ON $C_n^s(\kappa)$ AND THE JUHASZ-KUNEN QUESTION

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Abstract. We generalize the combinatorial principles $C_n(\kappa), C_n^s(\kappa)$ and $Princ(\kappa)$ introduced by various authors, and prove some of their properties and connections between them. We also answer a question asked by Juhasz-Kunen about the relation between these principles, by showing that $C_n(\kappa)$ does not imply $C_{n+1}(\kappa)$, for any $n > 2$. We also show the consistency of $C(\kappa) + \neg C^s(\kappa)$.

1. introduction

In this paper, we consider some combinatorial principles introduced in [1], [3], [4], [5], [6] and [7], present some generalization of them and prove some of their properties and connections between them. We also answer a question asked by Juhasz-Kunen [4] about the relation between these principles.

The work in this direction, has started by the work of Juhasz-Soukup-Szentmiklossy [5], where the authors introduced several combinatorial principles, which all hold in the Cohen-real generic extensions. Among other things, in particular they introduced the combinatorial principles $C^s(\kappa)$, $C(\kappa)$, and their restrictions $C^s_m(\kappa)$ and $C_m(\kappa)$, for $m < \omega$. The work of Juhasz-Kunen [4] has continued the work, by introducing some extra principles, like SEP, and discussing their relations. In particular, Juhasz and Kunen showed that $SEP \Rightarrow C^s_2(\aleph_2)$, while $C^s(\aleph_2) \nRightarrow SEP$. On the other hand, in [7], Shelah introduced a new combinatorial principle $Princ(\kappa)$, which is weaker than SEP, but still enough strong to imply $C^s(\kappa)$.

It turned out that these combinatorial principles are very useful, and have many applications, in particular in topology and the study of cardinal invariants, see [1], [3] and [6].

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The question of the difference between $C^s_n(\kappa)$ and $C^s_{n+1}(\kappa)$ remained open by Juhasz-Kunen [4] , and was also asked by Juhasz during the Beer-Sheva 2001 conference.

In this paper we consider some of these combinatorial principles, and present a natural generalization of them. We discuss the relation between them and also their consistency. We also address the above mentioned question of Juhasz-Kunen in section 5, and give a complete solution to it. In the last section, we discuss the relation between $C^s(\kappa)$ and $C(\kappa)$, and show the consistency of $C(\kappa) + \neg C^s(\kappa)$.

2. On $Princ(\kappa)$ and its generalizations

In this section, we consider the combinatorial principle $Prin(\kappa)$ introduced by Shelah [7], and present some of its generalizations.

Definition 2.1. *Let κ be regular uncountable, $A \supseteq \kappa$, and let D be a filter on $[A]^{<\kappa}$. D is called normal if*

1. *For all $a \in [A]^{<\kappa}, \{b \in [A]^{<\kappa} : a \subseteq b\} \in D$,*
2. *If for $x \in A, A_x \in D$, then $\triangle_{x\in A} A_x \in D$, where*

$$\triangle_{x\in A} A_x = \{a \in [A]^{<\kappa} : \forall x \in a, a \in A_x\}.$$

It is easily seen that if D is a normal filter on $[A]^{<\kappa}, X \neq \emptyset \mod D$ and if $F : X \to A$ is regressive, i.e., for all non-empty $a \in [A]^{<\kappa}, F(a) \in a$, then there are $Y \subseteq X, Y \neq \emptyset \mod D$ and $x \in A$ such that for all $a \in Y, F(a) = x$. To see this, assume on the contrary that for each $x \in A$, there exists $Y_x \in D$ such that $Y_x \cap \{a \in X : F(a) = x\} = \emptyset$. Let $Y = \triangle_{x\in A} Y_x$. Then $Y \in D$ and so $Y \cap X \neq \emptyset$ (as $X \neq \emptyset \mod D$). Let $a \in Y \cap X$ and $F(a) = x$. Then $a \in Y_x \cap \{a \in X : F(a) = x\}$, a contradiction.

Definition 2.2. *Let κ be regular uncountable. $\mathfrak{D}$ is a κ-definition of normal filters, if*

1. *For each $A \supseteq \kappa, \mathfrak{D}(A)$ is a normal filter on $[A]^{<\kappa}$,*
2. *If $\kappa \subseteq A_1 \subseteq A_2$, then $\mathfrak{D}(A_1) = \{\{a \cap A_1 : a \in X\} : X \in \mathfrak{D}(A_2)\}$.*

Definition 2.3. *Let κ be regular uncountable, $\theta < \lambda \leq \kappa$ and let $\chi > \kappa$ be large enough regular. Then*

(a) $\mathbf{N}^1_{\kappa,\lambda,\chi}$ *consists of those $N \prec (H(\chi), \in)$ such that:*

(1) $|N| \leq N \cap \kappa \in \kappa$,

(2) *For all* $a \in P(\omega)$, *there exists* $P \in N$, *such that* $P \subseteq P(\omega), |P| < \min\{|N|^+, \lambda\}$, *and for all* $b \in P(\omega) \cap N, a \subseteq b \Rightarrow \exists c \in P, a \subseteq c \subseteq b$ *(such a* P *is called an* N*-witness for* a*).*

(*b*) $\mathbf{N}^2_{\kappa,\lambda,\theta,\chi}$ *consists of those* $N \in \mathbf{N}^1_{\kappa,\lambda,\chi}$ *such that for any* θ*-sequence* $\langle a_\xi : \xi < \theta \rangle$ *of subsets of* ω, *there is some* $P \in N$, $P \subseteq P(\omega), |P| < \min\{|N|^+, \lambda\}$, *such that* P *is an* N*-witness for all* $a_\xi, \xi < \theta$ *simultaneously.*

(*c*) $\mathbf{N}^3_{\kappa,\lambda,\theta,\chi}$ *consists of those* $N \in \mathbf{N}^1_{\kappa,\lambda,\chi}$ *such that for each* $Y \in [N]^\theta$, *there exists some* $Z \in N, |Z| < \min\{|N|^+, \lambda\}$ *such that* $Y \subseteq Z$.

We now state our generalization of $Princ(\kappa)$

Definition 2.4. *Let* $\theta < \lambda \leq \kappa$ *and* $\mathfrak{D}$ *be as above.*

(*a*) $Princ_1(\kappa, \lambda, \mathfrak{D})$ *sates: for all large enough* $\chi > \kappa$,

$$\mathbf{N}^1_{\kappa,\lambda,\chi} \neq \emptyset \ mod\ \mathfrak{D}(H(\chi)).$$

(*b*) $Princ_{l,\theta}(\kappa, \lambda, \mathfrak{D})$ *(for* $l = 2, 3$*) sates: for all large enough* $\chi > \kappa$,

$$\mathbf{N}^l_{\kappa,\lambda,\theta,\chi} \neq \emptyset \ mod\ \mathfrak{D}(H(\chi)).$$

Remark 2.5. (*a*) *Let* κ *be regular uncountable, and for* $A \supseteq \kappa$ *let* $\mathfrak{D}(A)$ *be the club filter on* $[A]^{<\kappa}$. *Then our* $Princ_1(\kappa, \lambda, \mathfrak{D})$ *is just* $Princ(\kappa, \lambda)$ *from* [1]. *Also note that Shelah's* $Princ(\kappa)$ *is* $Princ(\kappa, \kappa)$.

(*b*) *If* $\theta < \lambda \leq \lambda' \leq \kappa$, *then* $Princ_1(\kappa, \lambda, \mathfrak{D}) \Rightarrow Princ_1(\kappa, \lambda', \mathfrak{D})$ *and* $Princ_{l,\theta}(\kappa, \lambda, \mathfrak{D}) \Rightarrow Princ_{l,\theta}(\kappa, \lambda', \mathfrak{D})$ *(for* $l = 2, 3$*).*

(*c*) *If* $\theta \leq \theta' < \lambda \leq \kappa$, *then* $Princ_{l,\theta'}(\kappa, \lambda, \mathfrak{D}) \Rightarrow Princ_{l,\theta}(\kappa, \lambda, \mathfrak{D})$ *(for* $l = 2, 3$*).*

(*d*) *If* $\lambda = \mu^+$ *is a successor cardinal, then we can replace* $\min\{|N|^+, \lambda\}$ *by* λ.

The next lemma follows from the definition, and the fact that we can code an ω-sequence of subsets of ω into a subset of ω.

Lemma 2.6. *Let* $\theta < \lambda \leq \kappa$ *and* $\mathfrak{D}$ *be as above. Then*

(*a*) $Princ_{3,\theta}(\kappa, \lambda, \mathfrak{D}) \Rightarrow Princ_{2,\theta}(\kappa, \lambda, \mathfrak{D}) \Rightarrow Princ_1(\kappa, \lambda, \mathfrak{D})$.

(*b*) $Princ_{2,\omega}(\kappa, \lambda, \mathfrak{D}) \Leftrightarrow Princ_1(\kappa, \lambda, \mathfrak{D})$.

Proof. (a) is by definition, let's prove (b). It suffices to show that

$$\mathbf{N}^2_{\kappa,\lambda,\omega,\chi} = \mathbf{N}^1_{\kappa,\lambda,\chi}.$$

Let $\Gamma : \omega \times \omega \to \omega$ be the Godel pairing function. Let $N \in \mathbf{N}^1_{\kappa,\lambda,\chi}$, and suppose that $\langle a_n : n < \omega \rangle$ is a sequence of subsets of ω. Let

$$a^* = \{\Gamma(i, n) : n < \omega, i \in a_n\}.$$

Let $P^* \in N$ be an N-witness for a^*. Let

$$P = \{\{i : \Gamma(i, n) \in b\} : n < \omega, b \in P^*\}.$$

We show that P is an N-witness for all $a_n, n < \omega$, simultaneously. Clearly $P \in N, P \subseteq P(\omega) \cap N$ and $|P| < \min\{|N|^+, \lambda\}$. Now let $n < \omega$, $b \in P(\omega) \cap N$ and assume $a_n \subseteq b$. Let $b^{[n]} = \{\Gamma(i, m) : i, m < \omega$ and $m = n \Rightarrow i \in b\}$. Clearly $b^{[n]} \in P(\omega) \cap N$ and $a^* \subseteq b^{[n]}$. Hence by the choice of P^*, there is $c \in P^*$ such that $a^* \subseteq c \subseteq b^{[n]}$. Let $c^{[n]} = \{i : \Gamma(i, n) \in c\}$. Then $c^{[n]} \in P$, and we can easily see that $a_n \subseteq c^{[n]} \subseteq b$. We are done. □

3. On $C^s(\kappa)$ and its generalizations

Recall that for a filter D on a set I, D^+ is defined by

$$D^+ = \{X \subseteq I : I \setminus X \notin D\}.$$

It is clear that $D \subseteq D^+$.

Definition 3.1. *Suppose κ is regular uncountable, D is a filter on κ, J is an ideal on ω and T is a subtree of $\theta^{<\omega}$.*

- (a) *The combinatorial principle $C^D_T(\kappa, J)$ states: for any $(\kappa \times \theta)$-matrix $\bar{A} = \langle a_{\alpha,\xi} : \alpha < \kappa, \xi < \theta \rangle$ of subsets of ω, one of the following holds:*
  - (α) *: There exists $S \in D^+$ such that for all $t \in T \cap \theta^n$ and all distinct $\alpha_0, \dots, \alpha_{n-1} \in S$, $\bigcap_{i<n} a_{\alpha_i,t(i)} \neq \emptyset$ mod J.*
  - (β) *: There are $t \in T \cap \theta^n$, for some $0 < n < \omega$, and $S_0, \dots, S_{n-1} \in D^+$ such that for all distinct $\alpha_i \in S_i, i < n$ we have $\bigcap_{i<n} a_{\alpha_i,t(i)} = \emptyset$ mod J.*
- (b) *$C^D(\kappa, J)$ is $C^D_T(\kappa, J)$ for all trees $T \subseteq \theta^{<\omega}$.*

(c) *For $m<\omega$, the combinatorial principles $C^D_{T,m}(\kappa, J)$ and $C^D_m(\kappa, J)$ are defined similarly, where we require $T \subseteq \theta^{\leq m}$.*

Remark 3.2. *Suppose that κ is regular uncountable and $m<\omega$.*

(a) *If D is the club filter on κ, and $J=\{\emptyset\}$, then $C^D(\kappa, J), C^D_m(\kappa, J)$ are respectively the principles $C^s(\kappa), C^s_m(\kappa)$ from [5].*

(b) *If D is the filter of co-bounded subsets of κ, and $J=\{\emptyset\}$, then $C^D(\kappa, J), C^D_m(\kappa, J)$ are respectively the principles $C(\kappa), C_m(\kappa)$ from [5].*

Theorem 3.3. *Assume $\theta<\kappa\leq 2^{\aleph_0}, \kappa$ is regular, J is an ideal on ω, T is a subtree of $\theta^{<\omega}$, and suppose that $Prin_{2,\theta}(\kappa,\kappa,\mathfrak{D})$ holds, where $\mathfrak{D}$ is a definition of κ-normal filters. Then $C^D_T(\kappa, J)$ holds, where D is any filter on κ satisfying: for $X\in D$ and $N\in \mathbf{N}^2_{\kappa,\lambda,\theta,\mathfrak{D},\chi}$ with $D\in N$, $X\in N \Rightarrow \delta(N)=N\cap\kappa\in X$.*

Remark 3.4. *If D is the club filter on κ or the filter of co-bounded subsets of κ, then D has the above mentioned property.*

Proof. Let $\bar{A}=\langle a_{\alpha,\xi}:\alpha<\kappa, \xi<\theta\rangle$ be a $(\kappa\times\theta)$-matrix of subsets of ω. Let $\chi>2^{\aleph_0}$ be large enough regular. By our assumption

$$\mathbf{N}^2_{\kappa,\kappa,\theta,\chi}\neq\emptyset \text{ mod } (\mathfrak{D}(H(\chi)))^+.$$

Hence by normality of the filter,

$$\mathcal{N}=\{N\in \mathbf{N}^2_{\kappa,\kappa,\theta,\chi}: D, \bar{A}\in N\}\in \mathfrak{D}(H(\chi))^+.$$

For $N\in\mathcal{N}$, set $\delta(N)=N\cap\kappa\in\kappa$. By our assumption, for each $N\in\mathcal{N}$, we can find $P_N\in N$ such that P_N is an N-witness for each $a_{\delta(N),\xi}, \xi<\theta$, simultaneously. Then the map $N\mapsto P_N$ is regressive on $\mathcal{N}$, so by the normality of the filter $\mathfrak{D}(H(\chi))$, we can find $\mathcal{N}_*\subseteq\mathcal{N}$ and P_* such that $\mathcal{N}_*\in\mathfrak{D}(H(\chi))^+$, and for all $N\in\mathcal{N}_*, P_N=P_*$. Let

$$S=\{\delta(N): N\in\mathcal{N}_*\}.$$

Claim 3.5. $S\in D^+$.

Proof. Suppose not; so $\kappa\setminus S\in D$. But then for all $N\in\mathcal{N}_*$,

$$\kappa \setminus S \in N \Rightarrow \delta(N) \in \kappa \setminus S.$$

On the other hand by normality of the filter $\mathfrak{D}(H(\chi))$, we have

$$\mathcal{N}_{**} = \{N \in \mathcal{N}_* : \kappa \setminus S \in N\} \in \mathfrak{D}(H(\chi))^+,$$

in particular $\mathcal{N}_{**} \neq \emptyset$. Let $N \in \mathcal{N}_{**}$. Then we have $\kappa \setminus S \in D$, which implies $\delta(N) \in \kappa \setminus S$. But on the other hand $N \in \mathcal{N}_*$ (as $\mathcal{N}_{**} \subseteq \mathcal{N}_*$), which implies $\delta(N) \in S$, a contradiction. □

If for all $t \in T \cap \theta^n$ and all distinct $\alpha_0, \ldots, \alpha_{n-1} \in S$, we have $\bigcap_{i<n} a_{\alpha_i, t(i)} \neq \emptyset$ mod J, then case (α) of Definition 3.1(a) holds and we are done. Otherwise, we can find $t \in T \cap \theta^n$ and distinct $\alpha_0, \ldots, \alpha_{n-1} \in S$, such that $\bigcap_{i<n} a_{\alpha_i, t(i)} = \emptyset$ mod J.

For each $i < n$, set $N_i \in \mathcal{N}_*$ be such that $\alpha_i = \delta(N_i)$. We also assume w.l.o.g. that $\alpha_0 < \cdots < \alpha_{n-1}$.

Claim 3.6. *There are $c_0, \ldots, c_{n-1} \in P_*$ such that:*

(1) $\bigcap_{i<n} c_i = \emptyset$ *mod* J,

(2) $i < n \Rightarrow a_{\alpha_i, t(i)} \subseteq c_i$ *mod* J.

Proof. We construct the sets $c_i, i < n$, by downward induction on i, so that for all $i < n$,

$(*)_i$ $$\bigcap_{j<i} a_{\alpha_j, t(j)} \cap \bigcap_{i \leq j < n} c_j = \emptyset \text{ mod } J.$$

For $i = n$, there is nothing to prove; thus suppose that $i < n$ and $c_{i+1} \in P_*$ is defined, so that $(*)_{i+1}$ is satisfied. It then follows that

$$a_{\alpha_i, t(i)} \subseteq b_i = \omega \setminus (\bigcap_{j<i} a_{\alpha_j, t(j)} \cap \bigcap_{i+1 \leq j < n} c_j) \text{ mod } J.$$

It is easily seen that $b_i \in N_i$, so as P_* is an N_i-witness for $a_{\alpha_i, t(i)}$, we can find $c_i \in P_*$ so that

$$a_{\alpha_i, t(i)} \subseteq c_i \subseteq b_i \text{ mod } J.$$

It is easily seen that $c_0, \ldots, c_{n-1}$ are as required. □

For $i < n$, set

$$S_i = \{\alpha \in \kappa : a_{\alpha, t(i)} \subseteq c_i \text{ mod } J\} \in N_i.$$

Claim 3.7. *For each $i < n, S_i \in D^+$.*

Proof. Suppose not; then $\kappa \setminus S_i \in D$. But as $\kappa \setminus S_i \in N_i$, we have $\alpha_i = \delta(N_i) \in \kappa \setminus S_i$, which is a contradiction. □

Now if $\beta_i \in S_i$ are distinct, then

$$\bigcap_{i<n} a_{\beta_i, t(i)} \subseteq \bigcap_{i<n} c_i = \emptyset \text{ mod } J,$$

and hence case (β) of Definition 3.1(a) holds and we are done. The theorem follows. □

Corollary 3.8. *Assume $\kappa \leq 2^{\aleph_0}$ is regular uncountable. Then $Princ(\kappa)$ implies $C^s(\kappa)$.*

4. Forcing $Princ_1(\kappa, \kappa, \mathfrak{D})$

In this section we consider the principles $Princ_1(\kappa, \lambda, \mathfrak{D})$ and $Princ_{2,\theta}(\kappa, \lambda, \mathfrak{D})$, where $\theta < \lambda \leq \kappa = cf(\kappa)$ and $\mathfrak{D}$ is a κ-definition of normal filters, and discuss their consistency. In fact, we will show that in the generic extension by the Cohen forcing $Add(\omega, \kappa)$ the above principles hold. We prove the result for $Princ(\kappa)$, as the other cases can be proved similarly.

Recall that the Cohen forcing $Add(\omega, I)$ for adding $|I|$-many new Cohen subsets of ω is defined as

$$Add(\omega, I) = \{p : \omega \times I \to 2 : |p| < \aleph_0\},$$

ordered by reverse inclusion.

For a nice name $\underset{\sim}{a} = \bigcup_{n<\omega} \{\check{n}\} \times A_n$, where each A_n is a maximal antichain in $Add(\omega, \lambda)$, set

$$\text{supp}(\underset{\sim}{a}) = \{\alpha \in \lambda : \exists n < \omega, \exists p \in A_n, \exists k \in \omega, (k, \alpha) \in \text{dom}(p)\}.$$

Note that, by the countable chain condition property of $Add(\omega, \lambda)$, $\text{supp}(\underset{\sim}{a})$ is a countable set, and $\underset{\sim}{a}$ can be considered as an $Add(\omega, \text{supp}(\underset{\sim}{a}))$-name. The following lemma follows easily by an absoluteness argument.

Lemma 4.1. *Assume $U \subseteq \lambda$, $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$ are $Add(\omega, U)$-names, $\phi(v_1, \ldots, v_n)$ is a Δ_1^{ZFC}-formula and $p \in Add(\omega, \lambda)$. Then*

$$p \Vdash_{Add(\omega,\lambda)} \text{“}\phi(\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n)\text{”} \iff p \restriction \omega \times U \Vdash_{Add(\omega,U)} \text{“}\phi(\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n)\text{”}.$$

We also need the following simple observation.

Lemma 4.2. *Let $\mathfrak{D}$ be defined by $\mathfrak{D}(A) =$the club filter on $[A]^{<\kappa}$. The following are equivalent:*

(a) *$Princ(\kappa)$.*

(b) *For all large enough $\chi > \kappa$ and $x \in H(\chi)$, there exists $N \in N^1_{\kappa,\kappa,\chi}$ such that $x \in N$.*

Proof. It is clear that $(a) \implies (b)$. To show that (b) implies (a), let $\chi > \kappa$ be large enough regular, $x \in H(\chi)$ and let $C \subseteq [H(\chi)]^{<\kappa}$ be a club set. We need to show that $N^1_{\kappa,\kappa,\chi} \cap C \neq \emptyset$. We assume $|M| \leq M \cap \kappa$ for all $M \in C$. Take $\chi' > \chi$ large enough regular. By the assumption, we can find $N' \in N^1_{\kappa,\kappa,\chi'}$ such that $x, C \in N'$. Let $N = N' \cap H(\chi)$. Then by elementarity, $N \in N^1_{\kappa,\kappa,\chi}$ and $N = \bigcup(N' \cap C)$. Since C is closed, $N \in C$ and so $N \in N^1_{\kappa,\kappa,\chi} \cap C$, as required. □

We are now ready to show that $Princ(\kappa)$ holds in the generic extension by Cohen forcing. We follow the proof in [2].

Theorem 4.3. *Assume $\lambda \geq \kappa = cf(\kappa) > 2^{\aleph_0}$. Then $\Vdash_{Add(\omega,\lambda)}$ "$Princ(\kappa)$".*

Proof. Let $\chi > \lambda$ and $p \in Add(\omega, \lambda)$ be such that

$$p \Vdash_{Add(\omega,\lambda)} \text{“}\underset{\sim}{X} \text{ has transitive closure of cardinality } < \chi\text{”}.$$

Let $\langle N_i : i < \delta \rangle$ be a sequence of elementary submodels of $(H(\chi), \in)$ such that:

(1) $\delta < \kappa, cf(\delta) > \aleph_0$,

(2) $\langle N_i : i < \delta \rangle$ is increasing continuous,

(3) $N_i \cap \kappa \in \kappa$,

(4) $\langle N_j : j \leq i \rangle \in N_{i+1}$,

(5) Each $N_{i+1}, i < \delta$, is closed under countable sequences,

(6) $|N_i| < \kappa$,

(7) $p, \lambda, \underset{\sim}{X} \in N_0$.

Let $N = \bigcup_{i<\delta} N_i$. As $cf(\delta) > \aleph_0$, it follows from clause (5) that N is closed under countable sequences.

We show that $p \Vdash$ "$\bigcup_{i<\delta} N_i[\dot{G}] \in N^1_{\kappa,\kappa,\chi}$". Let G be $Add(\omega, \lambda)$-generic over V with $p \in G$, $N^*_i = N_i[G], i < \delta$, and $N^* = \bigcup_{i<\delta} N^*_i$. Note that N^* is closed under countable sequences and $\underset{\sim}{x}[G] \in N^*$. We show that $N^* \in (N^1_{\kappa,\kappa,\chi})^{V[G]}$. This will complete the proof by the previous lemma.

Thus assume that $a \in P(\omega)^{V[G]}$ and let $\underset{\sim}{a}$ be a nice name for a. Let $U = \mathrm{supp}(\underset{\sim}{a})$.

Set $U_1 = U \cap N$ and $U_2 = U \setminus N$. Then $U_1 \in N$. Let $i < \delta$ be sufficiently large such that $U_1 \subseteq N_i$ and $|N \cap \lambda| = |N_i \cap \lambda|$[1] and set $M = N_{i+1}$. It follows from (5) that $U_1 \in M$.

Let $\pi : \lambda \simeq \lambda$ be a bijection such that $\pi[U] \subseteq M$, $\pi \restriction U_1 = id \restriction U_1$ and $\pi[\lambda \cap N] \subseteq M$. Using the homogeneity of the forcing $Add(\omega, \lambda)$, extend π to an isomorphism $\pi : Add(\omega, \lambda) \simeq Add(\omega, \lambda)$. Note that this also induces an isomorphism of the class of all $Add(\omega, \lambda)$-names, $V^{Add(\omega,\lambda)}$, that we still denote it by π. Note that $\pi[U], \pi(\underset{\sim}{a}) \in M$, as M is closed under countable sequences. Let

$$P = \{\underset{\sim}{c}_r[G] : r \in Add(\omega, \lambda \cap M \setminus U)\},$$

where for $r \in Add(\omega, \lambda \cap M \setminus U)$,

$$\Vdash_{\mathbb{P}} \text{“}\underset{\sim}{c}_r = \omega \setminus \{n \in \omega : \exists q \in \dot{G} \cap Add(\omega, U_1),\ r \cup q \Vdash \text{“}n \notin \pi(\underset{\sim}{a})\text{”}\}\text{”}.$$

Note that $\lambda \cap M \setminus U = \lambda \cap M \setminus U \cap M \in N$, so $Add(\omega, \lambda \cap M \setminus U) \in N$. Also $U_1 \in N$ so $Add(\omega, U_1) \in N$. It easily follows that $P \in N^*$. It is also clear that $P \subseteq P(\omega)^{V[G]}$ and $|P| \leq |Add(\omega, \lambda \cap M \setminus U)| = |M| < \kappa$.

To show that P is an N^*-witness for a, let $b \in P(\omega)^{V[G]} \cap N^*$ and $b \supseteq a$. Let $\underset{\sim}{b} \in N$ be a nice name for b and let $W = \mathrm{supp}(\underset{\sim}{b})$. Let $p^* \leq p$ be such that $p^* \Vdash \text{“}\underset{\sim}{b} \supseteq \underset{\sim}{a}\text{”}$. By Lemma 4.1, we may suppose that $p^* \in Add(\omega, U \cup W)$. Let

$$r = \pi(p^* \restriction \omega \times (\lambda \setminus U_1)).$$

Then $r \in Add(\omega, \lambda \cap M \setminus U)$. We complete the proof by showing that $a \subseteq \underset{\sim}{c}_r[G] \subseteq b$.

<u>$a \subseteq \underset{\sim}{c}_r[G]$</u> **:** Assume by contradiction that $n \in a \setminus \underset{\sim}{c}_r[G]$. Let $q \in G, q \leq p^*$ and

$$q \Vdash \text{“ } n \in \underset{\sim}{a} \text{ ”} .$$

Again we can suppose that $q \in Add(\omega, U \cup W)$. As $n \notin \underset{\sim}{c}_r[G]$, we can find $q^* \in G \cap Add(\omega, U_1)$ such that

$$r \cup q^* \Vdash \text{“ } n \notin \pi(\underset{\sim}{a}) \text{ ”}.$$

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[1]Such an i exists as $U = \mathrm{supp}(\underset{\sim}{a})$ is a countable set and $cf(\delta) > \aleph_0$.

This implies

$$\pi^{-1}(r) \cup \pi^{-1}(q^*) \Vdash \text{ “ } n \notin \underset{\sim}{a} \text{ ”}.$$

Note that $\pi^{-1}(r) \cup \pi^{-1}(q^*) = p^* \restriction \omega \times (\lambda \setminus U_1) \cup q^*$. As $q^*, q \restriction \omega \times U_1 \in G \cap Add(\omega, U_1)$, they are compatible, and we can easily conclude that q and $p^* \restriction \omega \times (\lambda \setminus U_1) \cup q^*$ are compatible, which is a contradiction, as they decide the statement “$n \in \underset{\sim}{a}$” in different ways.

$\underline{\underset{\sim}{c}_r[G] \subseteq b}$ **:** Suppose by contradiction that there is some $n \in \underset{\sim}{c}_r[G] \setminus b$. Let $q \in G, q \leq p^*$, be such that $q \Vdash$“$n \notin \underset{\sim}{b}$”. We can suppose that $q \in Add(\omega, U \cup W)$ and $q \restriction U_2 = p^* \restriction U_2$. As $p^* \Vdash$“$\underset{\sim}{b} \supseteq \underset{\sim}{a}$”, we have $q \Vdash$“$n \notin \underset{\sim}{a}$”, and hence $q \restriction U \Vdash$“$n \notin \underset{\sim}{a}$”. Applying π, we have

$$\pi(q \restriction U) \Vdash \text{“} n \notin \pi(\underset{\sim}{a}) \text{”}.$$

Hence

$$q \restriction U_1 \cup \pi(q \restriction U_2) \Vdash \text{“} n \notin \pi(\underset{\sim}{a}) \text{”},$$

which implies

$$q \restriction U_1 \cup \pi(p^* \restriction U_2) \Vdash \text{“} n \notin \pi(\underset{\sim}{a}) \text{”},$$

Now observe that $r \leq \pi(p^* \restriction U_2)$ and r is compatible with $q \restriction U_1$, so

$$r \cup q \restriction U_1 \Vdash \text{“} n \notin \pi(\underset{\sim}{a}) \text{”}.$$

Thus, $r \cup q \restriction U_1$ witnesses $n \notin \underset{\sim}{c}_r[G]$, which is a contradiction.

The theorem follows. □

The next theorem can be proved as in Theorem 4.3.

Theorem 4.4. *Assume $\theta < \lambda \leq \kappa = cf(\kappa)$ and $\mathfrak{D}$ is a κ-definition of normal filters. Then $\Vdash_{Add(\omega,\kappa)}$ “$Princ_1(\kappa, \lambda, \mathfrak{D}) + Princ_{2,\theta}(\kappa, \lambda, \mathfrak{D})$”. The same result holds with $\mathfrak{D}^+$ replaced with $\mathfrak{D}$.*

Remark 4.5. *In $V[G], \mathfrak{D}$ is defined as follows: For any large enough $\chi > \kappa, \mathfrak{D}(H^{V[G]}(\chi))$ is the filter generated by $\{N[G] : N \in X\}$, where $X \in \mathfrak{D}(H(\chi))$. Note that $N \prec H(\chi) \Rightarrow N[G] \prec H(\chi)[G] = H(\chi)^{V[G]}$, and so the above definition is well-defined.*

5. On a question of Juhasz- Kunen

In this section we answer a question of Juhasz- Kunen [4], by showing that for $n \geq 2$, $C_n(\aleph_2) \nRightarrow C_{n+1}(\aleph_2)$. In fact we prove the following stronger result.

Remark 5.1. *The results of this section are stated and proved for the ideal $J = [\omega]^{<\omega}$; but all of them are valid if we also assume $J = \{\emptyset\}$.*

Theorem 5.2. *Assume:*

(1) $\aleph_0 < \theta = \theta^{<\theta} < \kappa = cf(\kappa) < \chi$ *and* $\forall \alpha < \chi (|\alpha|^{<\theta} < \chi)$,

(2) *D is a κ-complete filter on κ which satisfies the Δ-system θ-property (see below for the definition) and contains the co-bounded subsets of κ, and $J = [\omega]^{<\omega}$,*

(3) $2 < n(*) < \omega$.

Then there is a cofinality preserving generic extension of the universe in which $C_n^D(\kappa, J)$ holds if $n < n(*)$, and fails if $n = n(*)$.[2]

The rest of this section is devoted to the proof of the above theorem. The forcing notion we define is of the form $\mathbb{P}_\chi * \underset{\sim}{\mathbb{Q}}_{\kappa,\mathcal{A}}$, where $\mathbb{P}_\chi$ is a suitable iteration of length χ, which adds a set $\mathcal{A} \subseteq [\kappa]^{<\aleph_0}$, which has nice enough properties. Then we use this added set $\mathcal{A}$ to define the forcing notion $\mathbb{Q}_{\kappa,\mathcal{A}}$.

In subsection 5.1 we define the notion of having the Δ-system θ-property for a filter D, and show that under suitable conditions, some filters have this property. In subsection 5.2 we define the forcing notion $\mathbb{P}_\chi$ and prove its basic properties. Subsection 5.3 is devoted to the definition of the forcing notion $\mathbb{Q}_{\kappa,\mathcal{A}}$. Finally in subsection 5.4 we complete the proof of the above theorem.

5.1. **Filters with the Δ-system θ-property.**

In this subsection we prove a generalized version of Δ-system lemma that will be used several times later.

Definition 5.3. *Let D be a filter on κ, and $\theta < \kappa$ be a cardinal. D has the Δ-system θ-property, if for any $Y \subseteq \kappa, Y \neq \emptyset \mod D$, and any sequence $\langle B_\alpha : \alpha \in Y \rangle$ of sets of cardinality $< \theta$, there exists $Z \subseteq Y, Z \neq \emptyset \mod D$ such that $\langle B_\alpha : \alpha \in Z \rangle$ forms a Δ-system, i.e., there is B^* such that for all $\alpha \neq \beta$, both in Z, $B_\alpha \cap B_\beta = B^*$.*

[2] When working in a forcing extension, we use D to denote the filter generated by D in that extension.

The following is essentially due to Erdos and Rado; we will present a proof for completeness.

Lemma 5.4. *Suppose κ is regular uncountable and $\forall \alpha < \kappa (|\alpha|^{<\theta} < \kappa)$.*

(a) If D is a normal filter on κ and $\{\delta < \kappa : cf(\delta) \geq \theta\} \in D$, then D has the the Δ-system θ-property.

(b) If D is the filter of co-bounded subsets of κ, then D has the Δ-system θ-property.

(c) If $D = \{S \subseteq \kappa : S \cup \{cf(\delta) < \theta\}$ contains a club$\}$, then D has the Δ-system θ-property.

Proof. (a) Let $Y \subseteq \kappa, Y \neq \emptyset \bmod D$, and suppose that $\langle B_\alpha : \alpha \in Y \rangle$ is a sequence of sets of cardinality $< \theta$. As $|\bigcup_{\alpha \in Y} B_\alpha| \leq \kappa$, we can assume that all B_α's, $\alpha \in Y$, are subsets of κ. Also as $\{\delta < \kappa : cf(\delta) \geq \theta\} \in D$, we can assume that $Y \subseteq \{\delta < \kappa : cf(\delta) \geq \theta\}$. Define the function g on Y by $g(\alpha) = \sup(B_\alpha \cap \alpha)$. Then for all $\alpha \in Y$, $g(\alpha) < \alpha$ (as $|B_\alpha| < \theta$ and $cf(\alpha) \geq \theta$), so by normality of D, we can find $Y_1 \subseteq Y, Y_1 \neq \emptyset \bmod D$, and $\xi < \kappa$ such that for all $\alpha \in Y_1, g(\alpha) = \xi$. Then

$$\alpha \in Y_1 \Rightarrow B_\alpha \cap \alpha = B_\alpha \cap \xi.$$

As there are only $|\xi|^{<\theta} < \kappa$ many subset of ξ of cardinality $< \theta$, and since D is normal, there are $Y_2 \subseteq Y_1, Y_2 \neq \emptyset \bmod D$ and a set B^* such that for all $\alpha \in Y_2, B_\alpha \cap \alpha = B_\alpha \cap \xi = B^*$. Let

$$X = \{\alpha < \kappa : \forall \xi \in Y_2 \cap \alpha \ (\sup(B_\xi) < \alpha)\}.$$

X is a club of κ, and hence $X \in D$ (as D contains the club filter by its normality). Set $Z = X \cap Y_2$. Then $Z \subseteq Y, Z \neq \emptyset \bmod D$, and $\langle B_\alpha : \alpha \in Z \rangle$ forms a Δ-system with root B^*.

(b) and (c) follow from (a). □

The following lemma will be used in the proof of Theorem 5.2.

Lemma 5.5. *The Δ-system θ-property is preserved under θ-closed θ^+-c.c. forcing notions.*

Proof. Suppose $\mathbb{P}$ is a θ-closed θ^+-c.c. forcing notion, G is $\mathbb{P}$-generic over V and let $D \in V$ be a filter on $\kappa > \theta$ which has the Δ-system θ-property. Let $\tilde{D}$ be the filter generated by D in $V[G]$. We are going to show that $\tilde{D}$ has the Δ-system θ-property.

Thus suppose that $\tilde{Y} \in V[G], \tilde{Y} \subseteq \kappa, \tilde{Y} \neq \emptyset$ mod $\tilde{D}$, and let $\langle B_\alpha : \alpha \in \tilde{Y}\rangle \in V[G]$ be a sequence of sets of cardinality $< \theta$. By the properties of the forcing notion, each $B_\alpha \in V$ and we can find $Y \in V$ with $Y \subseteq \tilde{Y}$ and $Y \neq \emptyset$ mod D such that $\langle B_\alpha : \alpha \in Y\rangle \in V$.

As D has the Δ-system θ-property, we can find $Z \subseteq Y, Z \neq \emptyset$ mod D such that $\langle B_\alpha : \alpha \in Z\rangle$ forms a Δ-system. Then $Z \neq \emptyset$ mod $\tilde{D}$ and so in $V[G], \tilde{D}$ has the Δ-system θ-property, as requested. □

5.2. **On the forcing notion $\mathbb{P}_\chi$.** Fix $n(*), \theta, \kappa, \chi$ and D as in Theorem 5.2. We describe a cofinality preserving forcing notion $\mathbb{P}_\chi$ which adds a set $\mathcal{A} \subseteq [\lambda]^{<\aleph_0}$ which has some nice properties.

Definition 5.6. $\mathbb{P}_\chi = \langle\langle\mathbb{P}_i : i \leq \chi\rangle, \langle\underset{\sim}{\mathbb{Q}}_i : i < \chi\rangle\rangle$ *is defined as a* $(< \theta)$*-support iteration of forcing notions such that:*

(1) $(\mathbb{Q}_0, \leq)$ *is defined by:*

(1-1) $p \in \mathbb{Q}_0$ *iff* $p = (w^p, \mathcal{A}^p)$*, where* $w^p \in [\kappa]^{<\theta}$ *and* $\mathcal{A} \subseteq [w]^{n(*)}$.

(1-2) $p \leq q \Leftrightarrow w^q \subseteq w^p$ *and* $\mathcal{A}^q = \mathcal{A}^p \cap [w^q]^{n(*)}$.

Also let $\underset{\sim}{\mathcal{A}} = \bigcup\{\mathcal{A}^p : p \in \dot{G}_{\mathbb{Q}_0}\}$.

(2) *Assume* $0 < i < \chi$*, and* $\mathbb{P}_i$ *is defined. Then for some* $\mathbb{P}_i$*-names* $\underset{\sim}{Y}_i$ *and* $\langle \underset{\sim}{w}^*_i, \langle \underset{\sim}{w}^i_\alpha : \alpha \in \underset{\sim}{Y}_i\rangle\rangle$ *we have:*

(2-1) $\Vdash_{\mathbb{P}_i}$ *"*$\underset{\sim}{Y}_i$ *is a subset of* $\kappa, \underset{\sim}{Y}_i \neq \emptyset$ *mod* D*",*

(2-2) $\Vdash_{\mathbb{P}_i}$ *"*$\langle \underset{\sim}{w}^i_\alpha : \alpha \in \underset{\sim}{Y}_i\rangle$ *is a* Δ*-system of subsets of* κ *with root* $\underset{\sim}{w}^*_i$*, each of cardinality* $\leq \aleph_0$*",*

(2-3) $\Vdash_{\mathbb{P}_i}$ *"*$\underset{\sim}{\mathbb{Q}}_i = \{u \subseteq \underset{\sim}{Y}_i : |u| < \theta$ *and if* $m < n(*), \alpha_0, \ldots, \alpha_{m-1} \in u$ *are distinct, then for all* $y \in \underset{\sim}{\mathcal{A}}, y \subseteq \bigcup_{l<m} \underset{\sim}{w}^i_{\alpha_l} \Rightarrow \exists l < m, y \subseteq \underset{\sim}{w}^i_{\alpha_l}\}$*".*

(2-4) $\Vdash_{\mathbb{P}_i}$ *"*$\leq_{\underset{\sim}{\mathbb{Q}}_i} = \supseteq$,

(3) *If* $\underset{\sim}{Y}$ *and* $\langle \underset{\sim}{w}^*, \langle \underset{\sim}{w}_\alpha : \alpha \in \underset{\sim}{Y}\rangle\rangle$ *are* $\mathbb{P}_i$*-names of objects as above, then for some* $j \in (i, \chi)$*, they are of the form* $\underset{\sim}{Y}_j$ *and* $\langle \underset{\sim}{w}^*_j, \langle \underset{\sim}{w}^j_\alpha : \alpha \in \underset{\sim}{Y}_j\rangle\rangle$.

Remark 5.7. (*a*) (3) *can be achieved by a bookkeeping argument, and using the fact that the forcing* $\mathbb{P}_\chi$ *satisfies the* θ^+*-c.c. (see below).*

(*b*) *It also follows from the* θ^+*-chain condition of the forcing that under the same assumptions as* (3)$, \Vdash_{\mathbb{P}_j}$ *"*$\underset{\sim}{Y} \neq \emptyset$ *mod* D*", so* $\mathbb{Q}_j$ *is well-defined.*

Lemma 5.8. *Let $\alpha \leq \chi$.*

(a) *$\mathbb{P}^*_\alpha$ is a dense subset of $\mathbb{P}_\alpha$, where $\mathbb{P}^*_\alpha$ consists of those $p \in \mathbb{P}_\alpha$ such that:*

(1) *$i \in \mathrm{dom}(p) \Rightarrow p(i)$ is an object (and not just a $\mathbb{P}_i$-name),*

(2) *$0 \in \mathrm{dom}(p)$ and for some w we have $w^{p(0)} = w$,*

(3) *If $i \in \mathrm{dom}(p)$, then for all $j \in p(i), p \restriction i$ decides $\underset{\sim}{w}^*_i, \underset{\sim}{w}^i_j$.*

(b) *If $\alpha < \chi$, then $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is θ^+-Knaster".*

(c) *Each $\mathbb{P}_\alpha$ satisfies the θ^+-c.c.*

Proof. (a) follows easily by induction on α, and using the fact that $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ is θ-closed". Let's present a proof for completeness.

Case 1. $\alpha = 0$**:** There is noting to prove.

Case 2. $\alpha + 1$ **is a successor ordinal:** Thus assume that $\mathbb{P}^*_\alpha$ is a dense subset of $\mathbb{P}_\alpha$, and let $p \in \mathbb{P}_{\alpha+1}$. Then $p \restriction \alpha \in \mathbb{P}_\alpha$, so for some $p_1 \in \mathbb{P}^*_\alpha, p_1 \leq_{\mathbb{P}_\alpha} p \restriction \alpha$. Since $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is θ-closed and $|p(\alpha)| < \theta$", we can find q_1 such that $p_1 \Vdash$ "$p(\alpha) = q_1$". As $|q_1| < \theta$, and again using $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is θ-closed", we can find $p_2 \leq_{\mathbb{P}_\alpha} p_1, p_2 \in \mathbb{P}^*_\alpha, q_2 \leq_{\mathbb{Q}_\alpha} q_1$ and w^*_α, w^α_j for $j \in q_2$ such that for all $j \in q_2, p_2 \Vdash$ "$\underset{\sim}{w}^*_\alpha = w^*_\alpha$ and $\underset{\sim}{w}^\alpha_j = w^\alpha_j$". Then $(p_2, q_2) \in \mathbb{P}^*_{\alpha+1}$ and $(p_2, q_2) \leq_{\mathbb{P}_{\alpha+1}} p$.

Case 3. α **is a limit ordinal,** $cf(\alpha) \geq \theta$**:** Let $p \in \mathbb{P}_\alpha$. Then as $|\mathrm{dom}(p)| < \theta$, we can find $\beta < \alpha$ such that $\mathrm{dom}(p) \subseteq \beta$, so $p \in \mathbb{P}_\beta$, and the induction applies.

Case 4. α **is a limit ordinal,** $cf(\alpha) < \theta$**:** Let $\langle \alpha_\xi : \xi < cf(\alpha) \rangle$ be a normal sequence cofinal in α. Let $p \in \mathbb{P}_\alpha$. By induction and the θ-closure of forcings, we can find a decreasing sequence $\langle q_\xi : \xi < cf(\alpha) \rangle$ of conditions such that: $q_\xi \in \mathbb{P}^*_{\alpha_\xi}$ and $q_\xi \leq_{\mathbb{P}_{\alpha_\xi}} p \restriction \alpha_\xi$. Let $p_1 = \bigcup_{\xi < cf(\alpha)} q_\xi$. Then $p_1 \in \mathbb{P}^*_\alpha$ and $p_1 \leq_{\mathbb{P}_\alpha} p$.

(b) can be proved easily by a Δ-system argument. To prove (c), it suffices, by (a), to show that $\mathbb{P}^*_\alpha$ satisfies the θ^+-c.c. Let $\{p_\beta : \beta < \theta^+\} \subseteq \mathbb{P}^*_\alpha$. We can assume that:

(1) $\langle \mathrm{dom}(p_\beta) : \beta < \theta^+ \rangle$ forms a Δ-system with root Δ,

(2) For each $i \in \Delta$, $\langle p_\beta(i) : \beta < \theta^+ \rangle$ are pairwise compatible in $\mathbb{Q}_i$ (using the fact that $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ is θ^+-Knaster").

Now let $\beta_1 < \beta_2 < \theta^+$. Let q be defined as follows:

- $\mathrm{dom}(q) = \mathrm{dom}(p_{\beta_1}) \cup \mathrm{dom}(p_{\beta_2})$,

- $q(0) = \langle w^{p_{\beta_1}} \cup w^{p_{\beta_2}}, \mathcal{A}^{p_{\beta_1}} \cup \mathcal{A}^{p_{\beta_2}} \rangle$,
- For all $i \in \mathrm{dom}(q), q(i) = p_{\beta_1}(i) \cup p_{\beta_2}(i)$ (where we assume $p_{\beta_k}(i) = \emptyset$, if $i \notin \mathrm{dom}(p_{\beta_k})$).

Clearly $q \in \mathbb{P}^*_\alpha$, and it extends both p_{β_1}, p_{β_2}. So $\{p_\beta : \beta < \theta^+\}$ is not an antichain. □

5.3. **On the forcing notion $\mathbb{Q}_{\lambda,\mathcal{A}}$.** In this subsection we describe a forcing notion $\mathbb{Q}_{\lambda,\mathcal{A}}$, which depends on a parameter $\mathcal{A} \subseteq [\lambda]^{<\aleph_0}$. For $\mathcal{A} \subseteq [\lambda]^{<\aleph_0}$, set

$$\mathcal{A}^+ = \{u \in [\lambda]^{<\aleph_0} : u \text{ includes some member of } \mathcal{A}\}.$$

Definition 5.9. *Assume λ is a cardinal and $\mathcal{A} \subseteq [\lambda]^{<\aleph_0}$. We define the forcing notion $(\mathbb{Q}_{\lambda,\mathcal{A}}, \leq)$ as follows:*

(*a*) *$p \in \mathbb{Q}_{\lambda,\mathcal{A}}$ iff p is a finite partial function from λ to $2^{n(p)}$, for some $n(p) < \omega$.*

(*b*) *For $p, q \in \mathbb{Q}_{\lambda,\mathcal{A}}, p \leq q$ (p is stronger than q) iff:*

$(b-1)$ $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$,

$(b-2)$ $\alpha \in \mathrm{dom}(q) \Rightarrow q(\alpha) \trianglelefteq p(\alpha)$,

$(b-3)$ *If $u \in \mathcal{A}$, $u \subseteq \mathrm{dom}(q)$ and $n(q) \leq k < n(p)$, then for some $\alpha \in u, p(\alpha)(k) = 0$.*

We also define the following $\mathbb{Q}_{\lambda,\mathcal{A}}$-names:

(α) $\underset{\sim}{\eta}_\alpha = \bigcup\{p(\alpha) : \alpha \in \mathrm{dom}(p) \text{ and } p \in \dot{G}_{\mathbb{Q}_{\lambda,\mathcal{A}}}\}$,

(β) $\underset{\sim}{a}_\alpha = \{k < \omega : \underset{\sim}{\eta}_\alpha(k) = 1\}$,

(γ) $\underset{\sim}{a}_{\alpha,n} = \underset{\sim}{a}_{\omega\cdot\alpha+n}$.

Remark 5.10. *Given any $w \subseteq \lambda$, let $\mathcal{A} \restriction w = \{u \in \mathcal{A} : u \subseteq w\}$. Then we define $\mathbb{Q}_{\lambda,\mathcal{A}} \restriction w$ to be $\mathbb{Q}_{\lambda,\mathcal{A}\restriction w}$ which is defined in the natural way. Then for disjoint $w, v \subseteq \lambda$ if*

$$\forall u \in \mathcal{A}(u \subseteq w \cup v \Rightarrow u \subseteq w \text{ or } u \subseteq v),$$

then we have a forcing isomorphism $\mathbb{Q}_{\lambda,\mathcal{A}} \restriction (w \cup v) \approx \{(p, q) \in (\mathbb{Q}_{\lambda,\mathcal{A}} \restriction w) \times (\mathbb{Q}_{\lambda,\mathcal{A}} \restriction v) : n(p) = n(q)\}$. But in general the above forcing isomorphism may not be true, if w, v do not satisfy the above requirement, as $(b-3)$ may fail.

We have the following easy lemma.

Lemma 5.11. *Let $\mathbb{Q} = \mathbb{Q}_{\lambda,\mathcal{A}}$. Then*

(*a*) *$\mathbb{Q}$ is a c.c.c. forcing notion,*

(b) $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta}_\alpha \in 2^\omega$ *and* $\underset{\sim}{a}_{\alpha,n} \subseteq \omega$",

(c) $\Vdash_{\mathbb{Q}}$ "$\bigcap_{i<n} \underset{\sim}{a}_{\alpha_i,m}$ *is finite" iff* $\{\omega \cdot \alpha_i + m : i < n\} \in \mathcal{A}^+$.

Proof. (a) follows by a simple Δ-system argument and (b) is clear. Let us prove (c). First assume that $\{\omega\cdot\alpha_i+m : i < n\} \in \mathcal{A}^+$. Then for some $u \in \mathcal{A}, u \subseteq \{\omega\cdot\alpha_i+m : i < n\}$, and since $\Vdash_{\mathbb{Q}}$ "$\bigcap_{i<n} \underset{\sim}{a}_{\alpha_i,m} \subseteq \bigcap_{\alpha\in u} \underset{\sim}{a}_{\alpha,m}$", so we can assume w.l.o.g. that $\{\omega\cdot\alpha_i+m : i < n\} \in \mathcal{A}$. Now let $p \in \mathbb{Q}$. By extending p, if necessary, we can assume that $\{\omega.\alpha_i + m : i < n\} \subseteq \text{dom}(p)$. But then by clause $(b-3)$, any $q \leq p$ forces "$\bigcap_{i<n} \underset{\sim}{a}_{\alpha_i,m} \subseteq n(p)$". The result follows immediately.

Conversely suppose that $\{\omega \cdot \alpha_i + m : i < n\} \notin \mathcal{A}^+$. Let $p \in \mathbb{Q}$ and $k < \omega$. We find $q \leq p$ and $k' > k$ such that $q \Vdash$"$k' \in \bigcap_{i<n} \underset{\sim}{a}_{\alpha_i,m}$". By extending p we may assume that $\text{dom}(p) \supseteq \{\omega \cdot \alpha_i + m : i < n\}$ and $n(p) > k$. Now define $q \leq p$ as follows:

- $\text{dom}(q) = \text{dom}(p)$.
- $n(q) = n(p) + 1$.
- If $\alpha \in \text{dom}(p) \setminus \{\omega \cdot \alpha_i + m : i < n\}$, then $q(\alpha) = p(\alpha)^\frown\langle(n(p), 0)\rangle$.
- If $i < n$, then $q(\omega \cdot \alpha_i + m) = p(\omega \cdot \alpha_i + m)^\frown\langle(n(p), 1)\rangle$.

q is easily seen to be well-defined and clearly

$$q \Vdash \text{“}k < k' \in \textstyle\bigcap_{i<n} \underset{\sim}{a}_{\alpha_i,m}\text{”},$$

where $k' = n(p)$. Let us show that $q \leq p$. It suffices to show that it satisfies clause $(b-3)$ of Definition 5.9. Thus let $u \in \mathcal{A}$ be such that $u \subseteq \text{dom}(p)$. We are going to find some $\alpha \in u$ such that $q(\alpha)(n_p) = 0$. As $\{\omega \cdot \alpha_i + m : i < n\} \notin \mathcal{A}^+$, $u \setminus \{\omega \cdot \alpha_i + m : i < n\} \neq \emptyset$. Let $\alpha \in u \setminus \{\omega \cdot \alpha_i + m : i < n\}$. Then by our definition, $q(\alpha)(n_p) = 0$, as requested. □

We now consider the combinatorial principle $C^D_T(\kappa, J)$ in the forcing extensions by $\mathbb{Q}_{\lambda,\mathcal{A}}$, and show that the truth or falsity of it depends on the choice of D and $\mathcal{A}$. For the rest of this subsection, let $J = [\omega]^{<\omega}$, the ideal of bounded subsets of ω. In the next lemma we discuss conditions on D and $\mathcal{A}$ which imply $\neg C^D_T(\kappa, J)$ in the forcing extensions by $\mathbb{Q}_{\lambda,\mathcal{A}}$.

Lemma 5.12. *Assume:*

(1) $\aleph_0 < \kappa = cf(\kappa) \leq \lambda$,

(2) *D is a filter on κ with the Δ-system $\aleph_0$-property,*

(3) *$\mathcal{A} \subseteq [\lambda]^{<\aleph_0}$ and T is a subtree of $\omega^{<\omega}$,*

(4) *There exists some $Y^* \in D^+$ such that;*

(a) *If $Y \subseteq Y^*, Y \neq \emptyset$ mod D, then there are $t \in T \cap \omega^n$ and distinct $\alpha_0, \dots, \alpha_{n-1} \in Y$ such that $\{\omega.\alpha_i + t(i) : i < n\} \in \mathcal{A}$.*

(b) *If $t \in T \cap \omega^n$ and for $i < n$, $Y_i \subseteq Y^*, Y_i \neq \emptyset$ mod D, then there are $\alpha_i \in Y_i$, for $i < n$ such that $\{\omega.\alpha_i + t(i) : i < n\} \notin \mathcal{A}$.*

Then $\Vdash_{\mathbb{Q}_{\lambda,\mathcal{A}}}$ "$\neg C^D_T(\kappa, J)$".

Proof. Let $\mathbb{Q} = \mathbb{Q}_{\lambda,\mathcal{A}}$, and suppose $\nVdash_{\mathbb{Q}}$ "$\neg C^D_T(\kappa, J)$". By Lemma 5.4, $\Vdash_{\mathbb{Q}}$ "$\langle \underset{\sim}{a}_{\alpha,n} : \alpha \in Y^*, n < \omega\rangle$ is a $(\kappa \times \omega)$-matrix for D (i.e., $Y^* \in D^+$)", so by our assumption one of the following holds:

Case 1. There are $p \in \mathbb{Q}$ and $\underset{\sim}{X}$ such that:

$p \Vdash$ "$\underset{\sim}{X} \subseteq Y^*, \underset{\sim}{X} \neq \emptyset$ mod D",

$p \Vdash$ "For every $t \in T \cap \omega^n$ and distinct $\alpha_0, \dots, \alpha_{n-1} \in \underset{\sim}{X}, \bigcap_{i<n} \underset{\sim}{a}_{\alpha_i,t(i)} \neq \emptyset$ mod J".

Let $X^* = \{\alpha \in Y^* : p \nVdash$ "$\alpha \notin \underset{\sim}{X}$"$\}$. Then $X^* \in V$ and $p \Vdash$ "$\underset{\sim}{X} \subseteq X^* \subseteq Y^*$", so $X^* \neq \emptyset$ mod D. For any $\alpha \in X^*$, let $p_\alpha \leq p$ be such that $p_\alpha \Vdash$ "$\alpha \in \underset{\sim}{X}$". As D has the Δ-system $\aleph_0$-property, we can find $X_1 \subseteq X^*, X_1 \neq \emptyset$ mod D such that $\{\mathrm{dom}(p_\alpha) : \alpha \in X_1\}$ forms a Δ-system with some root, say, Δ. Let $\Delta = \{\beta_0, \dots, \beta_{k^{**}-1}\}$, and for each $\alpha \in X_1$, let

$$\mathrm{dom}(p_\alpha) = \{\beta_{\alpha,j} : j < k_\alpha\},$$

where for $j < k^{**}, \beta_{\alpha,j} = \beta_j$. By shrinking X_1, and using the Δ-system $\aleph_0$-property of D, we can further suppose that:

(1) There is some $k^* < \omega$ such that $\alpha \in X_1 \Rightarrow k_\alpha = k^*$,

(2) $\alpha, \beta \in X_1 \Rightarrow p_\alpha \restriction \Delta = p_\beta \restriction \Delta$,

(3) $\{p_\alpha(\beta_{\alpha,j}) : \alpha \in X_1\}$ is constant, for each $j < k^*$.

Now by $(4-a)$, there are $t \in T \cap \omega^n$ and distinct $\alpha_0, \dots, \alpha_{n-1} \in X_1$ such that $\{\omega.\alpha_i + t(i) : i < n\} \in \mathcal{A}$. Let q be a common extension of $p_{\alpha_i}, i < n$, which exists by our above assumptions. Then $q \Vdash$ "$\alpha_0, \dots, \alpha_{n-1} \in \underset{\sim}{X}$", and by Lemma 5.10$(c)$,

$$q \Vdash \text{“}\textstyle\bigcap_{i<n} \underset{\sim}{a}_{\alpha_i,t(i)} \text{ is finite”},$$

which is a contradiction.

Case 2. There are $p \in \mathbb{Q}, t \in T \cap \omega^n$ and $\underset{\sim}{X}_0, \dots, \underset{\sim}{X}_{n-1}$ such that

$p \Vdash$"$\underset{\sim}{X}_i \subseteq Y^*, \underset{\sim}{X}_i \neq \emptyset \bmod D$" for all $i < n$,

$p \Vdash$"If $\alpha_i \in \underset{\sim}{X}_i$ are distinct, then $\bigcap_{i<n} \underset{\sim}{a}_{\alpha_i, t(i)} = \emptyset \bmod J$".

For $i < n$ set $X_i^* = \{\alpha \in Y^* : p \nVdash$"$\alpha \notin \underset{\sim}{X}_i$"$\}$. Then $X_i^* \in V$ and $p \Vdash$"$\underset{\sim}{X}_i \subseteq X_i^* \subseteq Y^*$", so $X_i^* \neq \emptyset \bmod D$. We now proceed by induction on $i < n$ and find $p_{i,\alpha} \leq p$ for $\alpha \in X_i^*$ so that:

(1) $p_{i,\alpha} \Vdash$"$\alpha \in \underset{\sim}{X}_i$",

(2) If $\alpha \in X_i^* \cap X_j^*$, for $i < j < n$, then $p_{i,\alpha} = p_{j,\alpha}$.

Now proceed as in case 1, and shrink each X_i^* to some $X_{i,1}$ so that:

(3) $X_{i,1} \neq \emptyset \bmod D$,

(4) $\{\mathrm{dom}(p_{i,\alpha}) : \alpha \in X_{i,1}\}$ forms a Δ-system with root, say, $\Delta_i = \{\beta_{i,0}, \dots, \beta_{i,k_i^{**}-1}\}$.

(5) For some $k_i^* < \omega$, $\mathrm{dom}(p_{i,\alpha}) = \{\beta_{i,\alpha,j} : j < k_i^*\}$, where for $j < k_i^{**}, \beta_{i,\alpha,j} = \beta_{i,j}$.

(6) $\alpha, \beta \in X_{i,1} \Rightarrow p_{i,\alpha} \restriction \Delta_i = p_{i,\beta} \restriction \Delta_i$,

(7) $\{p_{i,\alpha}(\beta_{i,\alpha,j}) : \alpha \in X_{i,1}\}$ is constant, for each $j < k_i^*$.

We again use the Δ-system argument successively on $X_{i,1}$'s to shrink them to some $X_{i,2}, i < n$, so that for all $i, j < n$:

(8) $X_{i,2} \neq \emptyset \bmod D$,

(9) For all $\alpha \in X_{i,2}, \beta \in X_{j,2}, \mathrm{dom}(p_{i,\alpha}) \cap \mathrm{dom}(p_{j,\beta}) = \Delta_{i,j}$, for some fixed set $\Delta_{i,j}$,

(10) For all $\alpha \in X_{i,2}, \beta \in X_{j,2}, p_{i,\alpha} \restriction \Delta_{i,j} = p_{j,\beta} \restriction \Delta_{i,j}$.

Now by $(4-b)$, there are $\alpha_i \in X_{i,2}, i < n$, such that $\{\omega.\alpha_i + t(i) : i < n\} \notin \mathcal{A}$. Let q be a common extension of $p_{i,\alpha_i}, i < n$, which exists by our above assumptions. Then $q \Vdash$"$\alpha_0, \dots, \alpha_{n-1} \in \underset{\sim}{X}$", and by Lemma 5.10$(c)$,

$$q \Vdash \text{“}\bigcap_{i<n} \underset{\sim}{a}_{\alpha_i, t(i)} \text{ is infinite”},$$

which is a contradiction. The lemma follows. □

we now discuss conditions on D and $\mathcal{A}$ which imply $C_T^D(\kappa, J)$ in the forcing extensions by $\mathbb{Q}_{\lambda, \mathcal{A}}$.

Lemma 5.13. *Assume:*

(1) *D is a κ-complete filter on κ, where $\kappa = cf(\kappa) > \aleph_0$ and $\forall \alpha < \kappa (|\alpha|^{\aleph_0} < \kappa)$,*

(2) *$T \subseteq \omega^{<n(*)}$ is a subtree, where $n(*) < \omega$,*

(3) *$\lambda \geq \kappa$ and $\mathcal{A} \subseteq [\lambda]^{<\aleph_0}$,*

(4) *If $Y \subseteq \kappa, Y \neq \emptyset$ mod D, and if $\langle w_\alpha : \alpha \in Y \rangle$ is such that $w_\alpha \in [\lambda]^{\leq \aleph_0}$, for $\alpha \in Y$, then there exists $X \subseteq Y, X \neq \emptyset$ mod D such that;*

(a) *$\langle w_\alpha : \alpha \in X \rangle$ form a Δ-system with root, say, w^* such that for all $\alpha \neq \beta$ in $X, w_\alpha \cap w_\beta = w^*$, and for all $\gamma \in w^*, otp(w_\alpha \cap \gamma) = otp(w_\beta) \cap \gamma$,*

(b) *If $\alpha_0, \ldots, \alpha_{n(*)-1} \in X$ are distinct, then*

$$\forall u (u \in \mathcal{A} \text{ and } u \subseteq \bigcup_{i<n(*)} w_{\alpha_i} \Rightarrow \exists i < n(*), u \subseteq w_{\alpha_i}).$$

Then $\Vdash_{\mathbb{Q}_{\lambda,\mathcal{A}}}$ "$C^D_T(\kappa, J)$".

Proof. Let $\mathbb{Q} = \mathbb{Q}_{\lambda,\mathcal{A}}$, and suppose G is $\mathbb{Q}$-generic over V. Assume on the contrary that $V[G] \models$"$\neg C^D_T(\kappa, J)$". Let $p \in G$, $\underset{\sim}{X}$ and $\langle \underset{\sim}{b}_{\alpha,n} : \alpha \in \underset{\sim}{X}, n < \omega \rangle$ be such that

$p \Vdash$"$\underset{\sim}{X} \subseteq \kappa, \underset{\sim}{X} \neq \emptyset$ mod D",

$p \Vdash$"$\langle \underset{\sim}{b}_{\alpha,n} : \alpha \in \underset{\sim}{X}, n < \omega \rangle$ is a counterexample to $C^D_T(\kappa, J)$.

Let $X_1 = \{\alpha < \kappa : p \nVdash$"$\alpha \notin \underset{\sim}{X}$"$\}$. Then $X_1 \in V$ and $p \Vdash$"$\underset{\sim}{X} \subseteq X_1$", so $X_1 \neq \emptyset$ mod D. For any $\alpha \in X_1$, let $p_\alpha \leq p$ be such that $p_\alpha \Vdash$"$\alpha \in \underset{\sim}{X}$". We may further assume that $\alpha \in \mathrm{dom}(p)$.

For each $\alpha \in X_1$, we can find $\langle q_{\alpha,n,m,k}, t_{\alpha,n,m} : n, m, k < \omega \rangle$ such that:

(1) $t_{\alpha,n,m} : \omega \to 2$,

(2) $\{q_{\alpha,n,m,k} : m < \omega\} \subseteq \mathbb{Q}$ is a maximal antichain below p,

(3) $q_{\alpha,n,m,k} \Vdash$"$k \in \underset{\sim}{b}_{\alpha,n}$"$\Leftrightarrow t_{\alpha,n,m}(k) = 1$.

We may note that then

$$p \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{b}_{\alpha,n} = \{\langle q_{\alpha,n,m,k}, k \rangle : m, k < \omega, t_{\alpha,n,m}(k) = 1\}\text{”},$$

and so from now on we assume $\underset{\sim}{b}_{\alpha,n}$ is of the form. Let

$$w_\alpha = \mathrm{dom}(p_\alpha) \cup \bigcup \{\mathrm{dom}(q_{\alpha,,n,m,k}) : n, m, k < \omega\} \cup \{\omega.\alpha + n : n < \omega\}.$$

Then each $w_\alpha \in [\lambda]^{\aleph_0}$. As D is κ-complete and $\kappa > 2^{\aleph_0}$, we can find $X_2, \bar{g}$ and $\bar{t}$ such that

(4) $X_2 \subseteq X_1, X_2 \neq \emptyset$ mod D,

(5) $\bar{t} = \langle t_{n,m} : n, m < \omega \rangle$ and $\forall \alpha \in X_2, t_{\alpha,n,m} = t_{n,m}$,

(6) For all $\alpha, \beta \in X_2$, $otp(w_\alpha) = otp(w_\beta)$,

(7) $\bar{g} = \langle g_{\alpha,\beta} : \alpha, \beta \in X_2 \rangle$,

(8) $g_{\alpha,\beta} : w_\beta \cong w_\alpha$ is an order preserving bijection,

(9) $g_{\alpha,\beta}(\beta) = \alpha$,

(10) $g_{\alpha,\beta}``[q_{\beta,n,m,k}] = q_{\alpha,n,m,k}$.

Consider $\langle w_\alpha : \alpha \in X_2 \rangle$. By our assumption, we can find X_3 and w^* such that

(11) $X_3 \subseteq X_2, X_3 \neq \emptyset \text{ mod } D$,

(12) For all $\alpha \neq \beta$ in $X_3, w_\alpha \cap w_\beta = w^*$, and for all $\gamma \in w^*, otp(w_\alpha \cap \gamma) = otp(w_\beta) \cap \gamma$,

(13) if $\alpha_0, \ldots, \alpha_{n(*)-1} \in X_3$ are distinct, then

$$\forall u (u \in \mathcal{A} \text{ and } u \subseteq \bigcup_{i<n(*)} w_{\alpha_i} \Rightarrow \exists i < n(*), u \subseteq w_{\alpha_i}).$$

Note that for $\alpha \neq \beta$ in $X_3, g_{\alpha,\beta} \restriction w^* = id \restriction w^*$ (by (12)). As the conclusion of the lemma fails, we can find $q \leq p, q \in G, t \in T$ and $\alpha_0, \ldots, \alpha_{n(*)-1} \in X_3$ such that

$$q \Vdash ``\bigcap_{i<n(*)} \underset{\sim}{b}_{\alpha_i, t(i)} = \emptyset \text{ mod } J".$$

We may suppose that $\text{dom}(q) \subseteq \bigcup_{i<n(*)} w_{\alpha_i}$.[3]

For $\beta \in X_3$ and $i < n(*)$ set

$$q_{i,\beta} = g_{\beta,\alpha_i}(q \restriction w_{\alpha_i}) \in \mathbb{Q} \restriction w_\beta.$$

Let $\underset{\sim}{Y}_i$ be such that

$$\Vdash_{\mathbb{Q}} ``\underset{\sim}{Y}_i = \{\beta \in X_3 : q_{i,\beta} \in \dot{G}_{\mathbb{Q}}\}".$$

Claim 5.14. *$\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{Y}_i \neq \emptyset$ mod D".*

Proof. Assume not; so there are $r \in G, r \leq q$ and $X \in D^+$ such that $r \Vdash ``X \cap \underset{\sim}{Y}_i = \emptyset"$. As $\text{dom}(r)$ is finite, we can find $\beta \in X$ so that $\text{dom}(r) \cap w_\beta \setminus w^* = \emptyset$. But then $r, q_{i,\beta}$ are compatible, and any common extension of them forces "$\beta \in X \cap \underset{\sim}{Y}_i$", which is impossible. □

We show that if $\Vdash_{\mathbb{Q}} ``\beta_i \in \underset{\sim}{Y}_i"$, for $i < n(*)$, then $\Vdash_{\mathbb{Q}} ``\bigcap_{i<n(*)} \underset{\sim}{b}_{\beta_i, t(i)} = \emptyset \text{ mod } J"$. So assume $V[G] \models ``\beta_i \in Y_i = \underset{\sim}{Y}_i[G]"$. Then $g = \bigcup_{i<n(*)} g_{\alpha_i,\beta_i}$ is an order preserving bijection from $\bigcup_{i<n(*)} w_{\beta_i}$ onto $\bigcup_{i<n(*)} w_{\alpha_i}$, and we can extend it to an automorphism of λ, in the natural way, so that its restriction to $\lambda \setminus (\bigcup_{i<n(*)} w_{\beta_i} \cup \bigcup_{i<n(*)} w_{\alpha_i})$ is identity. We denote

[3] This is because, by our representation of $\underset{\sim}{b}_{\alpha_i,t(i)}$, we can imagine each $\underset{\sim}{b}_{\alpha_i,t(i)}$ as a $\mathbb{Q} \restriction w_{\alpha_i}$-name.

the resulting function still by g. g easily extends to an automorphism $\hat{g} : \mathbb{Q} \cong \mathbb{Q}$ of $\mathbb{Q}$, which in turn also extends to an automorphism of nice names of $\mathbb{Q}$.

For $i < n(*), q_{i,\beta_i} \in G_{\mathbb{Q}}$, so

$$\hat{g}(q_{i,\beta_i}) = q \restriction w_{\alpha_i} \in \hat{g}``[G].$$

Hence, $q = \bigcup_{i<n(*)} q \restriction w_{\alpha_i} \in \hat{g}``[G]$. But it is easily seen that $\hat{g}(\underset{\sim}{b}_{\beta_i,t(i)}) = \underset{\sim}{b}_{\alpha_i,t(i)}$, and so

$$\hat{g}^{-1}(q) \Vdash ``\bigcap_{i<n(*)} \underset{\sim}{b}_{\beta_i,t(i)} = \emptyset \bmod J".$$

On the other hand $\hat{g}^{-1}(q) \in G$, and the result follows. $\square$

Remark 5.15. *Condition $(4-b)$ is implicitly used in the argument to guarantee that the restricted conditions and their union which we defined are well-defined. See also Remark 5.4.*

5.4. **Proof of Theorem 5.2.** Finally in this subsection we present the proof of theorem 5.2. Thus let $n(*), \theta, \kappa, \chi$ and D be as above and $J = [\omega]^{<\omega}$. Consider the forcing notion $\mathbb{P} = \mathbb{P}_\chi * \underset{\sim}{\mathbb{Q}}_{\kappa,\mathcal{A}}$, where $\mathcal{A} \subseteq [\kappa]^{<\aleph_0}$ is the set added by $\mathbb{P}_\chi$. It follows from Lemmas 5.7 and 5.10 that $\mathbb{P}$ is a cofinality preserving forcing notion.

Firs we show that $C_n^D(\kappa, J)$ holds for $n < n(*)$. It suffices to show that in $V^{\mathbb{P}_\chi}$, the pair $(n(*), D)$ satisfies the the demands in Lemma 5.12. Conditions $(1)-(3)$ from the lemma are clear. To prove $(4-a)$, let $\Vdash_{\mathbb{P}_\chi} ``\underset{\sim}{Y} \subseteq \kappa, \underset{\sim}{Y} \neq \emptyset \bmod D$, and $\langle \underset{\sim}{w}_\alpha : \alpha \in \underset{\sim}{Y}\rangle$ is a sequence of countable subsets of λ". Let $i < \chi$ be such that $\underset{\sim}{Y}$ and $\langle \underset{\sim}{w}_\alpha : \alpha \in \underset{\sim}{Y}\rangle$ are $\mathbb{P}_i$-names. By the fact that in $V^{\mathbb{P}_i}$, D has has the Δ-system θ-property (see Lemma 5.5), we can find $\mathbb{P}_i$-names $\underset{\sim}{X}$ and $\underset{\sim}{w}^*$ such that:

(1) $\Vdash_{\mathbb{P}_i} ``\underset{\sim}{X} \subseteq \underset{\sim}{Y}, \underset{\sim}{X} \neq \emptyset \bmod D"$,

(2) $\Vdash_{\mathbb{P}_i} ``\langle \underset{\sim}{w}_\alpha : \alpha \in \underset{\sim}{X}\rangle$ forms a Δ-system with root $\underset{\sim}{w}^*"$.

Then for some $j \in (i, \chi)$, $\underset{\sim}{X} = \underset{\sim}{Y}_j$ and $\langle \underset{\sim}{w}^*, \langle \underset{\sim}{w}_\alpha : \alpha \in \underset{\sim}{Y}\rangle\rangle = \langle \underset{\sim}{w}^*_j, \langle \underset{\sim}{w}^j_\alpha : \alpha \in \underset{\sim}{Y}_j\rangle\rangle$. Now by our definition of $\mathbb{Q}_j$, we can find $\underset{\sim}{Z} \in V^{\mathbb{P}_{j+1}}$ such that

(3) $\Vdash_{\mathbb{P}_{j+1}} ``\underset{\sim}{Z} \subseteq \underset{\sim}{Y}_j, \underset{\sim}{Z} \neq \emptyset \bmod D"$,

(4) $\Vdash_{\mathbb{P}_{j+1}}$ "If $\alpha_0, \dots \alpha_{n(*)-1} \in \underset{\sim}{Z}$ are distinct, then for all $u \in \mathcal{A}, u \subseteq \bigcup_{l<n(*)} \underset{\sim}{w}^j_{\alpha_l} \Rightarrow \exists l < n(*), u \subseteq \underset{\sim}{w}^j_{\alpha_l}$".

The result follows immediately, as then the above are also forced to be true by $\mathbb{P}_\chi$.

Now we show that $C^D_{n(*)}(\kappa, J)$ fails. Let $T = \omega^{n(*)}$. We show that

$$\Vdash_{\mathbb{P}_\chi * \mathbb{Q}_{\lambda,\mathcal{A}}} \text{``}\langle \underset{\sim}{a}_{\alpha,n} : \alpha < \kappa, n < \omega\rangle \text{ exemplify } \neg C^D_{n(*)}(\kappa, J)\text{''},$$

where the names $\underset{\sim}{a}_{\alpha,n}$ are defined just after Definition 5.8. To this end, we check conditions in Lemma 5.11. Conditions $(1)-(3)$ from the lemma are clear. For $(4-a)$, assume on the contrary that in $V^{\mathbb{P}_\chi}, Y \neq \emptyset$ mode D is given and $p \in \mathbb{P}_\chi, p \Vdash$"$\underset{\sim}{Y}$ is a counterexample for $(4-a)$". In V, let $X_1 = \{\delta < \kappa : p \nVdash \text{``}\delta \notin \underset{\sim}{Y}\text{''}\}$. Then $X_1 \in V$ and $X_1 \neq \emptyset$ mod $D$. For any $\delta \in X_1$, let $p_\delta \leq p$ be such that $p_\delta \Vdash \text{``}\delta \in \underset{\sim}{Y}\text{''}$.

As in the proof of Lemma 5.7(b), and using the fact that D has the Δ-system property, we can find $X_2 \subseteq X_1, X_2 \neq \emptyset$ mod D such that $\langle \text{dom}(p_\delta) : \delta \in X_2\rangle$ form a Δ-system with root Δ and for all $\delta, \gamma \in X_2, p_\delta \restriction \Delta \parallel p_\gamma \restriction \Delta$ ($p_\delta \restriction \Delta$ is compatible with $p_\gamma \restriction \Delta$) Now let $t \in \omega^{n(*)}$ and let $\delta_0, \ldots, \delta_{n(*)-1}$ be in X_2 such that for each l, $\delta_{l+1} > \sup\{w^{p_{\delta_j}(0)} : j \leq l\}$. Let q be an extension of all $p_{\delta_l}, l < n(*)$ such that

$$t \in \omega^{n(*)} \Rightarrow \{\omega.\delta_l + t(l) : l < n(*)\} \in \mathcal{A}^{q(0)},$$

and

$$u \in \mathcal{A}^{q(0)}, k < n(*), v \subseteq \bigcup\{w^{p_{\delta_l}} : l < n(*), l \neq k\} \Rightarrow (\exists l) v \subseteq w^{p_{\delta_l}}.$$

For example we can set

$q(0) = \langle \bigcup_{l<n(*)} w^{p_{\delta_l}(0)} \cup \{\omega \cdot \delta_l + t(l) : l < n(*)\}, \bigcup_{l<n(*)} \mathcal{A}^{p_{\delta_l}(0)} \cup \{\{\omega.\delta_l + t(l)\} : t \in \omega^{n(*)}\}\rangle.$

Then $q \leq p$ and $q \Vdash$"$\underset{\sim}{Y}$ can not be a counterexample to $(4-a)$", a contradiction.

For $(4-b)$, again assume for some $p \in \mathbb{P}_\chi, t \in T$ and $\underset{\sim}{Y}_i, i < n(*)$, we have $p \Vdash$"$t, \langle \underset{\sim}{Y}_i : i < n(*)\rangle$ are counterexample to $(4-b)$". For each $i < n(*)$, let $X^*_i = \{\delta < \kappa : p \nVdash \text{``}\delta \notin \underset{\sim}{Y}_i\text{''}\}$. Then $X^*_i \in V$ and $X^*_i \neq \emptyset$ mod $D$. For any $\delta \in X^*_i$, let $p_{i,\delta} \leq p$ be such that $p_{i,\delta} \Vdash \text{``}\delta \in \underset{\sim}{Y}_i\text{''}$. Now proceed as in the proof of Lemma 5.11, case 2, to shrink each X^*_i to some $X_{i,2}$, such that:

(5) $X_{i,2} \neq \emptyset$ mod D,

(6) $\langle \text{dom}(p_{i,\delta}) : \delta \in X_{i,2}\rangle$ form a Δ-system with root Δ_i,

(7) $\delta, \gamma \in X_{i,2}, p_{i\delta} \restriction \Delta \parallel p_{i,\gamma} \restriction \Delta$,

(8) For all $\delta \in X_{i,2}, \gamma \in X_{j,2}, \text{dom}(p_{i,\delta}) \cap \text{dom}(p_{j,\gamma}) = \Delta_{i,j}$, for some fixed set $\Delta_{i,j}$,

(9) For all $\delta \in X_{i,2}, \gamma \in X_{j,2}, p_{i,\delta} \restriction \Delta_{i,j} \parallel p_{j,\gamma} \restriction \Delta_{i,j}$.

Let $\delta_l \in X_{l,2}, l < n(*)$. Let q be an extension of all $p_{l,\delta_l}, l < n(*)$, such that

$$q(0) = \langle \bigcup_{l<n(*)} w^{p_{\delta_l}} \cup \{\omega \cdot \delta_l + n : n < \omega, l < n(*)\}, \bigcup_{l<n(*)} \mathcal{A}^{p_{\delta_l}} \}\rangle.$$

Then $q \le p$ and $\{\omega.\delta_l + t(l) : l < n(*)\} \notin \mathcal{A}^q$, so

$$q \Vdash \text{"}\{\omega.\delta_l + t(l) : l < n(*)\} \notin \underset{\sim}{\mathcal{A}}\text{"}.$$

So $q \Vdash$"$t, \langle \underset{\sim}{Y}_i : i < n(*)\rangle$ can not be counterexamples to $(4-b)$", a contradiction.

6. On $C^s(\kappa)$ v.s. $C(\kappa)$

In this section, we consider the difference between the combinatorial principles $C^s(\kappa)$ and $C(\kappa)$, and prove the consistency of "$C(\kappa)$ holds but $C^s_T(\kappa)$ fails for all non-trivial T".

Lemma 6.1. *Assume that:*

(1) $\kappa = cf(\kappa) > \aleph_0$,

(2) $S^* \subseteq \kappa$ *is a stationary subset of* κ,

(3) $\bar{C} = \langle C_\delta : \delta \in S^* \rangle$ *is such that:*

(*a*) *Each* C_δ *is a club of* δ,

(*b*) *For every club* E *of* κ, *the set* $\{\delta \in S^* : \sup(C_\delta \setminus E) = \delta\}$ *is not stationary.*

(4) $2 \le n(*) < \omega$

Then there is $\mathcal{A} \subseteq [\kappa]^{n(*)}$ *such that:*

(α) *If* $S_l \subseteq S^*$ *is stationary for* $l < n(*)$, *then we can find* $\alpha_l, \beta_l \in S_l$, *for* $l < n(*)$ *such that* $\alpha_0 < \cdots < \alpha_{n(*)-1} < \beta_0 < \cdots < \beta_{n(*)-1}$ *and* $\{\alpha_l : l < n(*)\} \in \mathcal{A}, \{\beta_l : l < n(*)\} \notin \mathcal{A}$

(β) *If* $Y \subseteq \kappa$ *is unbounded, then for some unbounded subset* $Z \subseteq Y$ *we have* $[Z]^{n(*)} \cap \mathcal{A} \in \{\emptyset, [Z]^{n(*)}\}$.

Proof. Let

$$\mathcal{A} = \{\{\alpha_0, \ldots, \alpha_{n(*)-1}\} : \alpha_{n(*)-1} \in S^* \text{ and } l < n(*) - 1 \Rightarrow otp(\alpha_l \cap C_{\alpha_{n(*)-1}}) \text{ is odd}\}.$$

Let's show that $\mathcal{A}$ is as required:

(α) should be clear; let's prove (β). So assume $Y \subseteq \kappa$ is unbounded. So there is $Z_1 \subseteq Y$ of size κ such that $Z_1 \subseteq S^*$ or $Z_1 \cap S^* = \emptyset$. If $Z_1 \cap S^* = \emptyset$, then obviously $[Z_1]^{n(*)} \cap \mathcal{A} = \emptyset$ and we are done; so assume $Z_1 \subseteq S^*$. Define the sequence $\langle \alpha_i : i < \kappa \rangle$, by induction on $i < \kappa$, such that:

(1) $\alpha_i \in Z_1, \alpha_i > \sup\{\alpha_j : j < i\}$,

(2) $\sup(C_{\alpha_i} \cap \bigcup_{j<i} \alpha_j)$ is minimal.

Let $E = \{\delta < \kappa : \delta = \sup_{j<\delta} \alpha_j$ is a limit ordinal$\}$; so that E is a club of κ. Set

$$W_1 = \{\delta \in E \cap S^* : \delta < \sup(C_{\alpha_\delta} \cap \delta)\} \subseteq S^*.$$

Then W_1 is a stationary subset of κ, as otherwise we can find a club $C \subseteq E$ which is disjoint from W_1 and we get a contradiction with $(3-b)$. It follows from Fodor's lemma that for some $\alpha^* < \kappa$, the set

$$W_2 = \{\delta \in W_1 : \sup(C_{\alpha_\delta} \cap \delta) = \alpha^* < \delta\}$$

is stationary. Again by Fodor's lemma, there exists $\delta^* < \kappa$ such that the set

$$W_3 = \{\delta \in W_2 : \sup(C_{\alpha_\delta} \cap \alpha^*) = \delta^*\}$$

is stationary. Let $Z = \{\alpha_\delta : \delta \in W_3\}$. Clearly Z is an unbounded subset of Y. We show that $[Z]^{n(*)} \cap \mathcal{A} \in \{\emptyset, [Z]^{n(*)}\}$. Thus suppose that $[Z]^{n(*)} \cap \mathcal{A} \neq \emptyset$. Let $\delta_0 < \cdots < \delta_{n(*)-1} \in W_3$. Then $\alpha_{\delta_{n(*)-1}} \in S^*$ and for $l < n(*)-1$ we have

$$\begin{aligned} otp(C_{\alpha_{\delta_{n(*)-1}}} \cap \alpha_{\delta_l}) &= otp(C_{\alpha_{\delta_{n(*)-1}}} \cap \alpha^*) + otp((C_{\alpha_{\delta_{n(*)-1}}} \setminus \alpha^*) \cap \alpha_{\delta_l}) \\ &= \delta^* + otp((C_{\alpha_{\delta_{n(*)-1}}} \setminus \delta_{n(*)-1}) \cap \alpha_{\delta_l}) \\ &= \delta^* + 1 \text{ (as } C_{\alpha_{\delta_{n(*)-1}}} \setminus \delta_{n(*)-1}) \cap \alpha_{\delta_l} = \{\alpha^*\}), \end{aligned}$$

which is odd. So $\{\alpha_0, \ldots, \alpha_{n(*)-1}\} \in \mathcal{A}$, as required. □

Remark 6.2. *We can replace* $(3-b)$ *with* (α) & (β), *where:*

(α) *For every club* E_1 *of* κ, *there exists a club* $E_2 \subseteq E_1$ *of* κ, *such that for every* $\delta \in S^* \cap E_2$, *we have* $\delta = \sup\{\alpha < \delta : otp(C_\delta \cap \alpha)$ *is even*$\} = \sup\{\alpha < \delta : otp(C_\delta \cap \alpha)$ *is odd*$\}$,

(β) *There is no increasing continuous sequence* $\langle \alpha_i : i < \kappa \rangle$ *of ordinals* $< \kappa$ *such that* $C_{\alpha_{2i+1}} \supseteq \{\alpha_{2j} : j < i\}$ (*note that this holds if* $\sup\{C_\delta : \delta \in S^*\} < \kappa$).

Remark 6.3. *We can force the existence of such an* S^* *and* $\bar{C}$ *by forcing.*

Theorem 6.4. *Assume* $\kappa = cf(\kappa) > \aleph_0$, $\forall \alpha < \kappa(|\alpha|^{\aleph_0} < \kappa)$, *and let* $\mathcal{A} \subseteq [\kappa]^{n(*)}$ *be as in the conclusion of Lemma 6.1. Then for any non-trivial tree* $T \subseteq \omega^{\leq n(*)}$, *we have* $V^{\mathbb{Q}_{\kappa,\mathcal{A}}} \models$ "$C_{T,n(*)}(\kappa) + \neg C^s_{T,n(*)}(\kappa)$".

Proof. That $C^s_{T,n(*)}(\kappa)$ fails in $V^{\mathbb{Q}_{\kappa,\mathcal{A}}}$ follows from Lemmas 5.11 and 6.1(α). Also, $C_{T,n(*)}(\kappa)$ holds in $V^{\mathbb{Q}_{\kappa,\mathcal{A}}}$ by Lemmas 5.12 and 6.1(β). □

The following lemma can be proved similar to the proof of Lemma 6.1.

Lemma 6.5. *Let S^* and $\bar{C}$ be as in Lemma 6.1, and assume any $\delta \in S^*$ has uncountable cofinality. Then there is $\mathcal{A} \subseteq [\kappa]^{<\omega}$ such that:*

(α) *If $n(*) < \omega$ and $S_l \subseteq S^*$ is stationary for $l < n(*)$, then we can find $\alpha_l, \beta_l \in S_l$, for $l < n(*)$ such that $\alpha_0 < \cdots < \alpha_{n(*)-1} < \beta_0 < \cdots < \beta_{n(*)-1}$ and $\{\alpha_l : l < n(*)\} \in \mathcal{A}, \{\beta_l : l < n(*)\} \notin \mathcal{A}$*

(β) *If $Y \subseteq \kappa$ is unbounded, then for some unbounded subset $Z \subseteq Y$ we have $[Z]^{<\omega} \cap \mathcal{A} \in \{\emptyset, [Z]^{<\omega}\}$.*

Finally, we have the following; whose proof is the same as the proof of Theorem 6.4, using Lemma 6.5, instead of Lemma 6.1.

Theorem 6.6. *Assume $\kappa = cf(\kappa) > \aleph_0$, $\forall \alpha < \kappa(|\alpha|^{\aleph_0} < \kappa)$, and let $\mathcal{A} \subseteq [\kappa]^{<\omega}$ be as in the conclusion of Lemma 6.5. Then $V^{\mathbb{Q}_{\kappa,\mathcal{A}}} \models$ "$C(\kappa)+$ For any non-trivial tree $T \subseteq \omega^{<\omega}, \neg C^s_T(\kappa)$".*

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