldots P_{n-1} \} \].

Also we denote each \( t \in \Xi \) as \( t = \langle P_0^t, \ldots, P_{\text{lh}(t)-1}^t \rangle \).

We now study the basic properties of the forcing notion \( (\mathbb{P}, \leq, \leq^*) \).

**Lemma 2.6.** \( (\mathbb{P}, \leq) \) satisfies the \( \kappa^{\omega^+} \)-c.c.

**Proof.** This follows easily using the fact that if \( p \) and \( q \) have the same lower part, then they are compatible, and that

\[ |\{ \langle P_0^p, \ldots, P_{\text{lh}(p)-1}^p \rangle : p \in \mathbb{P} \}| \leq \kappa^\omega. \]

\( \square \)

**Lemma 2.7.** \( (\mathbb{P}, \leq^*) \) is \( \kappa \)-closed.

**Proof.** By the \( \kappa \)-completeness of \( U_n \)'s. \( \square \)

We now show that \( (\mathbb{P}, \leq, \leq^*) \) is a Prikry type forcing notion.

**Lemma 2.8.** \( (\mathbb{P}, \leq, \leq^*) \) satisfies the Prikry property.

**Proof.** Let \( p \in \mathbb{P} \) and let \( \sigma \) be a statement of the forcing language \( (\mathbb{P}, \leq) \). We find \( q \leq^* p \) which decides \( \sigma \). Assume this is not true.

Call a lower part \( t = \langle P_0, \ldots, P_{n-1} \rangle \) indecisive if there is no tree \( T \) with trunk \( t \) such that \( p = \langle P_0, \ldots, P_{n-1}, T \rangle \in \mathbb{P} \) and \( p \) decides \( \sigma \). Otherwise \( t \) is called decisive. Note that by our assumption the lower part of \( p \) is indecisive.
Claim 2.9. If $t = \langle P_0, \ldots, P_{n-1} \rangle$ is indecisive, then
\[
\{ P \in K_n : t \prec \langle P \rangle \text{ is indecisive} \} \in U_n.
\]

Proof. Assume otherwise, so
\[
X = \{ P \in K_n : t \prec \langle P \rangle \text{ is decisive} \} \in U_n.
\]

For $P \in X$ pick a tree $T_P$ and $i < 2$ such that $q_P = \langle t \prec \langle P \rangle, T_P \rangle \in \mathbb{P}$ and $q_P \Vdash i \sigma$ (where $0 \sigma = \sigma$ and $1 \sigma = \neg \sigma$). Let $i < 2$ be such that
\[
Y = \{ P \in X : q_P \Vdash i \sigma \} \in U_n.
\]

Let $T$ be a tree with trunk $t$, so that $\text{Suc}_T(s) = Y$, and for each $P \in Y, T(t \prec \langle P \rangle) = T_P$. Let $p = \langle t, T \rangle$. Then $p \in \mathbb{P}$, and any extension of $p$ extends some $q_P, P \in Y$. It follows that $p \Vdash i \sigma$, hence $t$ is decisive, a contradiction. \hfill \Box

By the above claim and by induction, we can find a tree $T$ with trunk $\langle P^p_0, \ldots, P^p_{\text{lh}(p)-1} \rangle$ such that all nodes $t \in T, t \supset \langle P^p_0, \ldots, P^p_{\text{lh}(p)-1} \rangle$ are indecisive. Let $q = \langle P^p_0, \ldots, P^p_{\text{lh}(p)-1}, T \rangle$. Let $r \leq q$ and $r$ decides $\sigma$. Then $\langle P^r_0, \ldots, P^r_{\text{lh}(r)-1} \rangle \in T$ and it is decisive, a contradiction. The lemma follows. \hfill \Box

Let $G$ be $\mathbb{P}$-generic over $V$, and let $\langle P_i : i < \omega \rangle$ be the Prikry sequence added by $G$, where $P_i = P^p_i$, for some (and hence all) $p \in G$ with $\text{lh}(p) > i$. Then
\[
P_0 \prec P_1 \prec \cdots \prec P_i \prec \ldots.
\]

Lemma 2.10. For any $n \leq \omega$,
\[
\kappa_n = \bigcup \{ P_i \cap \kappa_n : i < \omega \},
\]
in particular all cardinals in $(\kappa, \kappa_\omega)$ are collapsed into $\kappa$.

Let us summarize the properties of forcing notion $\mathbb{P}$.

Theorem 2.11. Let $G$ be $\mathbb{P}$-generic over $V$. Then

(a) $\text{cf}^V[G](\kappa) = \omega$,

(b) $\kappa^+^V[G] = \kappa^+_\omega$,

(c) No bounded subsets of $\kappa$ are added, in particular all cardinals $\leq \kappa$ are preserved.
3. More on strongly compact diagonal Prikry forcing

In this section we prove some more properties of the forcing notion $P$ introduced in the previous section. Let $G$ be $P$-generic over $V$, and let $\langle P_i : i < \omega \rangle$ be the corresponding Prikry generic sequence. It is easily seen that

$$G = \{ p \in P : \langle P_0, \ldots, P_{\text{lh}(p) - 1} \rangle \text{ and } \forall i \geq \text{lh}(p), P_i \in \text{Suc}_{T_p}(\langle P_0, \ldots, P_{i-1} \rangle) \},$$

hence $V[G] = V[\langle P_i : i < \omega \rangle]$.

**Lemma 3.1.** (Diagonal intersection lemma) For each $t \in \Xi$, let $T^t$ be a $\bar{U}$-tree with trunk $t$ such that $\langle t, T^t \rangle \in P$. Then there is a $\bar{U}$-tree $S$ with trunk $\langle \rangle$, so that for each $t \in S$, $\langle t, S_t \rangle \leq \langle t, T^t \rangle$.

**Proof.** Define the tree $S$ by induction on levels so that for each $t \in S$,

$$\text{Suc}_S(t) = \bigcap_{i \leq \text{lh}(t)} \text{Suc}_{T^{S_t}}(t) \in K_{\text{lh}(t)}.$$

We show that $S$ is as required. Thus let $t \in S$. We need to show that $\langle t, S_t \rangle \leq \langle t, T^t \rangle$, i.e., $S_t \subseteq T^t$. Thus assume $t \unlhd s \in S$. Then

$$s \in \text{Suc}_S(s \upharpoonright \text{lh}(s) - 1) \subseteq \text{Suc}_{T^t}(s \upharpoonright \text{lh}(s) - 1),$$

so $s \in T^t$. \hfill $\Box$

**Lemma 3.2.** Assume $A \in V[G]$ is a set of ordinals of order type $\beta$, where $\omega < \beta = cf^V(\beta) < \kappa$. Then there exists an unbounded $B \subseteq A$ with $B \in V$.

**Proof.** For each $p \in G$ set $A_p = \{ \alpha : p \models \alpha \in A \}$. Then $A = \bigcup_{p \in G} A_p$. Note that in $V[G]$, $cf(\beta) = \beta > \omega$, so for some $n < \omega$, the set $A' = \bigcup_{p \in G, \text{lh}(p) = n} A_p$ is an unbounded subset of $A$.

Let $f \in V[G], f : \beta \to A'$ enumerate $A'$. For each $\alpha < \beta$ let $p_\alpha = \langle P_0, \ldots, P_{n-1}, T^\alpha \rangle \in P$ be such that $p_\alpha$ decides $f(\alpha)$, where $\langle P_0, \ldots, P_{i-1} \rangle$ is the generic Prikry sequence. Let $p$ be such that the lower part of $p$ is $\langle P_0, \ldots, P_{n-1} \rangle$ and for each $\langle P_0, \ldots, P_{n-1} \rangle \unlhd t \in T^p$, $\text{Suc}_{T^p}(t) = \bigcap_{\alpha < \beta} \text{Suc}_{T^{\alpha}}(t)$.

Then $p \in P$ and $p$ decides $\dot{f}$. The result follows immediately. \hfill $\Box$
Lemma 3.3. (Bounding lemma) Assume \( \forall n < \omega, \kappa_n = \kappa^{+n} \) (recall \( \langle \kappa_n : n < \omega \rangle \) is the sequence we fixed at the beginning). Let \( \eta : \omega \to \kappa \) be such that \( \eta(n) > n \) is a successor ordinal. Let \( \langle P_i : i < \omega \rangle \) be the Prikry generic sequence, and let \( h \in V[\langle P_i : i < \omega \rangle] \) with \( h \in \prod_{i<\omega} \kappa_{P_i}^{+\eta(i)} \). Then there exists \( \langle H_i : i < \omega \rangle \in V \), so that:

1. For each \( i \), \( \text{dom}(H_i) = K_i \),
2. For all \( Q \in \text{dom}(H_i) \), \( H_i(Q) < \kappa_Q^{+\eta(i)} \),
3. For all large \( i \), \( h(i) < H_i(P_i) \).

Proof. Assume for simplicity that the trivial condition forces \( \dot{h} \) is as in the statement of the lemma. For any \( t \in \Xi \), by the Prikry property, let \( q_t = \langle t, H^t \rangle \in \mathbb{P} \) be such that \( q_t \) decides \( \dot{h}(lh(t) - 1) \), say \( q_t \Vdash \dot{h}(lh(t) - 1) = g(t) < \kappa_{P_{lh(t)}}^{+\eta(lh(t)-1)} \).

By diagonal intersection lemma, we can find a tree \( S \) so that for each \( t \in S \), \( \langle t, S_t \rangle \leq q_t \).

Let \( p = \langle (\langle \rangle), S \rangle \). Then for any \( i < \omega \),

\[ p \Vdash \dot{h}(i) = g(\langle P_0, \ldots, P_i \rangle). \]

For any \( i < \omega \) let \( \text{dom}(H_i) = K_i \), and for \( Q \in K_i \) set

\[ H_i(Q) = \sup \{ g(t) : t \in \Xi, lh(t) = i + 1, P_{i}^t = Q \} + 1. \]

By a simple counting argument, \( H_i(Q) \leq \kappa_Q^{+i} < \kappa_Q^{+\eta(i)} \).

\[ \square \]

References


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