

# THE TREE PROPERTY AT ALL REGULAR EVEN CARDINALS

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ABSTRACT. Assuming the existence of a strong cardinal and a measurable cardinal above it, we construct a model of ZFC in which for every singular cardinal  $\delta$ ,  $\delta$  is strong limit,  $2^\delta = \delta^{+3}$  and the tree property at  $\delta^{++}$  holds. This answers a question of Friedman, Honzik and Stejskalova [8]. We also produce, relative to the existence of a strong cardinal and two measurable cardinals above it, a model of ZFC in which the tree property holds at all regular even cardinals. The result answers questions of Friedman-Halilovic [5] and Friedman-Honzik [6].

## 1. INTRODUCTION

Trees are combinatorial objects which are of great importance in contemporary set theory. Recall that for a regular cardinal  $\kappa$ , a  $\kappa$ -tree is a tree of height  $\kappa$  all of whose levels have size less than  $\kappa$ . A  $\kappa$ -tree is called  $\kappa$ -Aronszajn if it has no cofinal branches.

For a regular cardinal  $\kappa$  let the tree property at  $\kappa$ , denoted  $\text{TP}(\kappa)$ , be the assertion “there are no  $\kappa$ -Aronszajn trees”. The following ZFC results are known about Aronszajn trees (see [15]).

- The tree property holds at  $\aleph_0$  (König),
- The tree property fails at  $\aleph_1$  (Aronszajn),
- For an inaccessible cardinal  $\kappa$ , the tree property holds at  $\kappa$  if and only if  $\kappa$  is weakly compact.

The problem of getting the tree property at successor regular cardinals bigger than  $\aleph_1$  is more subtle and it is independent of ZFC (modulo some large cardinal assumptions). The major problem, due to Magidor, is to prove the consistency of the tree property at all regular cardinals  $\kappa > \aleph_1$ .

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In this paper, we are interested in the tree property at regular even cardinals, i.e., cardinals of the form  $\kappa = \aleph_\alpha$ , where  $\alpha$  is an even ordinal<sup>1</sup>.

First we consider the problem of getting the tree property at double successor of singular strong limit cardinals. The first result in this direction is due to Cummings and Foreman [3], who produced, starting from a supercompact cardinals  $\kappa$  and a weakly compact cardinal above it, a model of ZFC in which  $\kappa$  is a singular strong limit cardinal of countable cofinality and the tree property holds at  $\kappa^{++}$ . They also extended their result for  $\kappa = \aleph_\omega$ . Friedman and Halilovic [5] proved the same results from a cardinal  $\kappa$  which is  $H(\lambda)$ -hypermeasurable for some weakly compact cardinal  $\lambda > \kappa$ . In [10], Gitik produced a model of “ $\aleph_\omega$  is strong limit and the tree property holds at  $\aleph_{\omega+2}$ ” from optimal hypotheses. The papers [4], [7] and [8] have continued the work, where more results about the tree property at double successor of singular strong limit cardinals of countable cofinality are obtained.

In [13], singular cardinals of uncountable cofinality are considered, and in it, a model is constructed in which the tree property holds at double successor of a singular strong limit cardinal of any prescribed cofinality.

In [8], Friedman, Honzik and Stejskalova produced a model of ZFC in which  $\aleph_\omega$  is strong limit,  $2^{\aleph_\omega} = \aleph_{\omega+3}$  and the tree property holds at  $\aleph_{\omega+2}$ . They asked if we can replace  $\aleph_\omega$  by  $\aleph_{\omega_1}$ . We answer their question; in fact we prove the following global consistency result:

**Theorem 1.1.** *Assume  $\kappa$  is an  $H(\lambda^{++})$ -hypermeasurable cardinal where  $\lambda > \kappa$  is measurable. Then there is a generic extension  $W$  of  $V$  in which the following hold:*

- (a)  $\kappa$  is inaccessible.
- (b) For every singular cardinal  $\delta < \kappa$ ,  $\delta$  is strong limit and  $2^\delta = \delta^{+3}$ .
- (c) For every singular cardinal  $\delta < \kappa$ , the tree property at  $\delta^{++}$  holds.

*In particular the rank initial segment  $W_\kappa$  of  $W$  is a model of ZFC in which for every singular cardinal  $\delta$ , the tree property at  $\delta^{++}$  holds and  $2^\delta = \delta^{+3}$ .*

**Remark 1.2.** *Given any finite  $n \geq 2$ , we can replace  $2^\delta = \delta^{+3}$  with  $2^\delta = \delta^{+n}$ .*

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<sup>1</sup>Recall that each ordinal  $\alpha$  can be written uniquely as  $\alpha = 2 \cdot \beta + \xi$ , where  $\beta \leq \alpha$  is an ordinal and  $\xi < 2$ .  $\alpha$  is called even if  $\xi = 0$  and it is called odd if  $\xi = 1$ .

Then we consider the problem of getting the tree property at all regular even cardinals. In [23], Mitchell showed that starting from two weakly compact cardinals one can get the tree property at both  $\aleph_2$  and  $\aleph_4$ . Starting from infinitely many weakly compact cardinals, his result can be easily extended to get the tree property at all  $\aleph_{2n}$ 's,  $0 < n < \omega$ . The problem of getting the tree property at  $\aleph_{2n}$ 's,  $0 < n < \omega$  and  $\aleph_{\omega+2}$  while  $\aleph_\omega$  is strong limit has remained open. In [6], Friedman and Honzik produced a model in which the tree property holds at all even cardinals below  $\aleph_\omega$  where  $\aleph_\omega$  is strong limit and  $2^{\aleph_\omega} = \aleph_{\omega+2}$ . Unger [28] has extended this result to get the tree property at all  $\aleph_n$ 's,  $1 < n < \omega$ . None of these papers obtain the tree property at  $\aleph_{\omega+2}$ . We address this question and prove the following, which in particular answers a question of [6]:

**Theorem 1.3.** *Assume  $\eta > \lambda$  are measurable cardinals above  $\kappa$  and  $\kappa$  is  $H(\eta)$ -hypermeasurable.*

*Then there is a generic extension  $W$  of  $V$  in which:*

- (a)  $\kappa = \aleph_\omega$  is a strong limit cardinal.
- (b)  $\lambda = \aleph_{\omega+2}$ .
- (c) *The tree property holds at all  $\aleph_{2n}$ 's,  $0 < n < \omega$  and at  $\aleph_{\omega+2}$ .*

Then, we prove the following global consistency result, which is related to a question of Friedman and Halilovic [5].

**Theorem 1.4.** *Assume  $\eta > \lambda$  are measurable cardinals above  $\kappa$  and  $\kappa$  is  $H(\eta^+)$ -hypermeasurable.*

*Then there is a generic extension  $W$  of  $V$  in which:*

- (a)  $\kappa$  is inaccessible.
- (b) *The tree property holds at all regular even cardinals below  $\kappa$ .*

*In particular the rank initial segment  $W_\kappa$  of  $W$  is a model of ZFC in which the tree property holds at all regular even cardinals.*

The paper is organized as follows. Sections 2 and 3 are devoted to some preliminary results. In Section 2 we present some preservation lemmas and in Section 3 we review a generalization of Mitchell's forcing and prove some facts related to it. In Section 4, we prove Theorem 1.1. Section 5 is devoted to the proof of Theorem 1.3 and finally in Section 6 we prove Theorem 1.4.

We assume familiarity with forcing and large cardinals. For a forcing notion  $\mathbb{P}$  we use  $p \leq q$  to mean  $p$  gives more information than  $q$ , i.e.,  $p \Vdash "q \in \dot{G}"$ , where  $\dot{G}$  is the canonical  $\mathbb{P}$ -name for the generic filter.

## 2. SOME PRESERVATION LEMMAS

The standard way to construct models with the tree property in a small cardinal is to start with some large cardinal that has certain reflection properties, and collapse it to become a specific accessible cardinal. In order to show that the tree property holds in the generic extension, we pick a name for a  $\kappa$ -tree,  $\dot{T}$ , and assume towards a contradiction that it is forced to be an Aronszajn tree. Then, we use the reflection properties of the initial large cardinal and show that the fact that  $\dot{T}$  is Aronszajn in the generic extension, implies that the restriction of  $\dot{T}$  to some ordinal  $\alpha$  is Aronszajn tree at some intermediate stage of the forcing. Then we show that the rest of the forcing cannot add a cofinal branch to this Aronszajn tree and get a contradiction. So the standard way to construct models of the tree property in various small cardinals is to go through preservation lemmas that show that various forcing notions cannot add branches to certain trees.

The following three lemmas state that forcing notions with good enough chain condition do not add branches to Aronszajn trees

**Lemma 2.1** (Folklore). *Let  $\kappa$  be a regular cardinal and let  $T$  be a  $\kappa$ -Aronszajn tree. Let  $\mathbb{P}$  be a  $\eta$ -c.c. forcing notion, with  $\eta < \kappa$ . Then  $\mathbb{P}$  does not add a branch to  $T$ .*

*Proof.* Assume by contradiction that this is not the case and let  $\dot{b}$  be a name for a new branch. Let us define:

$$T' = \{t \in T \mid \exists p \in \mathbb{P}, p \Vdash \check{t} \in \dot{b}\}$$

Note that if  $x \leq_T y$  and  $y \in T'$  then  $x \in T'$  as well since  $\dot{b}$  is forced to be downward closed. Therefore  $T$  agrees with  $T'$  about the level of elements. Moreover, for every  $\alpha < \kappa$ , there is some  $t \in T'_\alpha$  since  $\dot{b}$  is forced to be unbounded.

We conclude that  $T'$  is a subtree of  $T$  of height  $\kappa$ . We claim that the width of the levels of  $T'$  is less than  $\eta$ . So let  $\alpha < \kappa$  and let  $\mathcal{A}$  be a maximal antichain in  $\mathbb{P}$  of elements that decide the value of  $\dot{b}$  at the level  $\alpha$ . By the chain condition of  $\mathbb{P}$ ,  $|\mathcal{A}| < \eta$  and therefore there

are less than  $\eta$  possible values for  $\dot{b}(\alpha)$ . But every element from  $T'_\alpha$  is a potential value for  $\dot{b}(\alpha)$ , so we conclude that  $|T'_\alpha| < \eta$ .

By a theorem of Kurepa (see [16] Proposition 7.9),  $T'$  has a branch.  $\square$

The following branch lemma is due to Kunen and Tall, (see [17]).

**Lemma 2.2.** *Let  $\kappa$  be a regular cardinal and let  $T$  be a  $\kappa$ -Aronszajn tree. Let  $\mathbb{P}$  be a  $\kappa$ -Knaster forcing notion. Then  $\mathbb{P}$  does not add a branch to  $T$ .*

*Proof.* Let  $\dot{b}$  be a name of a cofinal branch in  $T$ . For every  $\alpha < \kappa$ , pick a condition  $p_\alpha \in \mathbb{P}$  and an element  $t_\alpha \in T_\alpha$ , such that  $p_\alpha \Vdash t_\alpha \in \dot{b}$ . By the Knaster property, there is a cofinal subset  $I$  of  $\kappa$ , such that  $\forall \alpha, \beta \in I$   $p_\alpha$  is compatible with  $p_\beta$ .

Let us choose  $q \leq p_\alpha, p_\beta$ .  $q \Vdash t_\alpha, t_\beta \in \dot{b}$  and therefore it forces  $t_\alpha \leq_T t_\beta$ . But this is a  $\Delta_0$ -statement about elements of  $V$  so it holds in  $V$  as well. In particular,  $\tilde{b} = \{t \in T \mid \exists \alpha \in I, t \leq_T t_\alpha\}$  is a cofinal branch.  $\square$

The following lemma is due to Silver.

**Lemma 2.3.** *Let  $\kappa$  and  $\lambda$  be cardinals with  $\lambda$  regular. Suppose that  $2^\kappa \geq \lambda$ ,  $T$  be a  $\lambda$ -tree and  $\mathbb{P}$  be  $\kappa^+$ -closed. Then forcing with  $\mathbb{P}$  cannot add a new branch through  $T$ .*

We also need the following results of Unger (see [27]).

**Lemma 2.4.** *Assume  $\kappa$  is a regular cardinal and  $\mathbb{P}$  is a forcing notion such that  $\mathbb{P} \times \mathbb{P}$  is  $\kappa$ -c.c. Then forcing with  $\mathbb{P}$  adds no branches to  $\kappa$ -trees*

**Lemma 2.5.** *Let  $\kappa$  be a regular cardinal. Let  $\rho \leq \mu \leq \kappa$  be cardinals such that  $2^\rho \geq \kappa$  and  $2^{<\rho} < \kappa$ . Let  $\mathbb{P}$  be  $\mu$ -c.c. forcing notion and let  $\mathbb{Q}$  be  $\mu$ -closed forcing notion in the ground model. Let  $T$  be a  $\kappa$ -tree in  $V^\mathbb{P}$ . Then in  $V^\mathbb{P}, \mathbb{Q}$  does not add new branches to  $T$ .*

### 3. MITCHELL'S FORCING AND ITS PROPERTIES

In this section we present a version of Mitchell's forcing and discuss some of its properties. The forcing is essentially the same as Mitchell's forcing, but it allows us to blow up the power function as well.

**Definition 3.1.** Assume  $\alpha < \beta$  are regular cardinals and  $\gamma \geq \beta$  is an ordinal. Let  $\mathbb{M}(\alpha, \beta, \gamma)$  be the following forcing for making  $2^\alpha = |\gamma|$  and forcing the tree property at  $\beta = \alpha^{++}$ :

- (a) A condition in  $\mathbb{M}(\alpha, \beta, \gamma)$  is a pair  $(p, q)$ , where
- (1)  $p \in \text{Add}(\alpha, \gamma)$ ,
  - (2)  $\text{dom}(q)$  is a subset of  $\beta$  of size  $\leq \alpha$ ,
  - (3) For each  $\xi \in \text{dom}(q)$ ,  $1_{\text{Add}(\alpha, \xi)} \Vdash "q(\xi) \in \text{Add}(\alpha^+, 1)"$ .
- (b) For  $(p, q), (p', q') \in \mathbb{M}(\alpha, \beta, \gamma)$ , say  $(p', q') \leq (p, q)$  iff
- (1)  $p' \leq_{\text{Add}(\alpha, \gamma)} p$ ,
  - (2)  $\text{dom}(q') \supseteq \text{dom}(q)$ ,
  - (3) For all  $\xi \in \text{dom}(q)$ ,  $1_{\text{Add}(\alpha, \xi)} \Vdash "q'(\xi) \leq_{\text{Add}(\alpha^+, 1)} q(\xi)"$ .

In the case  $\gamma = \beta$  we obtain Mitchell's forcing.

**Definition 3.2.** Assume  $\alpha < \beta$  are regular cardinals. The Mitchell's forcing  $\mathbb{M}(\alpha, \beta)$  is defined by

$$\mathbb{M}(\alpha, \beta) = \mathbb{M}(\alpha, \beta, \beta).$$

We refer to [8] for more discussion about the forcing notion  $\mathbb{M}(\alpha, \beta, \beta)$  and only present some of its basic properties which are needed in this paper. Assume *GCH* holds and let  $\alpha < \beta \leq \gamma$  be such that  $\alpha$  is regular and  $\beta$  is a measurable cardinal.

- Lemma 3.3.**
- (a)  $\mathbb{M}(\alpha, \beta, \gamma)$  is  $\alpha$ -directed closed.
  - (b)  $\mathbb{M}(\alpha, \beta, \gamma)$  is  $\beta$ -Knaster.
  - (c) In the generic extension by  $\mathbb{M}(\alpha, \beta, \gamma)$ ,  $\alpha^+$  is preserved,  $2^\alpha \geq |\gamma|$ ,  $\beta = \alpha^{++}$ , and  $\text{TP}(\beta)$  holds.

Let  $\mathbb{T}(\alpha, \beta, \gamma)$  be the term forcing notion defined by

$$\mathbb{T}(\alpha, \beta, \gamma) = \{(\emptyset, q) : (p, q) \in \mathbb{M}(\alpha, \beta, \gamma)\}.$$

- Lemma 3.4.**
- (a)  $\mathbb{T}(\alpha, \beta, \gamma)$  is  $\alpha^+$ -closed
  - (b) There exists a projection from  $\text{Add}(\alpha, \gamma) \times \mathbb{T}(\alpha, \beta, \gamma)$  onto  $\mathbb{M}(\alpha, \beta, \gamma)$ .
  - (c)  $\mathbb{M}(\alpha, \beta, \gamma) \cong \text{Add}(\alpha, \gamma) * \dot{\mathbb{Q}}(\alpha, \beta, \gamma)$  for an  $\text{Add}(\alpha, \gamma)$ -name  $\dot{\mathbb{Q}}(\alpha, \beta, \gamma)$  which is forced by  $\text{Add}(\alpha, \gamma)$  to be  $\alpha^+$ -distributive.

The next lemma strengthens Lemma 3.3(c).

**Lemma 3.5.** *Assume  $\alpha < \beta < \gamma$ , where  $\alpha$  is regular and  $\beta, \gamma$  are measurable cardinals. Let  $G * H$  be an  $\mathbb{M}(\alpha, \beta) * \dot{\mathbb{M}}(\beta, \gamma)$ -generic filter over  $V$ . Then:*

- (a)  $\text{Card}^{V[G*H]} \cap [\alpha, \gamma] = \{\alpha, \alpha^+, \beta, \beta^+, \gamma\}$ .
- (b)  $V[G * H] \models "2^\alpha = \beta = \alpha^{++} \text{ and } 2^\beta = \gamma = \beta^{++}"$ .
- (c)  $V[G * H] \models "The \text{tree property holds at both } \beta \text{ and } \gamma"$ .

*Proof.* <sup>2</sup> Parts (a) and (b) can be proved easily using Lemmas 3.3 and 3.4. Let us prove (c). By Lemma 3.3(c),  $\text{TP}(\gamma)$  holds in  $V[G * H]$ , thus let us show that  $\text{TP}(\beta)$  holds in  $V[G * H]$ .

Let  $\dot{T}$  be an  $\mathbb{M}(\alpha, \beta) * \dot{\mathbb{M}}(\beta, \gamma)$ -name for a  $\beta$ -tree. We have

$$\mathbb{M}(\alpha, \beta) * \dot{\mathbb{M}}(\beta, \gamma) \cong \mathbb{M}(\alpha, \beta) * \dot{\text{Add}}(\beta, \gamma) * \dot{\mathbb{Q}}(\beta, \gamma, \gamma)$$

where

$$\Vdash_{\mathbb{M}(\alpha, \beta) * \dot{\text{Add}}(\beta, \gamma)} "\dot{\mathbb{Q}}(\beta, \gamma, \gamma) \text{ is } \beta^+ \text{ distributive}."$$

Thus  $\dot{T}$  is added by  $\mathbb{M}(\alpha, \beta) * \dot{\text{Add}}(\beta, \gamma)$ . By the chain condition and the homogeneity of  $\dot{\text{Add}}(\beta, \gamma)$ ,  $\dot{T}$  is equivalent to an  $\mathbb{M}(\alpha, \beta) * \dot{\text{Add}}(\beta, 1)$ -name.

Let  $j: V \rightarrow M$  be an elementary embedding with critical point  $\beta$ , and set  $\mathbb{Q} = \mathbb{M}(\alpha, \beta) * \dot{\text{Add}}(\beta, 1)$ . Then

$$j(\mathbb{Q}) = j(\mathbb{M}(\alpha, \beta)) * \dot{\text{Add}}(j(\beta), 1) \cong \mathbb{M}(\alpha, \beta) * \dot{\text{Add}}(\alpha^+, 1) * \dot{\mathbb{R}} * \dot{\text{Add}}(j(\beta), 1)$$

for some name  $\dot{\mathbb{R}}$ , which is forced to be  $\alpha^+$ -distributive. Since  $\dot{\text{Add}}(\alpha^+, 1)$  is forcing equivalent to  $\text{Col}(\alpha^+, 2^\alpha)$  and after forcing with  $\mathbb{M}(\alpha, \beta)$ ,  $2^\alpha = 2^{<\beta} = \beta$ , we have

$$\dot{\text{Add}}(\alpha^+, 1) \cong \text{Col}(\alpha^+, \beta) \cong \dot{\text{Add}}(\beta, 1) \times \text{Col}(\alpha^+, \beta) \cong \dot{\text{Add}}(\beta, 1) * \dot{\text{Col}}(\alpha^+, \beta)$$

and hence

$$\mathbb{M}(\alpha, \beta) * \dot{\text{Add}}(\alpha^+, 1) \cong \mathbb{M}(\alpha, \beta) * \dot{\text{Col}}(\alpha^+, \beta) \cong \mathbb{M}(\alpha, \beta) * \dot{\text{Add}}(\beta, 1) * \dot{\text{Col}}(\alpha^+, \beta).$$

Thus, we can represent  $j(\mathbb{Q})$  in the following way:

$$j(\mathbb{Q}) \cong \mathbb{Q} * \dot{\text{Col}}(\alpha^+, \beta) * \dot{\mathbb{R}} * \dot{\text{Add}}(j(\beta), 1).$$

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<sup>2</sup>The proof presented here is suggested by Yair Hayut, which is based on ideas of Unger [27]

Let  $J$  be the  $\mathbb{Q}$ -generic filter over  $V$  derived from  $G * H$ . Using the closure of  $\text{Add}(j(\beta), 1)$ , one can obtain a master condition and force a generic filter  $K$  over  $M[J]$  such that  $J * K$  is a generic filter for  $j(\mathbb{Q})$  and  $j''[J] \subseteq J * K$ . Therefore, in  $V[J * K]$ , we can extend  $j$  to an elementary embedding  $\tilde{j}: V[J] \rightarrow M[J * K]$ . In particular,  $\tilde{j}(\dot{T}^J)$  is a  $j(\beta)$ -tree and thus by taking any element from the  $\beta$ -th level of  $\tilde{j}(\dot{T}^J)$  we can obtain a branch  $b$  of  $\dot{T}^J$ .

Let us show that the forcing

$$\text{Col}(\alpha^+, \beta) * \dot{\mathbb{R}} * \dot{\text{Add}}(j(\beta), 1)$$

cannot add a branch to  $\dot{T}^J$ , so that  $b \in V[J] \subseteq V[G * H]$ .

By Lemma 3.4(b), there exists a projection from  $\text{Add}(\alpha, \beta) \times \mathbb{T}(\alpha, \beta, \beta)$  onto  $\mathbb{M}(\alpha, \beta)$ , and hence using  $j$ , we get a projection from  $\text{Add}(\alpha, j(\beta)) \times \mathbb{T}(\alpha, j(\beta), j(\beta))$  onto  $j(\mathbb{M}(\alpha, \beta))$ . Since  $j(\mathbb{M}(\alpha, \beta)) \cong \mathbb{M}(\alpha, \beta) * \dot{\text{Col}}(\alpha^+, \beta) * \dot{\mathbb{R}}$ , thus we get a projection from  $\text{Add}(\alpha, j(\beta)) \times \mathbb{T}(\alpha, j(\beta), j(\beta))$  onto  $\text{Col}(\alpha^+, \beta) * \dot{\mathbb{R}}$ . The forcing  $\mathbb{T}(\alpha, j(\beta), j(\beta)) * \dot{\text{Add}}(j(\beta), 1)$  is  $\alpha^+$ -closed of size  $j(\beta)$ , and hence

$$\text{Col}(\alpha^+, j(\beta)) \cong \text{Col}(\alpha^+, j(\beta)) \times (\mathbb{T}(\alpha, j(\beta), j(\beta)) * \dot{\text{Add}}(j(\beta), 1)).$$

Putting all things together, we get a projection

$$\pi : \text{Add}(\alpha, j(\beta)) \times \text{Col}(\alpha^+, j(\beta)) \rightarrow \text{Col}(\alpha^+, \beta) * \dot{\mathbb{R}} * \dot{\text{Add}}(j(\beta), 1).$$

But Lemma 2.5, the forcing  $\text{Add}(\alpha, j(\beta)) \times \text{Col}(\alpha^+, j(\beta))$  cannot add a branch to  $\dot{T}^J$ . It follows that  $\text{Col}(\alpha^+, \beta) * \dot{\mathbb{R}} * \dot{\text{Add}}(j(\beta), 1)$  does not add a branch to  $\dot{T}^J$ , as required.  $\square$

**Remark 3.6.** *The lemma is true if we assume  $\beta$  and  $\gamma$  are weakly compact cardinals. The argument is essentially the same, where instead of an embedding from the whole universe, we use a weakly compact embedding  $j : \mathcal{M} \rightarrow \mathcal{N}$ , where  $\mathcal{M}$  contains all relevant information.*

In a similar way, we can prove the following.

**Lemma 3.7.** *Assume  $\alpha_0 < \alpha_1 < \dots < \alpha_n$ , where  $\alpha_0$  is regular and  $\alpha_1, \dots, \alpha_n$  are measurable cardinals and let  $G = G_0 * G_1 * \dots * G_{n-1}$  be a generic filter over  $V$  for the forcing notion*

$$\mathbb{M}(\alpha_0, \alpha_1) * \dot{\mathbb{M}}(\alpha_1, \alpha_2) * \dots * \dot{\mathbb{M}}(\alpha_{n-1}, \alpha_n).$$

*Then*



- (a)  $\text{Card}^{V[G]} \cap [\alpha_0, \alpha_n] = \{\alpha_0, \alpha_0^+, \alpha_1, \alpha_1^+, \dots, \alpha_{n-1}, \alpha_{n-1}^+, \alpha_n\}$ .
- (b)  $V[G] \models$  “For each  $i < n$ ,  $2^{\alpha_i} = \alpha_{i+1} = \alpha_i^{++}$ ”.
- (c)  $V[G] \models$  “The tree property holds at each  $\alpha_i$ ,  $1 \leq i \leq n$ ”.

Assume that  $\eta > \lambda$  are measurable cardinals above  $\kappa$  and there exists an elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  such that  $H(\eta) \subseteq M$  and  $j$  is generated by a  $(\kappa, \eta)$ -extender. Suppose there exists  $\bar{g} \in V$  which is  $i(\text{Add}(\kappa, \lambda)_V)$ -generic over  $N$ , where  $U$  is the normal measure derived from  $j$  and  $i : V \rightarrow N \simeq \text{Ult}(V, U)$  is the ultrapower embedding.

Let

$$\mathbb{P} = \langle \langle \mathbb{P}_\alpha : \alpha \leq \kappa + 1 \rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha \leq \kappa \rangle \rangle$$

be the reverse Easton iteration, where

- (1) If  $\alpha \leq \kappa$  is a measurable limit of measurable cardinals, then

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{M}}(\alpha, \alpha_*) * \dot{\mathbb{M}}(\alpha_*, \alpha_{**})\text{”},$$

where for each  $\alpha \leq \kappa$ ,  $\alpha_* < \alpha_{**}$  are the first and the second measurable cardinals above  $\alpha$  respectively.

- (2) Otherwise,  $\Vdash_\alpha$  “ $\dot{\mathbb{Q}}_\alpha$  is the trivial forcing”.

Let

$$G = \langle \langle G_\alpha : \alpha \leq \kappa + 1 \rangle, \langle G(\alpha) : \alpha \leq \kappa \rangle \rangle$$

be  $\mathbb{P}$ -generic over  $V$ . Also let  $\mathbb{M} = \mathbb{M}(\aleph_0, \kappa)$ .

**Lemma 3.8.** *Forcing with  $\mathbb{P} \times \mathbb{M}$  forces “ $\kappa = \aleph_2$  and  $\text{TP}(\kappa)$  holds”.*

*Similarly, if we replace  $\mathbb{P}$  with  $\mathbb{P}_{(\lambda, \kappa+1]}$ , the tail iteration after  $\lambda$ , and  $\mathbb{M}$  with  $\mathbb{M}(\lambda, \kappa)$ , then the resulting product forces “ $\kappa = \lambda^{++}$  and  $\text{TP}(\kappa)$  holds”*

*Proof.* It is easily seen that the forcing notion  $\mathbb{P} \times \mathbb{M}$  preserves  $\aleph_1$  and  $\kappa$  and forces  $\kappa = \aleph_2$ .

Let us show that it forces the tree property at  $\kappa$ .

Let  $G \times H$  be  $\mathbb{P} \times \mathbb{M}$ -generic over  $V$  and assume  $T$  is a  $\kappa$ -tree in  $V[G \times H]$ . We have

$$j(\mathbb{P}) \cong \mathbb{P}_\kappa * \dot{\mathbb{M}}(\kappa, \lambda) * \dot{\mathbb{M}}(\lambda, \eta) * j(\mathbb{P})_{(\kappa+1, j(\kappa))} * \dot{\mathbb{M}}(j(\kappa), j(\lambda)) * \dot{\mathbb{M}}(j(\lambda), j(\eta))$$

where  $\Vdash_{\mathbb{P}_\kappa * \dot{M}(\kappa, \lambda) * \dot{M}(\lambda, \eta)}$  “ $j(\mathbb{P})_{(\kappa+1, j(\kappa))}$  is  $\kappa^+$ -closed and  $j(\kappa)$ -c.c.”. On the other hand,

$$\mathbb{M}(\kappa, \lambda) * \dot{M}(\lambda, \eta) \cong \text{Add}(\kappa, \lambda) * \dot{\mathbb{Q}},$$

where  $\dot{\mathbb{Q}}$  is forced to be  $\kappa^+$ -distributive. Thus

$$j(\mathbb{P}) \cong \mathbb{P}_\kappa * \text{Add}(\kappa, \lambda) * \dot{\mathbb{Q}} * j(\mathbb{P})_{(\kappa+1, j(\kappa))} * \text{Add}(j(\kappa), j(\lambda)) * j(\dot{\mathbb{Q}}).$$

Let us write  $G$  as

$$G = G_\kappa * g * h$$

which corresponds to  $\mathbb{P} = \mathbb{P}_\kappa * \text{Add}(\kappa, \lambda) * \dot{\mathbb{Q}}$ .

Let  $k : N \rightarrow M$  be the induced elementary embedding so that  $k \circ i = j$ . By standard arguments, we can lift  $k$  to  $N[G_\kappa]$  to get  $k : N[G_\kappa] \rightarrow M[G_\kappa]$ .

Let  $\bar{\lambda}$  be such that

$$N \models \text{“}\bar{\lambda} \text{ is the least measurable cardinal above } \kappa\text{”}.$$

Note that  $\kappa^+ < \bar{\lambda} < \kappa^{++}$ . Factor  $g$  as  $g_1 \times g_2$ , which corresponds to

$$\text{Add}(\kappa, \lambda)_{V[G_\kappa]} = \text{Add}(\kappa, \bar{\lambda})_{V[G_\kappa]} \times \text{Add}(\kappa, \lambda \setminus \bar{\lambda})_{V[G_\kappa]}.$$

We can further extend  $k$  to get  $k : N[G_\kappa][g_1] \rightarrow M[G_\kappa][g]$ .

By an argument as above, we can write  $i(\mathbb{P})$  as

$$i(\mathbb{P}) \cong \mathbb{P}_\kappa * \text{Add}(\kappa, \bar{\lambda}) * \dot{\mathbb{Q}}_1 * i(\mathbb{P})_{(\kappa+1, i(\kappa))} * \text{Add}(i(\kappa), i(\bar{\lambda})) * i(\dot{\mathbb{Q}}_1).$$

where  $\Vdash_{\mathbb{P}_\kappa * \text{Add}(\kappa, \bar{\lambda})}$  “ $\dot{\mathbb{Q}}_1$  is  $\kappa^+$ -distributive”. Let  $h_1$  be the filter generated by  $i''(h)$ . Then  $h_1$  is  $\dot{\mathbb{Q}}_1[G_\kappa * g_1]$ -generic over  $N[G_\kappa * g_1]$ .

By standard arguments, we can find  $K \in V[G_\kappa * g_1 * h_1]$ , which is  $i(\mathbb{P})_{(\kappa+1, i(\kappa))}$ -generic over  $N[G_\kappa * g_1 * h_1]$ . We transfer  $g_1 * h_1 * K$  along  $k$  to get

$$\begin{aligned} i : V[G_\kappa] &\rightarrow N[i(G_\kappa)], \\ k : N[i(G_\kappa)] &\rightarrow M[j(G_\kappa)], \end{aligned}$$

where the maps are defined in  $V[G_\kappa * g * h]$ . Since  $\mathbb{P}_\kappa$  has size  $\kappa$  and is  $\kappa$ -c.c., so the term forcing

$$\text{Add}(\kappa, \lambda)_{V[G_\kappa]} / \mathbb{P}_\kappa$$

is forcing isomorphic to  $\text{Add}(\kappa, \lambda)_V$  (see [1] Fact 2, §1.2.6). By our assumption, we have  $\bar{g} \in V$  which is  $i(\text{Add}(\kappa, \lambda)_V)$ -generic over  $N$ , and using it we can define  $g_a$  which is

$$\text{Add}(i(\kappa), i(\lambda))_{N[i(G_\kappa)]}$$

generic over  $N[i(G_\kappa)]$ .

Transfer  $g_a$  along  $k$  to get the new generic  $\bar{g}_a$ . Now using Woodin's surgery argument, we can alter the filter  $\bar{g}_a$  to find a generic filter  $h_a$  with the additional property  $j''[g] \subseteq h_a$ .

So we can build maps

$$\begin{aligned} j &: V[G_\kappa * g] \rightarrow M[j(G_\kappa * g)], \\ k &: N[i(G_\kappa * g)] \rightarrow M[j(G_\kappa * g)], \\ i &: V[G_\kappa * g] \rightarrow N[i(G_\kappa * g)]. \end{aligned}$$

The forcing  $\dot{\mathbb{Q}}[G_\kappa * g]$  is  $\kappa^+$ -distributive in  $V[G_\kappa * g]$ , so we can further extend the above embeddings and get

$$\begin{aligned} j &: V[G_\kappa * g * h] \rightarrow M[j(G_\kappa * g * h)], \\ k &: N[i(G_\kappa * g * h)] \rightarrow M[j(G_\kappa * g * h)], \\ i &: V[G_\kappa * g * h] \rightarrow N[i(G_\kappa * g * h)]. \end{aligned}$$

In particular, we have  $j : V[G] \rightarrow M[j(G)]$ . Now suppose  $j(H)$  is  $j(\mathbb{M})$ -generic over  $M[j(G)]$  such that  $j$  lifts to

$$\tilde{j} : V[G \times H] \rightarrow M[j(G) \times j(H)].$$

Then  $T \in M[G \times H]$  and  $T$  has a cofinal branch in  $M[j(G) \times j(H)]$ . But note that

$$j(\mathbb{P})/G \simeq j(\mathbb{P})_{(\kappa+1, j(\kappa))} * \dot{\text{Add}}(j(\kappa), j(\lambda)) * j(\dot{\mathbb{Q}})$$

is  $\kappa^+$ -closed. Also by the proof of Lemma 3.5,  $j(\mathbb{M})/H$  can not add a cofinal branch in  $T$ . Thus it follows that forcing with  $j(\mathbb{P})/G \times \mathbb{M}/H$  can not add a branch through  $T$ , which leads to a contradiction. The lemma follows.  $\square$

In fact one can say more. Let  $U$  be a normal measure on  $\kappa$ . Then for any  $\lambda < \kappa$ , one can show that

$$\Lambda_\lambda = \{\gamma \in (\lambda, \kappa) : \mathbb{P}_{(\lambda, \gamma)} \times \mathbb{M}(\lambda, \gamma) \Vdash \text{“} \gamma = \lambda^{++} + TP(\gamma) \text{”}\} \in U.$$

and hence

$$\Lambda = \Delta_{\lambda < \kappa} \Lambda_\lambda = \{\gamma < \kappa : \forall \lambda < \gamma, \gamma \in \Lambda_\lambda\} \in U.$$

In a similar way, we can prove the following lemma that will be used later.

**Lemma 3.9.** *Suppose  $\alpha < \beta < \kappa$  are such that  $\alpha$  is regular and  $\beta$  is measurable. Then*

$$\mathbb{M}(\alpha, \beta) * (\dot{\mathbb{P}}_{(\beta, \kappa+1]} \times \dot{\mathbb{M}}(\beta, \kappa))$$

*forces the tree property at both  $\beta$  and  $\kappa$ , where working in  $V^{\mathbb{M}(\alpha, \beta)}$ , the iteration  $\mathbb{P}$  is defined as before and  $\mathbb{P}_{(\beta, \kappa+1]}$  denotes the tail iteration after  $\beta$ .*

#### 4. THE TREE PROPERTY AT DOUBLE SUCCESSOR OF SINGULAR CARDINALS

In this section we prove Theorem 1.1. In Subsection 4.1 we define the notion of a measure sequence and then in Subsection 4.2, we assign to each measure sequence  $w$  a forcing notion  $\mathbb{R}_w$ , which is a version of Radin forcing which is needed for the proof of Theorem 1.1. In Subsection 4.3, we review some of the basic properties of the forcing notion  $\mathbb{R}_w$ . Then in Subsection 4.4 we define the required model and in Subsections 4.5 and 4.6 we complete the proof of Theorem 1.1.

**4.1. Measure sequences.** In this subsection we define a class  $\mathcal{U}_\infty$  of measure sequences which are needed for the proof of Theorem 1.1. Our presentation follows [11], but we present the details for completeness. During the Subsections 4.1, 4.2 and 4.3, we assume that the following conditions are satisfied:

- $\kappa$  is an  $H(\kappa^{++})$ -hypermeasurable cardinal.
- $2^\kappa = 2^{\kappa^+} = 2^{\kappa^{++}} = \kappa^{+3}$ .
- There is  $j : V \rightarrow M$  with critical point  $\kappa$  such that  $H(\kappa^{++}) \subseteq M$ .
- $j$  is generated by a  $(\kappa, \kappa^{+4})$ -extender.
- $\kappa^{+4} < j(\kappa) < \kappa^{+5}$ .
- If  $U$  is the normal measure derived from  $j$  and if  $i : V \rightarrow N \simeq \text{Ult}(V, U)$  is the ultrapower embedding, then there exists  $F \in V$  which is  $\text{Col}(\kappa^{+5}, < i(\kappa))_N$ -generic over  $N$ .

Let  $k : N \rightarrow M$  be the induced elementary embedding with  $j = k \circ i$ . Then  $\text{crit}(k) = \kappa_N^{+4} < \kappa_M^{+4} = \kappa^{+4}$ . Set

$$P^* = \{f : \kappa \rightarrow V_\kappa \mid \text{dom}(f) \in U \text{ and } \forall \alpha, f(\alpha) \in \text{Col}(\alpha^{+5}, < \kappa)\},$$

$$F^* = \{f \in P^* \mid i(f)(\kappa) \in F\}.$$

Then  $U$  can be read off  $F^*$  as

$$U = \{X \subseteq \kappa \mid \exists f \in F^*, X = \text{dom}(f)\}.$$

The following definitions are based on [1] and [11] with the modifications required for our purposes.

**Definition 4.1.** *A constructing pair is a pair  $(j, F)$ , where*

- $j : V \rightarrow M$  is a non-trivial elementary embedding into a transitive inner model, and if  $\kappa = \text{crit}(j)$ , then  $M^\kappa \subseteq M$ .
- $F$  is  $\text{Col}(\kappa^{+5}, < i(\kappa))_N$ -generic over  $N$ , where  $i : V \rightarrow N \simeq \text{Ult}(V, U)$  is the ultrapower embedding approximating  $j$ . Also factor  $j$  through  $i$ , say  $j = k \circ i$ .
- $F \in M$ .
- $F$  can be transferred along  $k$  to give a  $\text{Col}(\kappa^{+5}, < j(\kappa))_M$ -generic over  $M$ .

In particular note that the pair  $(j, F)$  constructed above is a constructing pair.

**Definition 4.2.** *If  $(j, F)$  is a constructing pair as above, then  $F^* = \{f \in P^* \mid i(f)(\kappa) \in F\}$ .*

**Definition 4.3.** *Suppose  $(j, F)$  is a constructing pair as above. A sequence  $w$  is constructed by  $(j, F)$  iff*

- $w \in M$ .
- $w(0) = \kappa = \text{crit}(j)$ .
- $w(1) = F^*$ .
- For  $1 < \beta < \text{lh}(w)$ ,  $w(\beta) = \{X \subseteq V_\kappa \mid w \upharpoonright \beta \in j(X)\}$ .
- $M \models |\text{lh}(w)| \leq w(0)^+$ .

**Remark 4.4.** *In [1], it is assumed that  $M \models |\text{lh}(w)| \leq w(0)^{++}$ , while here we just require that  $M \models |\text{lh}(w)| \leq w(0)^+$ . This is because we only need to preserve the inaccessibility*

of  $\kappa = w(0)$  and by results of Mitchell (see Lemma 4.18) it suffices that the length of the measure sequence to be  $\kappa^+$ .

If  $w$  is constructed by  $(j, F)$ , then we set  $\kappa_w = w(0)$ , and if  $\text{lh}(w) \geq 2$ , then we define

$$\begin{aligned} F_w^* &= w(1). \\ \mu_w &= \{X \subseteq \kappa_w \mid \exists f \in F_w^*, X = \text{dom}(f)\}. \\ \bar{\mu}_w &= \{X \subseteq V_{\kappa_w} \mid \{\alpha \mid \langle \alpha \rangle \in X\} \in \mu_w\}. \\ F_w &= \{[f]_{\mu_w} \mid f \in F_w^*\}. \\ \mathcal{F}_w &= \bar{\mu}_w \cap \bigcap \{w(\alpha) \mid 1 < \alpha < \text{lh}(w)\}. \end{aligned}$$

**Definition 4.5.** *Define inductively*

$$\begin{aligned} \mathcal{U}_0 &= \{w \mid \exists (j, F) \text{ such that } (j, F) \text{ constructs } w\}. \\ \mathcal{U}_{n+1} &= \{w \in \mathcal{U}_n \mid \mathcal{U}_n \cap V_{\kappa_w} \in \mathcal{F}_w\}. \\ \mathcal{U}_\infty &= \bigcap_{n \in \omega} \mathcal{U}_n. \end{aligned}$$

The elements of  $\mathcal{U}_\infty$  are called *measure sequences*.

Let  $u$  be the measure sequences constructed using the pair  $(j, F)$  above. It is easily seen that for each  $\alpha < \kappa^{++}$ ,  $u \upharpoonright \alpha$  exists and is in  $\mathcal{U}_\infty$ .

**4.2. Radin forcing with interleaved collapses.** In this subsection, we assign to each measure sequence  $w \in \mathcal{U}_\infty$  a forcing notion  $\mathbb{R}_w$ . The forcing  $\mathbb{R}_w$  adds a club  $C$  of ground model regular cardinals into  $\kappa_w$  in such a way that if  $\alpha < \beta$  are successive points in  $C$ , then it collapses all cardinals in the interval  $(\alpha^{+6}, \beta)$  into  $\alpha^{+5}$  and makes  $\beta = \alpha^{+6}$ . When  $\text{lh}(w) = \kappa_w^+$ , then the forcing preserves the inaccessibility of  $\kappa_w$  and all singular cardinals less than  $\kappa_w$  are limit points of  $C$ . As we will see in the next subsections, if we start with a suitably prepared model and force over it by  $\mathbb{R}_w$ , where  $\text{lh}(w) = \kappa_w^+$ , then in the resulting extension the tree property holds at double successor of every limit point of  $C$  and in particular, the rank initial segment of the final model at  $\kappa_w$  is a model in which the tree property holds at double successor of every singular cardinal.

First we define the building blocks of the forcing.

**Definition 4.6.** *Assume  $w \in \mathcal{U}_\infty$ . Then  $\mathbb{P}_w$  is the set of all tuples  $p = (w, \lambda, A, H, h)$ , where*

- (1)  $w$  is a measure sequence.

- (2)  $\lambda < \kappa_w$ .
- (3)  $A \in \mathcal{F}_w$ .
- (4)  $H \in F_w^*$  with  $\text{dom}(H) = \{\kappa_v > \lambda \mid v \in A\}$ .
- (5)  $h \in \text{Col}(\lambda^{+5}, < \kappa_w)$ .

Note that if  $\text{lh}(w) = 1$ , then the above tuple is of the form  $(w, \lambda, \emptyset, \emptyset, h)$ , where  $\lambda < \kappa_w$  and  $h \in \text{Col}(\lambda^{+5}, < \kappa_w)$ .

Given  $p \in \mathbb{P}_w$  as above, we denote it by

$$p = (w^p, \lambda^p, A^p, H^p, h^p).$$

The order on  $\mathbb{P}_w$  is defined as follows.

**Definition 4.7.** Assume  $p, q \in \mathbb{P}_w$ . Then  $p \leq^* q$  iff:

- (1)  $w^p = w^q$ .
- (2)  $\lambda^p = \lambda^q$ .
- (3)  $A^p \subseteq A^q$ .
- (4) For all  $v \in A^p$ ,  $H^p(\kappa_v) \leq H^q(\kappa_v)$ .
- (5)  $h^p \leq h^q$ .

Next we define the forcing notion  $\mathbb{R}_w$ .

**Definition 4.8.** If  $w$  is a measure sequence, then  $\mathbb{R}_w$  is the set of all finite sequences

$$p = \langle p_k \mid k \leq n \rangle,$$

where

- (1)  $p_k = (w_k, \lambda_k, A_k, H_k, h_k) \in \mathbb{P}_{w_k}$ , for each  $k \leq n$ .
- (2)  $w_n = w$ .
- (3) If  $k < n$ , then  $\lambda_{k+1} = \kappa_{w_k}$ .

Given  $p \in \mathbb{R}_w$  as above, we denote it by

$$p = \langle p_k \mid k \leq n^p \rangle$$

and call  $n^p$  the length of  $p$ . We also use  $w_k^p$  for  $w^{p_k}$  and so on (for  $k \leq n^p$ ). The direct extension relation  $\leq^*$  is defined on  $\mathbb{R}_w$  in the natural way:

**Definition 4.9.** Assume  $p, q \in \mathbb{R}_w$ . Then  $p \leq^* q$  iff

- (1)  $n^p = n^q$ .
- (2) For all  $k \leq n^p, p_k \leq^* q_k$  in  $\mathbb{P}_{w_k^p}$ .

The following definition is the key step towards defining the order relation  $\leq$  on  $\mathbb{R}_w$

**Definition 4.10.** (a) Assume  $p = (w, \lambda, A, H, h) \in \mathbb{P}_w$  and  $w' \in A$ . Then  $\text{Add}(p, w')$  is

the condition  $\langle p_0, p_1 \rangle \in \mathbb{R}_w$  defined by

- (1)  $p_0 = (w', \lambda, A \cap V_{\kappa_{w'}}, H \upharpoonright \kappa_{w'}, h)$ .
- (2)  $p_1 = (w, \kappa_{w'}, A \setminus V_\eta, H \upharpoonright \text{dom}(H) \setminus V_\eta, H(\kappa_{w'}))$ , where  $\eta = \text{sup range}(H(\kappa_{w'}))$ .

In the case that this does not yield a member of  $\mathbb{R}_w$ , then  $\text{Add}(p, w')$  is undefined.

(b) Suppose  $p = \langle p_0, \dots, p_n \rangle \in \mathbb{R}_w$ ,  $k \leq n$  and  $u \in A_k^p$ . Then  $\text{Add}(p, u)$  is the member

of  $\mathbb{R}_w$  obtained by replacing  $p_k$  with the two members of  $\text{Add}(p_k, u)$ , That is,

- $\text{Add}(p, u) \upharpoonright i = p \upharpoonright i$ .
- $\text{Add}(p, u)_i = \text{Add}(p_i, u)_0$ .
- $\text{Add}(p, u)_{i+1} = \text{Add}(p_i, u)_1$ .
- $\text{Add}(p, u) \upharpoonright [i+2, n+1] = p \upharpoonright [i+1, n]$ .

The next lemma shows that any condition  $p$  has a direct extension  $q$  such that  $\text{Add}(q, u)$  is well-defined for all  $k \leq n^q$  and all  $u \in A_k^q$ .

**Lemma 4.11.** (a) Suppose  $p = (w, \lambda, A, H, h) \in \mathbb{P}_w$ . Then

$$A' = \{w' \in A : \text{Add}(p, w') \text{ is well-defined}\} \in \mathcal{F}_w.$$

(b) Suppose  $p \in \mathbb{R}_w$ . Then there exists  $q \leq^* p$  such that for all  $k \leq n^q$  and all  $u \in A_k^q$ ,  $\text{Add}(q, u) \in \mathbb{R}_w$  is well-defined.

*Proof.* Clause (b) follows from (a), so let us prove (a). Let  $p = (w, \lambda, A, H, h) \in \mathbb{P}_w$ . We have to show that  $A' \in \bar{\mu}_w \cap \bigcap \{w(\alpha) \mid 1 < \alpha < \text{lh}(w)\}$ .

Suppose  $(j, F)$  constructs  $w$ , where  $j : V \rightarrow M$ , and let  $i : V \rightarrow N \simeq \text{Ult}(V, U)$  be the corresponding ultrapower embedding.

Let us first show that  $A' \in \bar{\mu}_w$ . We have

$$A' \in \bar{\mu}_w \iff \{\alpha : \langle \alpha \rangle \in A'\} \in U$$



$$\begin{aligned}
&\iff \kappa_w \in j(\{\alpha : \langle \alpha \rangle \in A'\}) \\
&\iff \langle \kappa_w \rangle \in j(A') \\
&\iff \text{Add}(j(p), \langle \kappa_w \rangle) \text{ is a well-defined condition in } \mathbb{R}_{j(w)}^M.
\end{aligned}$$

But we have

$$\text{Add}(j(p), \langle \kappa_w \rangle) = \langle \langle \kappa_w \rangle, \lambda, A, H, h \rangle, (j(w), \kappa_w, A^*, H^*, j(H)(\kappa_w)),$$

where  $A^* = j(A) \setminus V_{\kappa_w}$  and  $H^* = j(H) \upharpoonright \text{dom}(j(H)) \setminus V_{\kappa_w}$ . Thus  $\text{Add}(j(p), \langle \kappa_w \rangle)$  is well-defined, which implies  $A' \in \bar{\mu}_w$ .

Now let  $1 < \alpha < \text{lh}(w)$ . Then

$$\begin{aligned}
A' \in w(\alpha) &\iff w \upharpoonright \alpha \in j(A') \\
&\iff \text{Add}(j(p), w \upharpoonright \alpha) \text{ is a well-defined condition in } \mathbb{R}_{j(w)}^M.
\end{aligned}$$

By an argument as above, it is easily seen that  $\text{Add}(j(p), w \upharpoonright \alpha)$  is a well-defined condition in  $\mathbb{R}_{j(w)}^M$ , and hence  $A' \in w(\alpha)$ . Thus

$$A' \in \bar{\mu}_w \cap \bigcap \{w(\alpha) \mid 1 < \alpha < \text{lh}(w)\}$$

as required.  $\square$

By the above lemma we can always assume that  $\text{Add}(q, u)$  is well-defined for all  $q \in \mathbb{R}_w$ , all  $k \leq n^q$  and all  $u \in A_k^q$ .

**Definition 4.12.** (a) Suppose  $p, q \in \mathbb{R}_w$ . Then  $p$  is a one-point extension of  $q$ , denoted

$$p \leq^1 q, \text{ if there are } k \leq n^q \text{ and } u \in A_k^q \text{ such that } p \leq^* \text{Add}(q, u).$$

(b) Suppose  $p, q \in \mathbb{R}_w$ . Then  $p$  is an extension of  $q$ , denoted  $p \leq q$ , if there are  $n < \omega$  and conditions  $p_0, p_1, \dots, p_n$  such that

$$p = p_0 \leq^1 p_1 \leq^1 \dots \leq^1 p_n = q.$$

**4.3. Basic properties of the forcing notion  $\mathbb{R}_w$ .** We now state and prove some of the main properties of the forcing notion  $\mathbb{R}_w$ .

**Lemma 4.13.**  $(\mathbb{R}_w, \leq)$  satisfies the  $\kappa_w^+$ -c.c.

*Proof.* Assume on the contrary that  $A \subseteq \mathbb{R}_w$  is an antichain of size  $\kappa_w^+$ . We can assume that all  $p \in A$  have the same length  $n$ . Write each  $p \in A$  as  $p = d_p \widehat{\ } p_n$ , where  $d_p \in V_{\kappa_w}$  and

$p_n = (w, \lambda^p, A^p, H^p, h^p)$ . By shrinking  $A$ , if necessary, we can assume that there are fixed  $d \in V_{\kappa_w}$  and  $\lambda < \kappa_w$  such that for all  $p \in A$ ,  $d_p = d$  and  $\lambda^p = \lambda$ .

Note that for  $p \neq q$  in  $A$ , as  $p$  and  $q$  are incompatible, we must have  $h^p$  is incompatible with  $h^q$ . But  $\text{Col}(\lambda^{+5}, < \kappa_w)$  satisfies the  $\kappa_w$ -c.c., and we get a contradiction.  $\square$

The following factorization lemma can be proved easily.

**Lemma 4.14.** *(The factorization lemma) Assume that  $p = \langle p_0, \dots, p_n \rangle \in \mathbb{R}_w$ , where  $p_i = (w_i, \lambda_i, A_i, H_i, h_i)$  and  $m < n$ . Set  $p^{\leq m} = \langle p_0, \dots, p_m \rangle$  and  $p^{> m} = \langle p_{m+1}, \dots, p_n \rangle$ . Then*

(a)  $p^{\leq m} \in \mathbb{R}_{w_m}$ ,  $p^{> m} \in \mathbb{R}_w$  and there exists

$$i : \mathbb{R}_w/p \rightarrow \mathbb{R}_{w_m}/p^{\leq m} \times \mathbb{R}_w/p^{> m}$$

which is an isomorphism with respect to both  $\leq^*$  and  $\leq$ .

(b) If  $m+1 < n$ , then there exists

$$i : \mathbb{R}_w/p \rightarrow \mathbb{R}_{w_m}/p^{\leq m} \times \text{Col}(\kappa_{w_m}^{+5}, < \kappa_{w_{m+1}}) \times \mathbb{R}_w/p^{> m+1}$$

which is an isomorphism with respect to both  $\leq^*$  and  $\leq$ .  $\square$

**Lemma 4.15.**  $(\mathbb{R}_w, \leq, \leq^*)$  satisfies the Prikry property, i.e., for any  $p \in \mathbb{R}_w$  and any statement  $\sigma$  in the forcing language of  $(\mathbb{R}_w, \leq)$ , there exists  $q \leq^* p$  which decides  $\sigma$ .

*Proof.* We follow the argument given in [14]. We prove the lemma by induction on  $\kappa_w$ . Thus, assuming it is true for  $\mathbb{R}_u$  with  $\kappa_u < \kappa_w$ ; we prove it for  $\mathbb{R}_w$ .

Suppose that  $p \in \mathbb{R}_w$  and  $\sigma$  is a statement in the forcing language of  $(\mathbb{R}_w, \leq)$ . First, we assume that  $\text{lh}(p) = 1$ . So let us write it as  $p = (w, \lambda, A, H, h) \in \mathbb{P}_w$ .

Given  $q \in \mathbb{R}_w$ , we can write it as  $q = d_q \widehat{\langle q_{\text{lh}(q)} \rangle}$ , where  $d_q \in V_{\kappa_w}$  and  $q_{\text{lh}(q)} \in \mathbb{P}_w$ . We set  $\text{stem}(q) = d_q$  and call it the stem of  $q$ . Let  $\mathbf{L}$  be the set of stems of conditions in  $\mathbb{R}_w$  which extend  $p$ :

$$\mathbf{L} = \{\text{stem}(q) : q \in \mathbb{R}_w \text{ and } q \leq p\}.$$

Suppose  $v \in A$  and  $s = \langle q_k : k \leq n \rangle \in \mathbf{L}$ , where  $q_k = (w_k, \lambda_k, A_k, H_k, h_k)$  (for  $k \leq n$ ), are such that for some  $A_v, H_v, h_v, A', H'$  and  $h'$ ,

$$q = s \widehat{\langle v, \lambda, A_v, H_v, h_v \rangle} \widehat{\langle w, \kappa_v, A', H', h' \rangle} \in \mathbb{R}_w$$

and

$$q \leq \text{Add}(p, v).$$

Then  $\kappa_{q_n} = \kappa_{w_n^q} = \lambda$ , in particular there are less than  $\lambda^{+5}$ -many such stems  $s$ . For each  $v \in A$  and each stem  $s \in \mathbf{L}$  define the sets  $D^{\text{top}}(0, s, v)$  and  $D^{\text{top}}(1, s, v)$ , as follows:

- $D^{\text{top}}(0, s, v)$  is the set of all  $g \leq H(\kappa_v)$  for which there exist  $A_v, H_v, h_v, A'$  and  $H'$  such that  $s \frown \langle v, \lambda, A_v, H_v, h_v \rangle \frown \langle w, \kappa_v, A', H', g \rangle \leq \text{Add}(p, v)$  and it decides  $\sigma$ .
- $D^{\text{top}}(1, s, v)$  is the set of all  $g \leq H(\kappa_v)$  such that for all  $A_v, H_v, h_v, A', H'$  and  $g' \leq g$ ,  $s \frown \langle v, \lambda, A_v, H_v, h_v \rangle \frown \langle w, \kappa_v, A', H', g' \rangle$  does not decide  $\sigma$ .

Clearly,  $D^{\text{top}}(0, s, v) \cup D^{\text{top}}(1, s, v)$  is dense in  $\text{Col}(\kappa_v^{+5}, < \kappa_w)/H(\kappa_v)$ , and so by the distributivity of  $\text{Col}(\kappa_v^{+5}, < \kappa_w)$ , the intersection

$$D_v^{\text{top}} = \bigcap_{s \in \mathbf{L} \cap V_{\kappa_v}} (D^{\text{top}}(0, s, v) \cup D^{\text{top}}(1, s, v))$$

is also dense in  $\text{Col}(\kappa_v^{+5}, < \kappa_w)/H(\kappa_v)$ . Take  $\tilde{H} \in F_w^*$  such that

$$\tilde{A} = \{v \in A : \tilde{H}(\kappa_v) \in D_v^{\text{top}}\} \in \mathcal{F}_w.$$

Let  $H^* \in F_w^*$  extends both of  $H$  and  $\tilde{H}$ .

Next define the sets  $D^{\text{low}}(0, s, v)$  and  $D^{\text{low}}(1, s, v)$  as follows:

- $D^{\text{low}}(0, s, v)$  is the set of all  $g \leq h$  for which there exist  $A_v, H_v, A'$  and  $H'$  such that  $s \frown \langle v, \lambda, A_v, H_v, g \rangle \frown \langle w, \kappa_v, A', H', H^*(\kappa_v) \rangle \leq \text{Add}(p, v)$  and it decides  $\sigma$ .
- $D^{\text{low}}(1, s, v)$  is the set of all  $g \leq h$  such that for all  $A_v, H_v, A', H'$  and  $g' \leq H^*(\kappa_v)$ ,  $s \frown \langle v, \lambda, A_v, H_v, g \rangle \frown \langle w, \kappa_v, A', H', H^*(\kappa_v) \rangle$  does not decide  $\sigma$ .

The set  $D^{\text{low}}(0, s, v) \cup D^{\text{low}}(1, s, v)$  is dense in  $\text{Col}(\lambda^{+5}, < \kappa_v)/h$ , and so by the distributivity of  $\text{Col}(\lambda^{+5}, < \kappa_v)$ , the intersection

$$D_v^{\text{low}} = \bigcap_{s \in \mathbf{L} \cap V_{\kappa_v}} (D^{\text{low}}(0, s, v) \cup D^{\text{low}}(1, s, v))$$

is also dense in  $\text{Col}(\lambda^{+5}, < \kappa_v)/h$ . Take  $\tilde{h}_v \in D_v^{\text{low}}$ .

Now consider

$$p' = (w, \lambda, \tilde{A}, H^*, h) \leq p.$$

For any stem  $s$  of a condition in  $\mathbb{R}_w$  extending  $p'$  and every  $\alpha < \text{lh}(w)$ , let  $A(s, \alpha) \in \mathcal{F}_w$  be such that one of the following three possibilities holds for it:

(1<sub>s,α</sub>): For every  $v \in A(s, \alpha)$  there exists  $q' \leq p'$  such that  $q'$  forces  $\sigma$  and  $q'$  is of the form

$$q' = s \frown \langle v, \lambda, A'_{s,v}, H'_{s,v}, h'_{s,v} \rangle \frown \langle w, \kappa_v, A_{s,v}, H_{s,v}, h_{s,v} \rangle,$$

for some  $A'_{s,v}, H'_{s,v}, h'_{s,v} \leq \tilde{h}_v, A_{s,v}, H_{s,v}$  and  $h_{s,v} \leq H^*(\kappa_v)$ .

(2<sub>s,α</sub>): For every  $v \in A(s, \alpha)$  there exists  $q' \leq p'$  such that  $q'$  forces  $\neg\sigma$  and  $q'$  is of the above form.

(3<sub>s,α</sub>): For every  $v \in A(s, \alpha)$  there does not exist  $q' \leq p'$  of the above form such that  $q'$  decides  $\sigma$ .

For every  $v$ , we may suppose that  $H_{s,v}, h_{s,v}, H'_{s,v}$  and  $h'_{s,v}$ 's depend only on  $v$ , and so we denote them by  $H_v, h_v, H'_v$  and  $h'_v$  respectively. For each  $\alpha$  let  $A(\alpha) = \Delta_s A(s, \alpha)$  be the diagonal intersection of the  $A(s, \alpha)$ 's and set

$$A^* = A' \cap \bigcup_{\alpha < \text{lh}(w)} A(\alpha) \in \mathcal{F}_w.$$

Also let

$$p^* = (w, \lambda, A^*, H^*, h).$$

Note that if  $v \in A' \cap A(s, \alpha)$  and if one of the (1<sub>s,α</sub>) or (2<sub>s,α</sub>) happen, then we may take  $h_v = H^*(\kappa_v)$  and  $h'_v = \tilde{h}_v$ . This is because if one of these possibilities happen, then  $H^*(\kappa_v) \in D(0, s, v)$ , so there are  $\tilde{A}_v, \tilde{H}_v, \tilde{A}'$  and  $\tilde{H}'$  such that

$$q = s \frown \langle v, \lambda, \tilde{A}_v, \tilde{H}_v, \tilde{h}_v \rangle \frown \langle w, \kappa_v, \tilde{A}', \tilde{H}', H^*(\kappa_v) \rangle \leq \text{Add}(p, v)$$

and it decides  $\sigma$ . On the other hand, there exists

$$q' = s \frown \langle v, \lambda, A'_{s,v}, H'_{s,v}, h'_{s,v} \rangle \frown \langle w, \kappa_v, A_{s,v}, H_v, h_v \rangle \leq p'$$

which also decides  $\sigma$ . But the conditions  $q$  and  $q'$  are compatible and they decide the same truth value; hence we can take  $h_v = H^*(\kappa_v)$  and  $h'_v = \tilde{h}_v$ .

We show that there exists a direct extension of  $p^*$  which decides  $\sigma$ . Assume not and let  $r \leq p^*$  be of minimal length which decides  $\sigma$ , say it forces  $\sigma$ . Let us write

$$\text{stem}(r) = s \frown \langle u, \lambda, A^r, H^r, h^r \rangle,$$

where  $s \in V_{\kappa_u}$ . By our assumption, there exists  $\alpha < \text{lh}(w)$  such that  $A(s, \alpha) \in w(\alpha)$  satisfies  $(1_{s, \alpha})$ , so for every  $v \in A(s, \alpha)$ , there exists  $q'_v \leq p'$  such that  $q'_v$  forces  $\sigma$  and  $q'_v$  is of the form

$$q'_v = s \frown \langle v, \lambda, A'_v, H'_v, \tilde{h}_v \rangle \frown \langle w, \kappa_v, A_v, H_v, H^*(\kappa_v) \rangle,$$

for some  $A'_v, H'_v, A_v$  and  $H_v$ .

We show that there exists  $q^* \leq p^*$  with  $\text{stem}(q^*) = s$  such that every extension of  $q^*$  is compatible with  $q'_v$ , for some  $v \in A(s, \alpha)$ . This property implies that  $q^*$  forces  $\sigma$ , contradicting the minimal choice of  $\text{lh}(r)$ . We note that by the definition of extension in the forcing  $\mathbb{R}_w$ , we may assume from this point on that  $s$  is empty.

Consider the map  $\phi : A(\langle \rangle, \alpha) \rightarrow V$  which is defined by

$$\phi : v \mapsto (\phi_0(v), \phi_1(v)) = (A_\nu, H_\nu).$$

As  $A(\langle \rangle, \alpha) \in w(\alpha)$ , we have  $w \upharpoonright \alpha \in j(A(\langle \rangle, \alpha))$  (where  $j$  is the constructing embedding for  $w$ ). Let

$$(A^{<\alpha}, H^{<\alpha}) = j(\phi)(w \upharpoonright \alpha).$$

Also let

$$A^\alpha = \{v \in A(\langle \rangle, \alpha) : A^{<\alpha} \cap V_{\kappa_v} = A_v \text{ and } H^{<\alpha} \upharpoonright V_{\kappa_v} = H_v\}$$

and

$$A^{>\alpha} = \bigcup_{\alpha < \beta < \text{lh}(w)} \{v \in A^* : A^\alpha \cap V_{\kappa_v} \in w(\beta)\}.$$

Then  $A^{**} = A^{<\alpha} \cup A^\alpha \cup A^{>\alpha} \in \mathcal{F}_w$ . Set  $H^{**} = H^{<\alpha} \wedge H^*$  and finally set

$$q^* = \langle w, \lambda, A^{**}, H^{**}, h \rangle \leq p^*.$$

We show that  $q^*$  is as required. Thus let

$$q = \langle \langle w_k, \lambda_k, A_k, H_k, h_k \rangle : k \leq n \rangle$$

be an extension of  $q^*$ . There are various cases:

- (1) There is no index  $k$  such that  $\text{lh}(u_k) > 0$  and  $(A^\alpha \cup A^{>\alpha}) \cap V_{\kappa_{w_k}} \in \bigcup_{\beta < \text{lh}(w_k)} w_k(\beta)$ .

Then pick some non-trivial measure sequence  $v \in A^\alpha \cap A_n$ , and note that for all  $k < n$ ,  $A^{<\alpha} \cap A_k \in \bigcap_{\beta < \text{lh}(w_k)} w_k(\beta)$ . Then one can easily show that  $q$  is compatible with  $q'_v$ .

- (2) There is an index  $k$  with  $\text{lh}(u_k) > 0$  and  $(A^\alpha \cup A^{>\alpha}) \cap V_{\kappa_{w_k}} \in \bigcup_{\beta < \text{lh}(w_k)} w_k(\beta)$  and  $A^\alpha \in w_k(\beta)$  for some  $\beta < \text{lh}(w_k)$ . Let us pick  $k$  to be the least such an index. Let  $v \in A_k$  be such that  $A^{<\alpha} \cap A_k \in \bigcap_{\beta < \text{lh}(v)} v(\beta)$ . Then  $q$  is compatible with  $q'_v$ .
- (3) There is an index  $k$  with  $\text{lh}(u_k) > 0$  and  $(A^\alpha \cup A^{>\alpha}) \cap V_{\kappa_{w_k}} \in \bigcup_{\beta < \text{lh}(w_k)} w_k(\beta)$  and  $A^{>\alpha} \in w_k(\beta)$  for some  $\beta < \text{lh}(w_k)$ . Then by our choice of  $A^{>\alpha}$ , there is some  $v \in A_k$  that can be added to  $q$  such that we reduce to the case (2).

This completes the proof for the case  $\text{lh}(p) = 1$ . We now prove the lemma for an arbitrary condition  $p$ , by induction on  $\text{lh}(p)$ . Thus suppose that  $\text{lh}(p) \geq 2$ ; say

$$p = s \frown \langle (u, \lambda', A', H', h') \rangle \frown \langle (w, \lambda, A', H', h) \rangle.$$

By the factorization Lemma 4.14, we have

$$\mathbb{R}_w/p \simeq (\mathbb{R}_u/s \frown \langle (u, \lambda', A', H', h') \rangle) \times (\mathbb{R}_w/\langle (w, \lambda, A', H', h) \rangle).$$

Let  $\langle s_i : i < \kappa_u \rangle$  enumerate  $\mathbf{L} \cap V_{\kappa_u}$ , and define by recursion on  $i$  a  $\leq^*$ -decreasing chain  $\langle p_i : i \leq \kappa_u \rangle$  of conditions in  $\mathbb{R}_w/\langle (w, \lambda, A', H', h) \rangle$  as follows:

Set  $p_o = \langle (w, \lambda, A', H', h) \rangle$ . Given  $p_i$ , let  $p_{i+1} \leq^* p_i$  decide whether there is a condition in  $\mathbb{R}_u/s \frown \langle (u, \lambda', A', H', h') \rangle$  with stem  $s_i$  which decides  $\sigma$  and if so, then it forces one of  $\sigma$  or  $\neg\sigma$ . At limit ordinals  $i \leq \kappa_u$ , use the fact that  $(\mathbb{R}_w/\langle (w, \lambda, A', H', h) \rangle, \leq^*)$  is  $\kappa_w$ -closed to find a  $p_i$  which  $\leq^*$ -extends all  $p_j, j < i$ .

By our construction,

$$\Vdash_{\mathbb{R}_u/s \frown \langle (u, \lambda', A', H', h') \rangle} \text{“} p_{\kappa_u} \text{ decides } \sigma \text{”}.$$

By the induction hypothesis, there exists  $q \leq^* s \frown \langle (u, \lambda', A', H', h') \rangle$  which decides which way  $p_{\kappa_u}$  decides  $\sigma$ , and then  $q \frown p_{\kappa_u} \leq^* p$  decides  $\sigma$ .

The lemma follows.  $\square$

Now suppose that  $w = u \upharpoonright \kappa^+$ , where  $u$  is the measure sequence constructed by the pair  $(j, F)$  and let  $K \subseteq \mathbb{R}_w$  be a generic filter over  $V$ . Set

$$C = \{\kappa_u \mid \exists p \in K, \exists \xi < \text{lh}(p), p_\xi = (u, \lambda, A, H, h)\}.$$

By standard arguments,  $C$  is a club of  $\kappa$ , also we can suppose that  $\min(C) = \aleph_0$ . Let  $\langle \kappa_\xi : \xi < \kappa \rangle$  be the increasing enumeration of the club  $C$  and let  $\vec{u} = \langle u_\xi \mid \xi < \kappa \rangle$  be the

enumeration of

$$\{u \mid \exists p \in K, \exists \xi < \text{lh}(p), p_\xi = (u, \lambda, A, H, h)\}$$

such that for  $\xi < \zeta < \kappa$ ,  $\kappa_{u_\xi} = \kappa_\xi < \kappa_\zeta = \kappa_{u_\zeta}$ . Also let  $\vec{F} = \langle F_\xi \mid \xi < \kappa \rangle$  be such that each  $F_\xi$  is  $\text{Col}(\kappa_\xi^{+5}, < \kappa_{\xi+1})$ -generic over  $V$  produced by  $K$ .

**Lemma 4.16.** (a)  $V[K] = V[\vec{u}, \vec{F}]$ .

(b) For every limit ordinal  $\xi < \kappa$ ,  $\langle \vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi \rangle$  is  $\mathbb{R}_{u_\xi}$ -generic over  $V$ , and  $\langle \vec{u} \upharpoonright [\xi, \kappa), \vec{F} \upharpoonright [\xi, \kappa) \rangle$  is  $\mathbb{R}_w$ -generic over  $V[\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi]$ .

(c) For every  $\gamma < \kappa$  and every  $A \subseteq \gamma$  with  $A \in V[\vec{u}, \vec{F}]$ , we have  $A \in V[\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi]$ , where  $\xi$  is the least ordinal such that  $\gamma < \kappa_\xi$ .

*Proof.* (a) It suffices to show that  $K$  is definable from  $\vec{u}$  and  $\vec{F}$ . Let  $K'$  be the set of all conditions  $p \in \mathbb{R}_w$  such that

- For all measure sequences  $u \in V_\kappa$ , if  $u$  appears in  $p$ , then  $u = u_\xi$ , for some  $\xi < \kappa$ ,
- For all  $\xi < \kappa$ , there exists  $q \leq p$  such that  $u_\xi$  appears in  $q$ ,
- If  $f \in V_\kappa$  appears in  $p$ , then  $f \subset F_\xi$ , for some  $\xi < \kappa$ ,
- For all  $\xi < \kappa$  and all  $f \in \mathcal{P}(F_\xi) \cap \text{Col}(\kappa_\xi^{+5}, < \kappa_{\xi+1})$ , there exists  $q \leq p$  such that  $f$  appears in  $q$ .

It is clear that  $K' \in V[\vec{u}, \vec{F}]$ . It is also easily seen that  $K'$  is a filter which includes  $K$ . It follows from the genericity of  $K$  that  $K = K'$ . So  $K \in V[\vec{u}, \vec{F}]$ , as required.

(b) Follows from (a) and the factorization lemma 4.14.

(c) First note that  $\nu$  is not a limit ordinal, so assume  $\nu = \xi + 1$  is a successor ordinal (if  $\nu = 0$ , then the proof is similar). Let  $p \in K$  be such that  $p$  mentions both  $u_\xi$  and  $u_{\xi+1}$ , say  $u_\xi = u^{p_m}$  and  $u_{\xi+1} = u^{p_{m+1}}$ . By the Factorization Lemma 4.14,

$$\mathbb{R}_w/p \simeq \mathbb{R}_{u_\xi}/p^{\leq m} \times \text{Col}(\kappa_\xi^{+5}, < \kappa_{\xi+1}) \times \mathbb{R}_w/p^{\geq m+1}.$$

Let  $\dot{A}$  be an  $\mathbb{R}_w$ -name for  $A$  such that  $\Vdash_{\mathbb{R}_w} \dot{A} \subseteq \gamma$ . Let  $\dot{B}$  be an  $\mathbb{R}_w/p^{\geq m+1}$ -name for a subset of  $\mathbb{R}_{u_\xi}/p^{\leq m} \times \text{Col}(\kappa_\xi^{+5}, < \kappa_{\xi+1}) \times \gamma$  such that

$$\Vdash_{\mathbb{R}_w/p^{\geq m+1}} \forall \eta < \gamma, ((r, f, \eta) \in \dot{B} \iff (r, f) \Vdash_{\mathbb{R}_{u_\xi}/p^{\leq m} \times \text{Col}(\kappa_\xi^{+5}, < \kappa_{\xi+1})} \eta \in \dot{A}).$$

Let  $\langle y_\alpha : \alpha < \kappa_{\xi+1} \rangle$  be an enumeration of  $\mathbb{R}_{u_\xi}/p^{\leq m} \times \text{Col}(\kappa_\xi^{+5}, \kappa_{\xi+1}) \times \gamma$ . Define a  $\leq^*$ -decreasing sequence  $\langle q_\alpha \mid \alpha < \kappa_{\xi+1} \rangle$  of conditions in  $\mathbb{R}_w/p^{> m+1}$  such that for all  $\alpha, q_\alpha$  decides “ $y_\alpha \in \dot{B}$ ”. This is possible as  $(\mathbb{R}_w/p^{> m+1}, \leq^*)$  is  $\kappa_{\xi+1}^+$ -closed and by Lemma 4.15 it satisfies the Prikry property. Let  $q \leq^* q_\alpha$  for all  $\alpha < \kappa_{\xi+1}$ . Then  $q$  decides each “ $y_\alpha \in \dot{B}$ ”. It follows that  $A \in V[\vec{u} \upharpoonright \nu, \vec{F} \upharpoonright \nu]$   $\square$

We now state a geometric characterization of generic filters for  $\mathbb{R}_w$ . Such a characterization was first given by Mitchell [24] for Radin forcing. The characterization given below is essentially due to Cummings [1].

**Lemma 4.17.** (*Geometric characterization*) *The pair  $(\vec{u}, \vec{F})$  is  $\mathbb{R}_w$ -generic over  $V$  if and only if it satisfies the following conditions:*

- (1) *If  $\xi < \kappa$  and  $\text{lh}(u_\xi) > 1$ , then the pair  $(\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi)$  is  $\mathbb{R}_{u_\xi}$ -generic over  $V$ .*
- (2) *For all  $A \in V_{\kappa+1}$  ( $A \in \mathcal{F}_w \iff \exists \alpha < \kappa \forall \xi > \alpha, u_\xi \in A$ ).*
- (3) *For all  $f \in w(1)$  there exists  $\alpha < \kappa$  such that  $\forall \xi > \alpha, f(\kappa_\xi) \in F_\xi$ .*

As  $\text{lh}(w) = \kappa^+$ , it follows from Mitchell [24] (see also [9]) that

**Lemma 4.18.**  *$\kappa$  remains strongly inaccessible in  $V[K]$ .*

*Proof.* We follow Cummings [1]. Suppose not and let  $p \in \mathbb{R}_w, \delta < \kappa$  and  $\dot{f}$  be such that

$$p \Vdash \text{“}\dot{f} : \delta \rightarrow \kappa \text{ is cofinal”}.$$

Let  $\theta > \kappa$  be large enough regular such that  $p, \dot{f}, w, \mathbb{R}_w \in H(\theta)$  and let  $\mathbf{X} \prec H(\theta)$  be such that

- (1)  $p, \kappa^+, \dot{f}, w, \mathbb{R}_w \in \mathbf{X}$ .
- (2)  $V_\kappa \subseteq \mathbf{X}$ .
- (3)  ${}^{<\kappa}\mathbf{X} \subseteq \mathbf{X}$ .
- (4)  $|\mathbf{X}| = \kappa$ .

Let  $\pi : \mathbf{X} \rightarrow \mathbf{N}$  be the Mostowski collapse of  $\mathbf{X}$  onto a transitive model  $\mathbf{N}$ . Note that  $\pi \upharpoonright \mathbf{X} \cap V_{\kappa+1} = \text{id} \upharpoonright \mathbf{X} \cap V_{\kappa+1}$ .



Let  $v = \pi(w)$  and  $\beta = \pi(\kappa^+)$ . Then

$$\pi(\mathcal{F}_w) = \mathcal{F}_w \cap \mathbf{X} = \mathcal{F}_w \cap \mathbf{N}$$

and

$$\forall \alpha \in \mathbf{X} \cap \kappa^+, \pi(v(\alpha)) = v(\pi(\alpha)) = v(\alpha) \cap \mathbf{X} = w(\alpha) \cap \mathbf{N}.$$

Let  $\bar{\beta} = \sup(\mathbf{X} \cap \kappa^+) < \kappa^+$ . Using Lemma 4.17, if  $G$  is  $\mathbb{R}_{w \upharpoonright \gamma}$ -generic over  $V$ , where  $\bar{\beta} \leq \gamma < \kappa^+$ , then  $G$  is  $\pi(\mathbb{R}_w)$ -generic over  $N$ . But note that for any limit ordinal  $\gamma$  as above with  $\text{cf}(\gamma) < \kappa$ , we have

$$\Vdash_{\mathbb{R}_{w \upharpoonright \gamma}} \text{“cf}(\kappa) = \text{cf}(\gamma)\text{”}.$$

We get a contradiction and the lemma follows.  $\square$

It follows that

$$CARD^{V[K]} \cap \kappa = \bigcup_{\alpha \in C} \{\alpha, \alpha^+, \alpha^{++}, \alpha^{+3}, \alpha^{+4}, \alpha^{+5}\}.$$

As every limit point of  $C$  is singular in  $V[K]$ , it follows that  $\kappa$  is the least inaccessible cardinal. Also note that  $\text{lim}(C)$ , the set of limit points of  $C$ , is exactly the set of all singular cardinals below  $\kappa$  in  $V[K]$ .

**4.4. The final model.** Suppose that  $GCH$  holds and  $\kappa$  is an  $H(\lambda^{++})$ -hypermeasurable cardinal where  $\lambda$  is a measurable cardinal above  $\kappa$ . We define a generic extension  $W$  of  $V$  which satisfies  $W \models \text{“}\kappa \text{ is inaccessible and for all singular cardinals } \delta < \kappa, 2^\delta = \delta^{+3} \text{ and TP}(\delta^{++}) \text{ holds”}$ .

We will next give a vague and incomplete description of the way the model  $W$  is constructed. Thus we start with  $GCH$  and an  $H(\lambda^{++})$ -hypermeasurable embedding  $j : V \rightarrow M$  with  $i : V \rightarrow N$  being its ultrapower embedding.

We first define a generic extension  $V^1$  of  $V$  in which  $\kappa$  remains  $H(\lambda^{++})$ -hypermeasurable as witnessed by an elementary embedding  $j^1 : V^1 \rightarrow M^1$  which extends  $j$  and in which there exists a generic filter for a suitably chosen forcing notion defined in  $\text{Ult}(V^1, U^1)$ , where  $U^1$  is the normal measure on  $\kappa$  derived from  $j^1$ .

We then define a generic extension  $V^2$  and  $V^1$  in which  $\kappa$  remains  $H(\kappa^{++})$ -hypermeasurable witnessed by an elementary embedding  $j^2 : V^2 \rightarrow M^2$  which extends  $j^1$  and such that if  $U^2$  is the normal measure derived from  $j^2$ , then for  $U^2$ -measure one many  $\delta < \kappa$  we have  $\delta$  is measurable,  $2^\delta = \delta^{+3}$  and  $\text{TP}(\delta^{++})$  holds. Further the model  $V^2$  satisfies the hypotheses at the beginning of Subsection 4.1.

Working in  $V^2$  we force with the forcing notion  $\mathbb{R}_w$ , for a suitably chosen measure sequence  $w$ , to find a generic extension  $V^3$  of  $V^2$ . We show that in  $V^3$  the tree property holds at double successors of the limit points of the Radin club and using it we conclude that  $W = V^3$  is the required model.

Thus suppose that  $V$  satisfies  $GCH$  and let  $\kappa$  be an  $H(\lambda^{++})$ -hypermeasurable cardinal in it where  $\lambda$  is the least measurable cardinal above  $\kappa$ . Also let  $f : \kappa \rightarrow \kappa$  be defined by

$$f(\alpha) = (\min\{\beta > \alpha : \beta \text{ is a measurable cardinal}\})^+.$$

Let  $j : V \rightarrow M$  witness the  $H(\lambda^{++})$ -hypermeasurability of  $\kappa$  and suppose  $j$  is generated by a  $(\kappa, \lambda^{++})$ -extender, i.e.,

$$M = \{j(g)(\alpha) : g : \kappa \rightarrow V, \alpha < \lambda^{++}\}.$$

Then  $j(f)(\kappa) = \lambda^+$ . Also let  $U$  be the normal measure derived from  $j$ ;  $U = \{X \subseteq \kappa : \kappa \in j(X)\}$  and let  $i : V \rightarrow N \simeq \text{Ult}(V, U)$  be the induced ultrapower embedding. Let  $k : N \rightarrow M$  be elementary so that  $j = k \circ i$ .

**Notation 4.19.** (a) For each infinite cardinal  $\alpha \leq \kappa$  let  $\alpha_*$  denote the least measurable cardinal above  $\alpha$ . Note that  $\kappa_* = \lambda$ .

(b) For an infinite cardinal  $\alpha \leq \kappa$  let  $\mathbb{M}_\alpha = \mathbb{M}(\alpha, \alpha_*, \alpha_*^+)$ .

We start with the following lemma.

**Lemma 4.20.** *There exists a cofinality preserving generic extension  $V^1$  of  $V$  satisfying the following conditions:*

- (a) *There is  $j^1 : V^1 \rightarrow M^1$  with critical point  $\kappa$  such that  $H(\lambda^{++}) \subseteq M^1$  and  $j^1 \upharpoonright V = j$ .*
- (b)  *$j^1$  is generated by a  $(\kappa, \lambda^{++})$ -extender.*

- (c) If  $U^1$  is the normal measure derived from  $j^1$  and if  $i^1 : V^1 \rightarrow N^1 \simeq \text{Ult}(V^1, U^1)$  is the ultrapower embedding, then there exists  $\bar{g} \in V^1$  which is  $i^1(\text{Add}(\kappa, \lambda^+)_{V^1})$ -generic over  $N^1$ . Further  $i^1 \upharpoonright V = i$ .

*Proof.* See [8] Theorem 3.1 and Remark 3.6. □

Let  $V^1$  be the model constructed above.

**Lemma 4.21.** *Work in  $V^1$ . There exists a forcing iteration  $\mathbb{P}_\kappa$  of length  $\kappa$  such that if  $G_\kappa * g$  is  $\mathbb{P}_\kappa * \dot{\mathbb{M}}(\kappa, \lambda, \lambda^+)$ -generic over  $V^1$  and  $V^2 = V^1[G_\kappa * g]$ , then the following conditions hold:*

- (a) *There is  $j^2 : V^2 \rightarrow M^2$  with critical point  $\kappa$  such that  $H(\kappa^{++}) \subseteq M^2$  and  $j^2 \upharpoonright V^1 = j^1$ .*
- (b)  *$j^2$  is generated by a  $(\kappa, \lambda^+)$ -extender.*
- (c)  *$V^2 \models \text{“}\lambda = \kappa^{++} + 2^\kappa = \lambda^+ = \kappa^{+3} + \text{TP}(\lambda)\text{”}$ .*
- (d) *If  $U^2$  is the normal measure derived from  $j^2$  and if  $i^2 : V^2 \rightarrow N^2 \simeq \text{Ult}(V^2, U^2)$  is the ultrapower embedding, then there exists  $F \in V^2$  which is  $\text{Col}(\kappa^{+5}, < i(\kappa))_{N^2}$ -generic over  $N^2$ .*

*Proof.* Work in  $V^1$ . Factor  $j^1$  in two steps through the models

$$N' = \text{the transitive collapse of } \{j^1(f)(\kappa) : f : \kappa \rightarrow V^1\}$$

$$\bar{N}' = \text{the transitive collapse of } \{j^1(f)(\alpha) : f : \kappa \rightarrow V^1, \alpha < \lambda^+\}.$$

$N'$  is the familiar ultrapower approximating  $M^1$ , while  $\bar{N}'$  corresponds to the extender of length  $\lambda^+$ . We have maps

$$\begin{aligned} i' &: V^1 \rightarrow N', \\ k' &: N' \rightarrow M^1, \\ \bar{i}' &: N' \rightarrow \bar{N}', \\ \bar{k}' &: \bar{N}' \rightarrow M^1 \end{aligned}$$

such that

$$k' \circ i' = j^1 \ \& \ \bar{k}' \circ \bar{i}' = k'.$$

Let

$$\mathbb{P}_\kappa = \langle\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle\rangle$$

be the reverse Easton iteration of forcing notions such that

- (1) If  $\alpha < \kappa$  is a measurable limit of measurable cardinals, then  $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \dot{\mathbb{M}}(\alpha, \alpha_*, \alpha_*^+)$ .
- (2) Otherwise,  $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$  is the trivial forcing.

Let  $G_k * g$  be  $\mathbb{P}_\kappa * \dot{\mathbb{M}}(\kappa, \lambda, \lambda^+)$ -generic over  $V^1$ .

Note that we can factor  $\mathbb{M}(\kappa, \lambda, \lambda^+)$  as

$$\mathbb{M}(\kappa, \lambda, \lambda^+) = \text{Add}(\kappa, \lambda^+) * \dot{\mathbb{Q}},$$

where  $\dot{\mathbb{Q}}$  is forced to be  $\kappa^+$ -distributive. Let us factor  $g$  as  $g = g(0) * g(1)$ .

As  $\mathbb{P}_\kappa$  is computed in all of the models the same and the embeddings  $\bar{i}', k', \bar{k}'$  have critical point bigger than  $\kappa$ , we can easily lift them to get

$$\begin{aligned} k' &: N'[G_\kappa] \rightarrow M^1[G_\kappa], \\ \bar{i}' &: N'[G_\kappa] \rightarrow \bar{N}'[G_\kappa], \\ \bar{k}' &: \bar{N}'[G_\kappa] \rightarrow M^1[G_\kappa], \end{aligned}$$

where  $\bar{k}' \circ \bar{i}' = k'$ . The models  $\bar{N}'[G_\kappa]$  and  $M^1[G_\kappa]$  are closed under  $\kappa$ -sequences in  $V^1[G_\kappa]$  and they compute the cardinals up to  $\lambda^+$  in the correct way, in particular, the least measurable above  $\kappa$  in these models is  $\lambda$ , and so if we set  $\mathbb{Q}_\kappa = \text{Add}(\kappa, \lambda^+)_{V^1[G_\kappa]}$ , then it is computed in the same way in the models  $\bar{N}'[G_\kappa]$  and  $M^1[G_\kappa]$ , i.e.,

$$\mathbb{Q}_\kappa = (\mathbb{Q}_\kappa)_{\bar{N}'[G_\kappa]} = (\mathbb{Q}_\kappa)_{M^1[G_\kappa]}.$$

On the other hand

$$(\mathbb{Q}_\kappa)_{N'[G_\kappa]} = \text{Add}(\kappa, \bar{\lambda})_{V^1[G_\kappa]},$$

where  $\bar{\lambda} = (i'(f)(\kappa)^+)_{N'}$ . Note that  $\kappa^+ < \bar{\lambda} < \kappa^{++}$ . Factor  $g(0)$  as  $g(0)_1 \times g(0)_2$ , which corresponds to

$$\text{Add}(\kappa, \lambda^+)_{V^1[G_\kappa]} = \text{Add}(\kappa, \bar{\lambda})_{V^1[G_\kappa]} \times \text{Add}(\kappa, \lambda^+ \setminus \bar{\lambda})_{V^1[G_\kappa]}.$$

We build further extensions

$$\begin{aligned} k' &: N'[G_\kappa][g(0)_1] \rightarrow M^1[G_\kappa][g(0)], \\ \bar{i}' &: N'[G_\kappa][g(0)_1] \rightarrow \bar{N}'[G_\kappa][g(0)], \\ \bar{k}' &: \bar{N}'[G_\kappa][g(0)] \rightarrow M^1[G_\kappa][g(0)], \end{aligned}$$

still preserving the relation  $\bar{k}' \circ \bar{i}' = k'$ .

Let us now write  $i'(\mathbb{P}_\kappa)$  as

$$i'(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \text{Add}(\kappa, \bar{\lambda}) * \dot{Q}' * i'(\mathbb{P}_\kappa)_{(\kappa+1, i'(\kappa))} * \text{Add}(i'(\kappa), i'(\bar{\lambda})) * i'(\dot{Q}'),$$

where  $\Vdash_{\mathbb{P}_\kappa * \text{Add}(\kappa, \bar{\lambda})} \dot{Q}'$  is  $\kappa^+$ -distributive". Let  $g(1)'$  be the filter generated by  $i''(g(1))$ . Then  $g(1)'$  is  $\dot{Q}'[G_\kappa * g(0)_1]$ -generic over  $N'[G_\kappa * g(0)_1]$ .

By standard arguments, we can find  $H \in V^1[G_\kappa][g(0)_1]$ , which is  $i'(\mathbb{P}_\kappa)_{(\kappa+1, i'(\kappa))}$ -generic over  $N'[G_\kappa][g(0)_1]$  and hence we can get

$$i' : V^1[G_\kappa] \rightarrow N'[i'(G_\kappa)].$$

Also transfer  $H$  along  $\bar{i}', k'$  to get

$$k' : N'[i'(G_\kappa)] \rightarrow M^1[j^1(G_\kappa)],$$

$$\bar{i}' : N'[i'(G_\kappa)] \rightarrow \bar{N}'[\bar{i}' \circ i'(G_\kappa)],$$

$$\bar{k}' : \bar{N}'[\bar{i}' \circ i'(G_\kappa)] \rightarrow M^1[j^1(G_\kappa)],$$

where all the maps are defined in  $V^1[G_\kappa][g(0)]$ . Let  $l = \bar{i}' \circ i'$ . Since  $\mathbb{P}_\kappa$  has size  $\kappa$  and is  $\kappa$ -c.c., so the term forcing

$$\text{Add}(\kappa, \lambda^+)_{V^1[G_\kappa]}/\mathbb{P}_\kappa$$

is forcing isomorphic to  $\text{Add}(\kappa, \lambda^+)_{V^1}$  (see [1] Fact 2, §1.2.6). By our assumption, we have  $\bar{g} \in V^1$ , which is  $i'(\text{Add}(\kappa, \lambda^+)_{V^1})$ -generic over  $N'$ , and using it we can define  $g_a$  which is

$$\text{Add}(i(\kappa), i(\lambda^+))_{N'[i'(G_\kappa)]}$$

generic over  $N'[i'(G_\kappa)]$ . Using the fact that

$$V^1[G_\kappa][g(0)_1] \models \text{“} {}^\kappa N'[i'(G_\kappa)] \subseteq N'[i'(G_\kappa)] \text{”}$$

we also build  $F$ , which is  $\text{Col}(\kappa^{+5}, < i'(\kappa))_{N'[i'(G_\kappa)]}$  generic over  $N'[i'(G_\kappa)]$ . Note that  $g_a$  and  $F$  are mutually generic.

Transfer  $g_a$  and  $F$  along  $\bar{i}'$  to get new generics  $\bar{g}_a$  and  $\bar{F}$ . Now using Woodin's surgery argument, we can alter the filter  $\bar{g}_a$  to find a generic filter  $h_a$  with the additional property  $l''[g(0)] \subseteq h_a$ . Also  $h_a$  is easily seen to be mutually generic with  $\bar{F}$ .

We now transfer  $h_a$  along  $\bar{k}'$  to get  $H_a$  which is  $j^1(\mathbb{Q}_\kappa)$ -generic over  $M^1$ . Further,  $j^{1''}[g(0)] \subseteq H_a$ , so we can build maps

$$\bar{j} : V^1[G_\kappa * g(0)] \rightarrow M^1[j^1(G_\kappa * g(0))],$$

$$\bar{k}' : \bar{N}'[l(G_\kappa * g(0))] \rightarrow M^1[j^1(G_\kappa * g(0))],$$

$$l : V^1[G_\kappa * g(0)] \rightarrow \bar{N}'[l(G_\kappa * g(0))].$$

such that  $\bar{j} = \bar{k}' \circ l$ .

Now let us look at  $\dot{\mathbb{Q}}[G_\kappa * g(0)]$ . It is  $\kappa^+$ -distributive in  $V^1[G_\kappa * g(0)]$ , so we can further extend the above embeddings and get

$$\begin{aligned} \bar{j} &: V^1[G_\kappa * g(0) * g(1)] \rightarrow M^1[j^1(G_\kappa * g(0) * g(1))], \\ \bar{k}' &: \bar{N}'[l(G_\kappa * g(0) * g(1))] \rightarrow M^1[j^1(G_\kappa * g(0) * g(1))], \\ l &: V^1[G_\kappa * g(0) * g(1)] \rightarrow \bar{N}'[l(G_\kappa * g(0) * g(1))]. \end{aligned}$$

Let

$$\begin{aligned} V^2 &= V^1[G_\kappa * g(0) * g(1)], \\ M^2 &= M^1[j^1(G_\kappa * g(0) * g(1))] \end{aligned}$$

and

$$N^2 = \bar{N}'[l(G_\kappa * g(0) * g(1))].$$

Also let  $j^2 = \bar{j}$ . We argue

$$\text{Ult}(V[G_\kappa * g(0) * g(1)], U^2) \simeq N^2,$$

where  $U^2$  is the normal measure derived from  $j^2$ . To see this, factor  $l$  through  $l^\dagger : V^2 \rightarrow N^\dagger \simeq \text{Ult}(V^2, U')$ , where  $U'$  is the normal measure derived from  $l$ . Also let  $k^\dagger : N^\dagger \rightarrow N^2$ . Then  $P(\kappa)_{V^2} \subseteq N^\dagger$  and  $N^\dagger \models "2^\kappa = \lambda^+"$ . So  $\text{crit}(k^\dagger) > \lambda^+$ . Since  $\lambda^+ \subseteq \text{range}(k^\dagger)$  and  $N^2$  is generated by a  $(\kappa, \lambda^+)$ -extender, we have  $N^2 = N^\dagger$  and we are done.

So if we let  $i^2 = l$ , then

$$i^2 : V^2 \rightarrow N^2$$

is the ultrapower embedding. Finally note that  $F$  is generic for the appropriate collapse ordering. The lemma follows.  $\square$

Note that in the model  $V^2 = V^1[G_\kappa * g]$ , the following conditions are satisfied:

- $V^2 \models " \lambda = \kappa^{++} + 2^\kappa = \lambda^+ = \kappa^{+3} "$ .
- There is  $j^2 : V^2 \rightarrow M^2$  with critical point  $\kappa$  such that  $H(\kappa^{++}) \subseteq M^2$  and  $j^2 \upharpoonright V^1 = j^1$ .
- $j^2$  is generated by a  $(\kappa, \lambda^+)$ -extender.

- If  $U^2$  is the normal measure derived from  $j^2$  and if  $i^2 : V^2 \rightarrow N^2 \simeq \text{Ult}(V^2, U^2)$  is the ultrapower embedding, then there exists  $F \in V^2$  which is  $\text{Col}(\kappa^{+5}, < i(\kappa))_{N^2}$ -generic over  $N^2$ .

Thus the hypotheses of the beginning of Subsection 4.1 are satisfied, and so, working in  $V^2$ , we can construct the pair  $(j, F)$ . Let  $u$  be the measure sequence constructed from it. Set  $w = u \upharpoonright \kappa^+$  and let  $\mathbb{R}_w$  be the corresponding forcing notion as in Definition 4.8. Also let  $K$  be  $\mathbb{R}_w$ -generic over  $V^2$ . Build the sequences  $\vec{\kappa} = \langle \kappa_\xi : \xi < \kappa \rangle$ ,  $\vec{u} = \langle u_\xi : \xi < \kappa \rangle$  and  $\vec{F} = \langle F_\xi : \xi < \kappa \rangle$  from  $K$ , as in Subsection 4.3.

**4.5.  $\text{TP}(\kappa^{++})$  holds in  $V^1[G_\kappa * g * K]$ .** In this subsection we show that  $\text{TP}(\kappa^{++})$  holds in  $V^1[G_\kappa * g * K]$ , and then in the next subsection, we complete the proof of Theorem 1.1 by showing that

$$V^1[G_\kappa * g * K] \models \text{“TP}(\alpha^{++}) \text{ holds for all singular cardinals } \alpha < \kappa \text{”}.$$

As  $V^1[G_\kappa * g * K] \models \kappa^{++} = \lambda$ , it suffices to prove the following:

**Theorem 4.22.**  $V^1[G_\kappa * g * K] \models \text{“TP}(\lambda) \text{”}$ .

The rest of this subsection is devoted to the proof of the above theorem. The proof we present follows ideas of [8], but is more involved as instead of working with the Prikry collapse forcing of [8] we are working with the Radin forcing  $\mathbb{R}_w$ .

Let  $\dot{\mathbb{M}}$  be such that  $\Vdash_{\mathbb{P}_\kappa} \text{“}\dot{\mathbb{M}} = \dot{\mathbb{M}}(\kappa, \lambda, \lambda^+) \text{”}$ .

**Lemma 4.23.** *The forcing  $\mathbb{P}_\kappa * \dot{\mathbb{M}} * \dot{\mathbb{R}}_w$  satisfies the  $\lambda$ -c.c.*

*Proof.* The forcing  $\mathbb{P}_\kappa$  is  $\kappa$ -c.c. Now the result follows from the facts that  $\mathbb{P}_\kappa$  forces “ $\dot{\mathbb{M}}$  is  $\lambda$ -c.c.” (by Lemma 3.3) and  $\mathbb{P}_\kappa * \dot{\mathbb{M}}$  forces “ $\dot{\mathbb{R}}_w$  is  $\kappa^+$ -c.c.” (by Lemma 4.13).  $\square$

Assume towards contradiction that  $\text{TP}(\lambda)$  fails in  $V^1[G_\kappa * g][\vec{u}, \vec{F}]$  and let  $\dot{T} \in V^1[G_\kappa]$  be an  $\mathbb{M} * \dot{\mathbb{R}}_w$ -name for a  $\lambda$ -Aronszajn tree in  $V^1[G_\kappa * g][\vec{u}, \vec{F}]$ . Suppose for simplicity that the trivial condition forces that  $\dot{T}$  is a  $\lambda$ -Aronszajn tree and let us view it as a nice name for a subset of  $\lambda$ ; so that  $\dot{T} = \bigcup_{\xi < \lambda} \{\check{\xi}\} \times A_\xi$ , where each  $A_\xi$  is a maximal antichain in  $\mathbb{M} * \dot{\mathbb{R}}_w$ . By Lemma 4.23, each  $A_\xi$  has size less than  $\lambda$ .

Recall from the remarks after Lemma 3.3 that the forcing  $\mathbb{M}$  is forcing isomorphic to  $\text{Add}(\kappa, \lambda^+) * \dot{\mathbb{Q}}$ , where  $\dot{\mathbb{Q}}$  is some  $\text{Add}(\kappa, \lambda^+)$ -name for a forcing notion which is forced to be  $\kappa^+$ -distributive.

**Lemma 4.24.** *Work in  $V^1[G_\kappa]$ . The set*

$\{r = ((p, \dot{q}), \check{d}^\frown \langle w, \lambda, \dot{A}, \dot{H}, \check{h} \rangle) \in \mathbb{M} * \dot{\mathbb{R}}_w : d, h \in V^1[G_\kappa] \text{ and } \dot{A}, \dot{H} \text{ are } \text{Add}(\kappa, \lambda^+) \text{-names}\}$   
is dense in  $\mathbb{M} * \dot{\mathbb{R}}_w$ .

*Proof.* Recall that a condition in  $\mathbb{R}_w$  is of the form  $p = d^\frown \langle w, \lambda, A, H, h \rangle$  where

- (1)  $d \in V_\kappa$ .
- (2)  $\langle w, \lambda, A, H, h \rangle \in \mathbb{P}_w$ .
- (3)  $h \in \text{Col}(\lambda^{+5}, < \kappa)$ .

As  $\mathbb{M}$  does not add bounded subsets to  $\kappa$ , so any condition  $((p, \dot{q}), \dot{d}^\frown \langle w, \lambda, \dot{A}, \dot{H}, \dot{h} \rangle)$  has an extension of the form  $((p', \dot{q}'), \check{d}'^\frown \langle \kappa, \lambda, \dot{A}', \dot{H}', \check{h}' \rangle)$ , where  $d', h' \in V^1[G_\kappa]$ .

Also note that all conditions in  $\mathbb{P}_w$  and hence in  $\mathbb{R}_w$  exist already in the extension by  $\text{Add}(\kappa, \lambda^+)$ , the Cohen part of  $\mathbb{M}$  (though the definition of  $\mathbb{R}_w$  may require the whole  $\mathbb{M}$ ). Thus we can further extend  $((p', \dot{q}'), \check{d}'^\frown \langle \kappa, \lambda, \dot{A}', \dot{H}', \check{h}' \rangle)$  to another condition

$$((p'', \dot{q}''), \check{d}''^\frown \langle w, \lambda, \dot{A}'', \dot{H}'', \check{h}'' \rangle)$$

where  $\dot{A}''$  and  $\dot{H}''$  are forced to be  $\text{Add}(\kappa, \lambda^+)$ -names (over  $V^1[G_\kappa]$ ). The result follows immediately.  $\square$

From now on, we assume that all the conditions in  $\mathbb{M} * \dot{\mathbb{R}}_w$  are of the above form. This is useful in some of the arguments below (see for example Lemma 4.25(a)). Let us define

$$\mathbb{C} = \{((p, \emptyset), r) : ((p, \emptyset), r) \in \mathbb{M} * \dot{\mathbb{R}}_w\}$$

and

$$\mathbb{T} = \{(\emptyset, q) : (\emptyset, q) \in \mathbb{M}\}.$$

Let  $\tau : \mathbb{C} \times \mathbb{T} \rightarrow \mathbb{M} * \dot{\mathbb{R}}_w$  be defined by

$$\tau(((p, \emptyset), r), (\emptyset, q)) = ((p, q), r).$$



**Lemma 4.25.** (a)  $\tau$  is a projection from  $\mathbb{C} \times \mathbb{T}$  onto  $\mathbb{M} * \dot{\mathbb{R}}_w$ .

(b)  $\mathbb{T}$  is  $\kappa^+$ -closed in  $V^1[G_\kappa]$ .

(c)  $\mathbb{C}$  is  $\kappa^+$ -c.c. in  $V^1[G_\kappa]$ .

*Proof.* (a) It is clear that  $\tau$  is order preserving. Suppose that

$$(p', q', r') \leq_{\mathbb{M} * \dot{\mathbb{R}}_w} \tau(\langle\langle (p, \emptyset), r \rangle, (\emptyset, q) \rangle) = \langle\langle (p, q), r \rangle\rangle.$$

We are going to find  $p^*, q^*$  and  $r^*$  such that  $\langle\langle (p^*, \emptyset), r^* \rangle, (\emptyset, q^*) \rangle \leq_{\mathbb{C} \times \mathbb{T}} \langle\langle (p, \emptyset), r \rangle, (\emptyset, q) \rangle$  and

$$\tau(\langle\langle (p^*, \emptyset), r^* \rangle, (\emptyset, q^*) \rangle) = \langle\langle (p^*, q^*), r^* \rangle\rangle \leq_{\mathbb{M} * \dot{\mathbb{R}}_w} \langle\langle (p', q'), r' \rangle\rangle.$$

Let  $p^* = p'$ . Let  $q^*$  be a name such that

- $p^* \Vdash \text{“}q^* = q'\text{”}$ .
- If  $\tilde{p}$  is incompatible with  $p'$ , then  $\tilde{p} \Vdash \text{“}q^* = \dot{q}\text{”}$ .

Also set  $r^* = r'$ . Then  $p^*, q^*$  and  $r^*$  are as required.

(b) follows from the fact that  $1_{\text{Add}(\kappa, \lambda^+)} \Vdash \text{“Add}(\kappa^+, 1) \text{ is } \kappa^+ \text{-closed”}$ .

(c) follows from Lemma 4.13 and the fact that  $\text{Add}(\kappa, \lambda^+)$  is  $\kappa^+$ -c.c. □

Let  $k : V^1 \rightarrow N^1$  witness the measurability of  $\lambda$  in  $V^1$ . As  $|\mathbb{P}_\kappa| = \kappa < \lambda$ , so by the Levy-Solovay's theorem [19], we can lift  $k$  to  $k : V^1[G_\kappa] \rightarrow N^1[G_\kappa]$ .

Let  $\mathbb{M}^* * \dot{\mathbb{R}}_w^* = k(\mathbb{M} * \dot{\mathbb{R}}_w)$ . The next lemma follows from Lemma 4.23.

**Lemma 4.26.** (in  $V^1[G_\kappa]$ )  $k \upharpoonright \mathbb{M} * \dot{\mathbb{R}}_w : \mathbb{M} * \dot{\mathbb{R}}_w \rightarrow \mathbb{M}^* * \dot{\mathbb{R}}_w^*$  is a regular embedding.

*Proof.* It is clear that  $k$  is order preserving and if  $p \perp_{\mathbb{M} * \dot{\mathbb{R}}_w} q$  ( $p$  is incompatible with  $q$  in  $\mathbb{M} * \dot{\mathbb{R}}_w$ ), then  $k(p) \perp_{\mathbb{M}^* * \dot{\mathbb{R}}_w^*} k(q)$  ( $k(p)$  and  $k(q)$  are incompatible in  $\mathbb{M}^* * \dot{\mathbb{R}}_w^*$ ). Now suppose that  $A \subseteq \mathbb{M} * \dot{\mathbb{R}}_w$  is a maximal antichain in  $\mathbb{M} * \dot{\mathbb{R}}_w$ . By Lemma 4.23,  $|A| < \lambda$  and so by elementarity of  $k$ ,  $k''[A] = k(A)$  is a maximal antichain in  $\mathbb{M}^* * \dot{\mathbb{R}}_w^*$ . □

Thus let  $g^* * K^*$  be  $\mathbb{M}^* * \dot{\mathbb{R}}_w^*$ -generic over  $V^1[G_\kappa]$  such that  $k''[g^* K] \subseteq g^* * K^*$ . It follows that we can lift  $k$  to

$$k : V^1[G_\kappa * g^* K] \rightarrow N^1[G_\kappa * g^* * K^*].$$

Hence, in  $V^1[G_\kappa]$ , by Lemma 4.26, there is a projection

$$\pi : \mathbb{M}^* * \dot{\mathbb{R}}_w^* \rightarrow RO(\mathbb{M} * \dot{\mathbb{R}}_w),$$

where  $RO(\mathbb{M} * \dot{\mathbb{R}}_w)$  denotes the Boolean completion of  $\mathbb{M} * \dot{\mathbb{R}}_w$ .

Given a condition  $((p, q), r) \in \mathbb{M}^* * \dot{\mathbb{R}}_w^*$ , let us identify  $\pi(p) = \pi((p, \emptyset), 1_{\mathbb{R}_w^*})$  with

$$(k^{-1})''[p] = p \upharpoonright (\kappa \times \lambda) \cup \{((\gamma, \alpha), i) : \gamma < \kappa, \alpha \geq \lambda, i \in \{0, 1\}, ((\gamma, k(\alpha)), i) \in p\}.$$

Let  $\mathbb{Q}_\pi$  be the quotient forcing determined by  $\pi$ :

$$\mathbb{Q}_\pi = \{((p, \dot{q}), \dot{r}) \in \mathbb{M}^* * \dot{\mathbb{R}}_w^* : \pi((p, \dot{q}), \dot{r}) \in g * K\}.$$

Let us define

$$\mathbb{C}_\pi = \{((p, \emptyset), r) : ((p, \emptyset), r) \in \mathbb{Q}_\pi\}$$

where the ordering is the one inherited from  $\mathbb{Q}_\pi$ , and let

$$\mathbb{T}_\pi = \{(\emptyset, q) \in \mathbb{M}^* : (\emptyset, q) \in g\},$$

with the ordering inherited from  $\mathbb{M}^*$ . Also define  $\tau_\pi : \mathbb{C}_\pi \times \mathbb{T}_\pi \rightarrow \mathbb{Q}_\pi$  by

$$\tau_\pi(\langle((p, \emptyset), r), (\emptyset, q)\rangle) = ((p, q), r).$$

This is well-defined.

**Lemma 4.27.**  $\tau_\pi$  is a projection from  $\mathbb{C}_\pi \times \mathbb{T}_\pi$  onto  $\mathbb{Q}_\pi$ .

*Proof.* The proof is similar to the proof of Lemma 4.25(a). Clearly  $\tau_\pi$  is order preserving.

Suppose that

$$\langle(p', \dot{q}'), \dot{r}'\rangle \leq_{\mathbb{Q}_\pi} \tau(\langle((p, \emptyset), \dot{r}), (\emptyset, \dot{q})\rangle) = ((p, \dot{q}), \dot{r}).$$

We are going to find  $p^*, \dot{q}^*$  and  $\dot{r}^*$  such that  $\langle((p^*, \emptyset), \dot{r}^*), (\emptyset, \dot{q}^*)\rangle \leq_{\mathbb{C}_\pi \times \mathbb{T}_\pi} \langle((p, \emptyset), \dot{r}), (\emptyset, \dot{q})\rangle$

and

$$\tau(\langle((p^*, \emptyset), \dot{r}^*), (\emptyset, \dot{q}^*)\rangle) = (p^*, \dot{q}^*), \dot{r}^* \leq_{\mathbb{Q}_\pi} (p', \dot{q}'), \dot{r}'.$$

Let  $p^* = p'$ . Let  $\dot{q}^*$  be a name such that

- $p^* \Vdash \dot{q}^* = \dot{q}'$ .
- If  $\tilde{p}$  is incompatible with  $p'$ , then  $\tilde{p} \Vdash \dot{q}^* = \dot{q}$ .

Also set  $\dot{r}^* = \dot{r}'$ . Then  $p^*, \dot{q}^*$  and  $\dot{r}^*$  are as required. □

**Lemma 4.28.**  $\mathbb{T}_\pi$  is  $\kappa^+$ -closed in  $N^1[G_\kappa * g]$ .

*Proof.* Similar to the proof of Lemma 4.25(b). □

Also, as in the proof of 4.25(c), one can show that  $\mathbb{C}_\pi$  is  $\kappa^+$ -c.c. in  $N^1[G_\kappa * g * K]$ . Here we prove something stronger, which is needed for the proof of Theorem 4.22.

**Lemma 4.29.**  $\mathbb{C}_\pi \times \mathbb{C}_\pi$  is  $\kappa^+$ -c.c. in  $N^1[G_\kappa * g * K]$ .

*Proof.* Assume towards a contradiction that  $A \in N^1[G_\kappa * g * K]$  is an antichain in  $\mathbb{C}_\pi \times \mathbb{C}_\pi$  of size  $\kappa^+$ . Let  $\langle (a_i^1, a_i^2) : i < \kappa^+ \rangle$  be an enumeration of  $A$ , and for  $i < \kappa^+$  and  $k \in \{1, 2\}$  let us write  $a_i^k$  as

$$a_i^k = ((p_i^k, \emptyset), \check{d}_i^k \frown \langle w, \lambda_i^k, \dot{A}_i^k, \dot{H}_i^k, \check{h}_i^k \rangle).$$

By shrinking  $A$  in necessary, we assume that there is a condition  $((p, \dot{q}), \check{d} \frown \langle w, \lambda, \dot{A}, \dot{H}, \check{h} \rangle) \in g * K$  which forces the following:

- (1)  $\dot{A}$  is an antichain.
- (2) There exists  $t_1$  such that all  $i < \kappa^+$ ,  $d_i^1 = t_1$ .
- (3) There exists  $t_2$  such that all  $i < \kappa^+$ ,  $d_i^2 = t_2$ .
- (4) For some fixed  $\eta_1 < \kappa$  and all  $i < \kappa^+$ ,  $\lambda_i^1 = \eta_1$ .
- (5) For some fixed  $\eta_2 < \kappa$  and all  $i < \kappa^+$ ,  $\lambda_i^2 = \eta_2$ .
- (6) For some  $f_1 \in \text{Col}(\eta_1^{+5}, < \kappa)$  and all  $i < \kappa^+$ ,  $h_i^1 = f_1$ .
- (7) For some  $f_2 \in \text{Col}(\eta_2^{+5}, < \kappa)$  and all  $i < \kappa^+$ ,  $h_i^2 = f_2$ .

For each  $i$ , choose  $((p_i, \dot{q}_i), \check{d}_i \frown \langle w, \lambda, \dot{A}_i, \dot{H}_i, \check{h}_i \rangle) \in g * K$  which extends  $((p, \dot{q}), \check{d} \frown \langle w, \lambda, \dot{A}, \dot{H}, \check{h} \rangle)$  and decides both  $a_i^1$  and  $a_i^2$ , say it forces (for  $k \in \{1, 2\}$ )

$$a_i^k = ((p_i^k, \emptyset), \check{t}_1 \frown \langle w, \eta_k, \dot{A}_i^k, \dot{H}_i^k, \check{f}_k \rangle).$$

By further shrinking and extending the conditions, we may assume that for some  $s$  and all  $i < \kappa^+$ ,  $d = d_i = s$ .

Let

$$((p_i^{*1}, \dot{q}_i^{*1}), \check{s} \frown \langle w, \lambda, \dot{A}_i^{*1}, \dot{H}_i^{*1}, \check{h}_i^{*1} \rangle)$$

extends  $a_i^1$ ,  $((p_i, \dot{q}_i), \check{d}_i \frown \langle w, \lambda, \dot{A}_i, \dot{H}_i, \check{h}_i \rangle)$  and  $((p, \dot{q}), \check{d} \frown \langle w, \lambda, \dot{A}, \dot{H}, \check{h} \rangle)$  and such that  $\pi(p_i^{*1}) = (k^{-1})''[p_i^{*1}]$  is in the Cohen part of  $g * K$ . Similarly let

$$((p_i^{*2}, \dot{q}_i^{*2}), \check{s} \frown \langle w, \lambda, \dot{A}_i^{*2}, \dot{H}_i^{*2}, \check{h}_i^{*2} \rangle)$$

extends  $a_i^2$ ,  $((p_i, \dot{q}_i), \check{d}_i \frown \langle w, \lambda, \dot{A}_i, \dot{H}_i, \check{h}_i \rangle)$  and  $((p, \dot{q}), \check{d} \frown \langle w, \lambda, \dot{A}, \dot{H}, \check{h} \rangle)$  and such that  $\pi(p_i^{*2}) = (k^{-1})''[p_i^{*2}]$  is in the Cohen part of  $g * K$ . Note that in particular  $\pi(p_i^{*1}) \parallel \pi(p_i^{*2})$  ( $\pi(p_i^{*1})$  is compatible with  $\pi(p_i^{*2})$ ).

By  $\Delta$ -system arguments, we can find  $i < j$  such that  $p_i^{*1} \parallel p_j^{*1}$  and  $p_i^{*2} \parallel p_j^{*2}$ . Let

$$g^1 = ((p_i^{*1}, \dot{q}_i^{*1}), \check{s} \frown \langle w, \lambda, \dot{A}_i^{*1}, \dot{H}_i^{*1}, \check{h}_i^{*1} \rangle) \wedge ((p_j^{*1}, \dot{q}_j^{*1}), \check{s} \frown \langle w, \lambda, \dot{A}_j^{*1}, \dot{H}_j^{*1}, \check{h}_j^{*1} \rangle)$$

be the greatest lower bound of

$$((p_i^{*1}, \dot{q}_i^{*1}), \check{s} \frown \langle w, \lambda, \dot{A}_i^{*1}, \dot{H}_i^{*1}, \check{h}_i^{*1} \rangle)$$

and

$$((p_j^{*1}, \dot{q}_j^{*1}), \check{s} \frown \langle w, \lambda, \dot{A}_j^{*1}, \dot{H}_j^{*1}, \check{h}_j^{*1} \rangle).$$

Similarly let

$$g^2 = ((p_i^{*2}, \dot{q}_i^{*2}), \check{s} \frown \langle w, \lambda, \dot{A}_i^{*2}, \dot{H}_i^{*2}, \check{h}_i^{*2} \rangle) \wedge ((p_j^{*2}, \dot{q}_j^{*2}), \check{s} \frown \langle w, \lambda, \dot{A}_j^{*2}, \dot{H}_j^{*2}, \check{h}_j^{*2} \rangle).$$

Let

$$p' = \pi(p_i^{*1}) \cup \pi(p_j^{*1}) \cup \pi(p_i^{*2}) \cup \pi(p_j^{*2}),$$

which is well-defined. Let

$$g = ((p', \emptyset), \emptyset) \wedge ((p, \dot{q}), \check{d} \frown \langle w, \lambda, \dot{A}, \dot{H}, \check{h} \rangle) \bigwedge_{l=i,j} ((p_l, \dot{q}_l), \check{d}_l \frown \langle w, \lambda, \dot{A}_l, \dot{H}_l, \check{h}_l \rangle),$$

be the greatest lower bound of the conditions considered. To continue, we need the following two claims:

**Claim 4.30.** *Assume that*

$$r = ((p, \dot{q}), \check{d} \frown \langle w, \lambda, \dot{A}, \dot{H}, \check{h} \rangle) \in \mathbb{M} * \dot{\mathbb{R}}_w$$

and

$$r^* = ((p^*, \emptyset), \check{d}^* \frown \langle w, \lambda, \dot{A}^*, \dot{H}^*, \check{h}^* \rangle) \in \mathbb{M}^* * \dot{\mathbb{R}}_w^*$$

and the following conditions are satisfied:

- (1)  $r \leq \pi(r^*)$ .
- (2) Suppose that  $d = \langle d_0, \dots, d_{n-1} \rangle$  and  $d^* = \langle d_0^*, \dots, d_{m-1}^* \rangle$ . Then  $n = m$  and for all  $k < n$ ,  $\kappa^{d_k} = \kappa^{d_k^*}$  and  $\lambda^{d_k} = \lambda^{d_k^*}$ .

(3) For all  $k < n$ ,  $h^{d_k} \leq h^{d_k^*}$ .

(4)  $h \leq h^*$ .

Then  $r$  does not force  $r^*$  out the quotient  $\mathbb{C}_\pi$ .

*Proof.* Consider the conditions  $r$  and  $r^*$ . The above conditions imply that they are compatible, so let  $r \wedge r^*$  be a common extension of them. Let  $\bar{g} \times \bar{K}$  be  $\mathbb{M}^* * \dot{\mathbb{R}}_w^*$ -generic over  $V^1[G_\kappa]$  such that  $r \wedge r^* \in \bar{g} \times \bar{K}$ . But then  $\pi(r) \in \langle \pi''[\bar{g} \times \bar{K}] \rangle$ , the filter on  $\mathbb{M}^* \dot{\mathbb{R}}_w^*$  generated by  $\pi''[\bar{g} \times \bar{K}]$ . The result follows immediately.  $\square$

**Claim 4.31.** *Assume  $r$  and  $r^*$  are as in Claim 4.30. Then there exists  $\bar{r} \leq^* r$  such that  $\bar{r}$  forces “ $r^* \in \mathbb{C}_\pi$ ”.*

*Proof.* By Lemma 4.15, there exists  $\bar{r} \leq^* r$  which decides “ $r^* \in \mathbb{C}_\pi$ ”. By Claim 4.30,  $\bar{r}$  cannot force “ $r^* \notin \mathbb{C}_\pi$ ”. So  $\bar{r} \Vdash “r^* \in \mathbb{C}_\pi”$ .  $\square$

Note that conditions  $g$  and  $g^1$  satisfy the conditions in Claim 4.30, hence by Claim 4.31, there exists  $\bar{g}_1 \leq^* g$  which forces “ $g^1 \in \mathbb{C}_\pi$ ”. Then  $\bar{g}_1$  and  $g^2$  satisfy the conditions in Claim 4.30, so again by Claim 4.31, there exists  $\bar{g}_2 \leq^* \bar{g}_1$  which forces “ $g^2 \in \mathbb{C}_\pi$ ”. It follows that

$$\bar{g}_2 \Vdash “g^1, g^2 \in \mathbb{C}_\pi”.$$

But then

- $\bar{g}_2 \Vdash “(g^1, g^2) \in \mathbb{C}_\pi \times \mathbb{C}_\pi”$ .
- $\bar{g}_2 \Vdash “(g^1, g^2) \leq (a_i^1, a_i^2), (a_j^1, a_j^2)”$ .
- $\bar{g}_2 \leq ((p, \dot{q}), \dot{d} \smallfrown \langle \kappa, \lambda, \dot{A}, \dot{f}, \dot{F} \rangle)$ .

It follows that  $\bar{g}_2 \Vdash “\dot{A}$  is an antichain”, and from the above, we get a contradiction.  $\square$

We are now ready to complete the proof of Theorem 4.22. Note that by our assumption  $\dot{T} \in V^1[G_\kappa]$  is an  $\mathbb{M}^* \dot{\mathbb{R}}_w^*$  name such that  $T = \dot{T}[g^* K]$  is a  $\lambda$ -Aronszajn tree in  $V^1[G_\kappa * g^* K]$ . Also note that  $\dot{T} \in N^1[G_\kappa]$ .

By standard arguments,  $k(T)_{<\lambda} = T$  and so  $T$  has a cofinal branch in  $N^1[G_\kappa * g^* K^*] \subseteq V^1[G_\kappa * g^* K^*]$ .

Suppose that  $X \times Y$  is  $\mathbb{C}_\pi \times \mathbb{Q}_\pi$ -generic over  $V^1[G_\kappa * g^* K]$  so that  $N^1[G_\kappa * g^* K^*] \subseteq N^1[G_\kappa * g^* K][X \times Y]$ , which is possible by Lemma 4.27. It follows that  $T$  has a cofinal

branch in  $N[G * H * K][X \times Y]$ . Now lemmas 2.4, 2.5 4.28 and 4.29 can be used to show that forcing with  $\mathbb{C}_\pi \times \mathbb{Q}_\pi$  over  $N^1[G_\kappa * g * K]$  does not add cofinal branches to  $T$  (see [8] for details). We get a contradiction and Theorem 4.22 follows.

**4.6. Completing the proof of Theorem 1.1.** In this subsection we complete the proof of Theorem 1.1, by showing that in the model  $V^1[G_\kappa * g * K]$ ,  $\text{TP}(\alpha^{++})$  holds for all singular cardinals  $\alpha < \kappa$ .

Recall that  $C = \{\kappa_i : i < \kappa\}$  is the Radin club added by  $K$ . Let us also assume that  $\min(C) = \aleph_0$ . Recall that  $G_\kappa$  is assumed to be  $\mathbb{P}_\kappa$ -generic over  $V^1$ . Let us write it as

$$G_\kappa = \langle \langle G_\alpha : \alpha \leq \kappa \rangle, \langle G(\alpha) : \alpha < \kappa \rangle \rangle,$$

which corresponds to the iteration

$$\mathbb{P}_\kappa = \langle \langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle \rangle.$$

By simple reflecting arguments, we have the following lemma.

**Lemma 4.32.** *The set  $X \in \mathcal{F}_w$ , where  $X$  consists of all those  $u \in \mathcal{U}_\infty$  such that  $\alpha = \kappa_u$  satisfies the following conditions:*

- $\alpha$  is a measurable cardinal.
- $\mathbb{P}_\alpha$  is  $\alpha$ -c.c. and of size  $\alpha$ .
- $\alpha$  remains measurable after forcing with  $\mathbb{P}_\alpha$  and  $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{M}}(\alpha, \alpha_*, \alpha_*^+)$ .
- Some elementary embedding  $j : V^1 \rightarrow M^1$  with  $\text{crit}(j) = \alpha$  can be extended to

$$j : V^1[G_\alpha] \rightarrow M^1[j(G_\alpha)]$$

and then to

$$j : V^1[G_\alpha * G(\alpha)] \rightarrow M^1[j(G_\alpha * G(\alpha))].$$

- $\mathbb{P}_{\alpha+1} \Vdash$  “ $\dot{\mathbb{P}}_{(\alpha+1, \kappa)}$  does not add any new subsets to  $\alpha_*$ ”.

*Proof.* It suffices to show that  $\forall \alpha < \kappa^+, w \upharpoonright \alpha \in j(X)$ , which can be easily checked.  $\square$

Thus we can assume that

$$\aleph_0 < \alpha \in C \implies \alpha \in X.$$

On the other hand, if  $\alpha < \kappa$  is a limit cardinal in  $V^1[G_\kappa * g * K]$ , then  $\alpha \in \lim(C)$ , the set of limit points of  $C$ , and  $2^\alpha = \alpha^{+3}$ . Thus the following completes the proof:

**Theorem 4.33.** *Assume  $\alpha \in \lim(C)$ . Then  $V^1[G_\kappa * g * K] \models \text{“TP}(\alpha^{++})\text{”}$ .*

*Proof.* Fix  $\alpha \in \lim(C)$ , and let  $\xi < \kappa$  be such that  $\alpha = \kappa_\xi$ . Note that  $\xi$  is a limit ordinal.

We have

$$V^1[G_\kappa * g * K] = V^1[G_{\alpha+1}[G_{(\alpha+1, \kappa)}][g][[\vec{u} \restriction \xi, \vec{F} \restriction \xi][\vec{u} \restriction [\xi, \kappa), \vec{F} \restriction [\xi, \kappa)].$$

and the following hold:

- (1)  $V^1[G_\kappa * g * K]$  is a generic extension of  $V^1[G_\kappa * g][\vec{u} \restriction \xi, \vec{F} \restriction \xi]$  by a forcing notion which does not add any new subsets to  $\alpha_*$ <sup>3</sup>.
- (2) Forcing with  $\mathbb{P}_{(\alpha+1, \kappa)} * \dot{\mathbb{M}}$  does not add any subsets to  $\alpha_*$ ; in particular, the forcing notion  $\mathbb{R}_{u_\xi}$  is defined in the same way in the models  $V^1[G_\kappa * g]$  and  $V^1[G_{\alpha+1}]$ .

It follows that  $V^1[G_\kappa * g * K]$  is a generic extension of  $V^1[G_{\alpha+1}][\vec{u} \restriction \xi, \vec{F} \restriction \xi]$ , by a forcing notion which does not add any new subsets to  $\alpha_*$ . Also note that

- (3)  $V^1[G_{\alpha+1}][\vec{u} \restriction \xi, \vec{F} \restriction \xi] \models \text{“}\alpha^{++} = \alpha_*\text{”}$ .

Thus it suffices to prove the following:

**Lemma 4.34.** *Tree property at  $\alpha_*$  holds in the generic extension  $V^1[G_{\alpha+1}][\vec{u} \restriction \xi, \vec{F} \restriction \xi]$ , which is obtained using the forcing notion*

$$\mathbb{P}_{\alpha+1} * \dot{\mathbb{R}}_{u_\xi} = \mathbb{P}_\alpha * \dot{\mathbb{M}}_\alpha * \dot{\mathbb{R}}_{u_\xi}.$$

*Proof.* The proof is very similar to the proof of Theorem 4.22. □

This completes the proof of Theorem 4.33. □

Theorem 1.1 follows.

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<sup>3</sup>Recall that  $\alpha_*$  is the least measurable cardinal above  $\alpha$ .

5. TREE PROPERTY AT  $\aleph_{2n}$ 'S AND  $\aleph_{\omega+2}$ 

In this section we prove Theorem 1.3. Thus assume that  $GCH$  holds and  $\eta > \lambda$  are measurable cardinals above  $\kappa$ . We assume that they are the least such cardinals. Suppose  $\kappa$  is an  $H(\eta)$ -hypermeasurable cardinal as witnessed by the elementary embedding  $j : V \rightarrow M \supseteq H(\eta)$ . We may assume that it is generated by a  $(\kappa, \eta)$ -extender. Let  $i : V \rightarrow N$  be the ultrapower embedding derived from  $j$  and let  $k : N \rightarrow M$  be such that  $j = k \circ i$ .

The next lemma can be proved as in Lemma 4.20

**Lemma 5.1.** *Then there exists a cofinality preserving generic extension  $V^1$  of  $V$  satisfying the following conditions:*

- (a)  $V^1 \models "GCH"$ .
- (b) *There is  $j^1 : V^1 \rightarrow M^1$  with critical point  $\kappa$  such that  $H(\eta) \subseteq M^1$  and  $j^1 \upharpoonright V = j$ .*
- (c)  *$j^1$  is generated by a  $(\kappa, \eta)$ -extender.*
- (d) *If  $U^1$  is the normal measure derived from  $j^1$  and if  $i^1 : V^1 \rightarrow N^1 \simeq \text{Ult}(V^1, U^1)$  is the ultrapower embedding, then there exists  $\bar{g} \in V^1$  which is  $i^1(\text{Add}(\kappa, \lambda)_{V^1})$ -generic over  $N^1$ . Further  $i^1 \upharpoonright V = i$ .*

Let  $V^1$  be the model constructed above. We need the following lemma which is an analogue of Lemma 4.21.

**Lemma 5.2.** *Work in  $V^1$ . There exists a forcing iteration  $\mathbb{P}_\kappa$  of length  $\kappa$  such that if  $G_\kappa * g * h$  is  $\mathbb{P}_\kappa * \dot{\mathbb{M}}(\kappa, \lambda) * \dot{\mathbb{M}}(\lambda, \eta)$ -generic over  $V^1$ , then in  $V^2 = V^1[G_\kappa * g * h]$ , the following holds:*

- (a) *There is  $j^2 : V^2 \rightarrow M^2$  with critical point  $\kappa$  such that  $j^2 \upharpoonright V^1 = j^1$ .*
- (b)  *$j^2$  is generated by a  $(\kappa, \eta)$ -extender.*
- (c)  $V^2 \models "\lambda = \kappa^{++} + \eta = \kappa^{+4} + TP(\lambda) + TP(\eta)"$ .
- (d) *If  $U^2$  is the normal measure derived from  $j^2$  and if  $i^2 : V^2 \rightarrow N^2 \simeq \text{Ult}(V^2, U^2)$  is the ultrapower embedding, then there exists  $F \in V^2$  which is  $\dot{\mathbb{M}}(\kappa^{+4}, i^2(\kappa))_{N^2}$ -generic over  $N^2$ .*

*Proof.* We follow the proof of Lemma 4.21. For an ordinal  $\alpha \leq \kappa$  let  $\alpha_*$  and  $\alpha_{**}$  denote the first and second measurable cardinals above  $\alpha$ . Note that  $\kappa_* = \lambda$  and  $\kappa_{**} = \eta$ .



Work in  $V^1$ . Factor  $j^1$  in two steps through the models

$$N' = \text{the transitive collapse of } \{j^1(f)(\kappa) : f : \kappa \rightarrow V^1\}$$

$$\bar{N}' = \text{the transitive collapse of } \{j^1(f)(\alpha) : f : \kappa \rightarrow V^1, \alpha < \lambda\}.$$

Again, note that  $N'$  is the familiar ultrapower approximating  $M^1$ , while  $\bar{N}'$  corresponds to the extender of length  $\lambda$ . We have maps

$$i' : V^1 \rightarrow N',$$

$$k' : N' \rightarrow M^1,$$

$$\bar{i}' : N' \rightarrow \bar{N}',$$

$$\bar{k}' : \bar{N}' \rightarrow M^1$$

such that

$$k' \circ i' = j^1 \ \& \ \bar{k}' \circ \bar{i}' = k'.$$

Let

$$\mathbb{P}_\kappa = \langle\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle\rangle$$

be the reverse Easton iteration, where

- (1) If  $\alpha < \kappa$  is a measurable limit of measurable cardinals, then  $\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{M}}(\alpha, \alpha_*) * \dot{\mathbb{M}}(\alpha_*, \alpha_{**})\text{”}$ .
- (2) Otherwise,  $\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ is the trivial forcing”}$ .

Let

$$G_\kappa = \langle\langle G_\alpha : \alpha \leq \kappa \rangle, \langle G(\alpha) : \alpha < \kappa \rangle\rangle$$

be be  $\mathbb{P}_\kappa$ -generic over  $V^1$ .

Note that we can factor  $\mathbb{P}_\kappa * \dot{\mathbb{M}}(\kappa, \lambda) * \dot{\mathbb{M}}(\lambda, \eta)$  as

$$\mathbb{P}_\kappa * \dot{\mathbb{M}}(\kappa, \lambda) * \dot{\mathbb{M}}(\lambda, \eta) = \mathbb{P}_\kappa * \text{Add}(\kappa, \lambda) * \dot{\mathbb{Q}},$$

where  $\dot{\mathbb{Q}}$  is forced to be  $\kappa^+$ -distributive. So the arguments of the proof of Lemma 4.21 can be used to get the embeddings

$$\bar{j} : V^1[G_\kappa * g * h] \rightarrow M^1[j^1(G_\kappa * g * h)],$$

$$\bar{k}' : \bar{N}'[l(G_\kappa * g * h)] \rightarrow M^1[j^1(G_\kappa * g * h)],$$

$$l : V^1[G_\kappa * g * h] \rightarrow \bar{N}'[l(G_\kappa * g * h)],$$

together a filter  $F$  which is  $\mathbb{M}(\kappa^+4, i^2(\kappa))_{N'[i'(G_\kappa)]}$  generic over  $N'[i'(G_\kappa)]$ .

Let

$$\begin{aligned} V^2 &= V^1[G_\kappa * g * h], \\ M^2 &= M^1[j^1(G_\kappa * g * h)] \end{aligned}$$

and

$$N^2 = \bar{N}'[l(G_\kappa * g * h)].$$

Also let  $j^2 = \bar{j}$ . We argue

$$\text{Ult}(V[G_\kappa * g * h], U^2) \simeq N^2,$$

where  $U^2$  is the normal measure derived from  $j^2$ . To see this, factor  $l$  through  $l^\dagger : V^2 \rightarrow N^\dagger \simeq \text{Ult}(V^2, U')$ , where  $U'$  is the normal measure derived from  $l$ . Also let  $k^\dagger : N^\dagger \rightarrow N^2$ . Then  $P(\kappa)_{V^2} \subseteq N^\dagger$  and  $N^\dagger \models "2^\kappa = \lambda"$ . So  $\text{crit}(k^\dagger) > \lambda$ . Since  $\lambda \subseteq \text{range}(k^\dagger)$  and  $N^2$  is generated by a  $(\kappa, \lambda)$ -extender, we have  $N^2 = N^\dagger$  and we are done. So if we let  $i^2 = l$ , then

$$i^2 : V^2 \rightarrow N^2$$

is the ultrapower embedding. Finally note that  $F$  is generic for the appropriate ordering. The lemma follows.  $\square$

Also note that  $F \in M^2$ . Now, working in  $V^2 = V^1[G_\kappa * g * h]$ , we would like to define a version of Prikry forcing. Set

$$\begin{aligned} P^* &= \{f : \kappa \rightarrow V_\kappa^2 \mid \text{dom}(f) \in U^2 \text{ and } \forall \alpha, f(\alpha) \in \mathbb{M}(\alpha^{+4}, \kappa)\}, \\ F^* &= \{f \in P^* \mid i(f)(\kappa) \in F\}. \end{aligned}$$

Now define the notion of a constructing pair as in Definition 4.2, where the forcing notions  $\text{Col}(\kappa^{+5}, < i(\kappa))_N$  and  $\text{Col}(\kappa^{+5}, < j(\kappa))_M$  are replaced by  $\mathbb{M}(\kappa^{+4}, i(\kappa))_N$  and  $\mathbb{M}(\kappa^{+4}, j(\kappa))_M$  respectively. Then definitions 3.3-3.5 go in the same way.

We now define our forcing notion as in Section 4, but using the different guiding generic filters that we obtained above. For this aim, and as before, we define forcing notions  $\mathbb{P}_w, w \in \mathcal{U}_\infty$ , which are the building blocks of our main forcing notion.

**Definition 5.3.** *If  $w \in \mathcal{U}_\infty$ , then  $\mathbb{P}_w$  is the set of tuples  $p = \langle w, \lambda, A, H, h \rangle$  such that*

- (1)  $w$  is a measure sequence.
- (2)  $\lambda < \kappa_w$ .
- (3)  $A \in \mathcal{F}_w$ .

(4)  $H \in F_w^*$  with  $\text{dom}(H) = \{\kappa_v > \lambda \mid v \in A\}$ .

(5)  $h \in \mathbb{M}(\lambda^{+4}, \kappa_w)$ .

Note that if  $\text{lh}(w) = 1$ , then the above tuple is of the form  $(w, \lambda, \emptyset, \emptyset, h)$  (where  $\lambda < \kappa_w$  and  $h \in \mathbb{M}(\lambda^{+4}, \kappa_w)$ ).

The forcing notion  $\mathbb{R}_w$  is defined in the same way as before:

**Definition 5.4.** *If  $w$  is a measure sequence, then  $\mathbb{R}_w$  is the set of finite sequences*

$$p = \langle p_k \mid k \leq n \rangle,$$

where

(1)  $p_k = (w_k, \lambda_k, A_k, H_k, h_k) \in \mathbb{P}_w$ , for each  $k \leq n$ .

(2)  $w_n = w$ .

(3) If  $k < n$ , then  $\lambda_{k+1} = \kappa_{w_k}$ .

Given  $p \in \mathbb{R}_w$  as above,  $n$  is called the length of  $p$  and we denote it by  $\text{lh}(p)$ . The order relations  $\leq^*$  and  $\leq$  are defined as before.

Now assume that  $u$  is the measure sequence constructed using  $(j^2, F)$  and set  $w = u \upharpoonright 2$ . Let  $\mathbb{R} = \mathbb{R}_w$ . Let  $K$  be  $\mathbb{R}$ -generic over  $V^1[G_\kappa * g * h]$ . Let  $C$  be the  $\omega$ -sequence added by  $K$  and let  $\vec{\kappa} = \langle \kappa_n : n < \omega \rangle$  enumerate  $C$  in the increasing order and note that  $\sup_{n < \omega} \kappa_n = \kappa$ . We may further assume that  $\kappa_0 = \aleph_0$ . Also let  $\vec{F} = \langle F_n : n < \omega \rangle$  be the  $\omega$ -sequence added by  $K$ , where each  $F_n$  is  $\mathbb{M}(\kappa_n^{+4}, \kappa_{n+1})$ -generic over  $V^1[G_\kappa * g * h]$ . The following lemma summarizes the basic properties of  $\mathbb{R}$ .

**Lemma 5.5.** (a)  $(\mathbb{R}, \leq)$  satisfies the  $\kappa^+$ -c.c.

(b) Assume  $p \in \mathbb{R}$  and  $m < n^p$ . Then

$$\mathbb{R}/p \simeq \prod_{i \leq m} \mathbb{M}(\kappa_i^{+4}, \kappa_{i+1}) \times \mathbb{R}/p^{>m},$$

where  $p^{>m} = \langle p_{m+1}, \dots, p_{\text{lh}(p)} \rangle$ .

(c)  $(\mathbb{R}, \leq, \leq^*)$  satisfies the Prikry property.

(d)  $V^1[G_\kappa * g * h * K] = V^1[G_\kappa * g * h][\vec{F}]$ .

(e) In  $V^1[G_\kappa * g * h * K]$ ,  $\kappa = \aleph_\omega$ ,  $\lambda = \aleph_{\omega+2}$  and  $\eta = \aleph_{\omega+4}$ .

$$(f) \text{Card}^{V^1[G_\kappa * g * h * K]} \cap \kappa = \{\kappa_n, \kappa_n^+, \kappa_n^{++}, \kappa_n^{+3}, \kappa_n^{+4}, \kappa_n^{+5} : n < \omega\}.$$

Recall that, given a cardinal  $\alpha \leq \kappa$ , we are using  $\alpha_*$  to denote the least measurable cardinal above  $\alpha$  and  $\alpha_{**}$  to denote the second measurable cardinal above  $\alpha$ ; so that  $\alpha_{**} = (\alpha_*)_*$ . The next lemma can be proved by the same arguments as in [8] (and using Lemma 3.5); see also Theorem 4.22:

**Lemma 5.6.** *In  $V^1[G_\kappa * g * h * K]$ , the tree property holds at  $\aleph_{\omega+2}$ .*

We now show that the tree property holds at all  $\aleph_{2n}$ 's,  $0 < n < \omega$ . The next lemma can be proved by simple reflection arguments.

**Lemma 5.7.** *The set  $X \in \mathcal{F}_w$ , where  $X$  consists of cardinals  $\alpha < \kappa$  such that*

- (1)  $\mathbb{P}_\alpha$  is  $\alpha$ -c.c. and of size  $\alpha$ .
- (2)  $\mathbb{P}_\alpha \Vdash$  “  $\mathbb{P}(\alpha) = \mathbb{M}(\alpha^{+4}, \alpha_*) * \dot{\mathbb{M}}(\alpha_*^{+4}, \alpha_{**})$  ”.
- (3)  $\alpha$  remains measurable after forcing with  $\mathbb{P}_\alpha$  and  $\mathbb{P}_{\alpha+1}$ .
- (4) Some elementary embedding  $j : V^1 \rightarrow M^1$  with  $\text{crit}(j) = \alpha$  can be extended to

$$j : V^1[G_\alpha] \rightarrow M^1[j(G_\alpha)]$$

and then to

$$j : V^1[G_\alpha * G(\alpha)] \rightarrow M^1[j(G_\alpha * G(\alpha))].$$

- (5)  $\mathbb{P}_{\alpha+1} \Vdash$  “  $\dot{\mathbb{P}}_{(\alpha+1, \kappa)}$  does not add any new subsets to  $\alpha_*^{+4}$  ”.
- (6)  $\forall \gamma < \alpha, \mathbb{P}_{(\gamma, \alpha)} \times \mathbb{M}(\gamma, \alpha) \Vdash$  “  $\alpha = \gamma^{++} + TP(\alpha)$  ”.

So we assume that each  $\kappa_n \in X$ .

The next lemma follows from Lemma 5.5(f).

**Lemma 5.8.** *In  $V^1[G_\kappa * g * h * K]$ ,*

$$\{\aleph_{2n} : n < \omega\} = \{\kappa_m : m < \omega\} \cup \{\kappa_m^{++} : m < \omega\} \cup \{\kappa_m^{+4} : m < \omega\}.$$

We now prove, in a sequence of lemmas that the tree property holds at all  $\aleph_{2n}$ 's,  $n < \omega$ . The case  $\kappa_0 = \aleph_0$  follows from König's theorem stated in the introduction. We start with the simple case of the tree property at  $\kappa_{m+1}$ .

**Lemma 5.9.** *For each  $m$ ,  $V^1[G_\kappa * g * h * K] \models \text{“TP}(\kappa_{m+1})\text{”}$ .*

*Proof.* We can write

$$V^1[G_\kappa * g * h * K] = V^1[G_\kappa * g * h][\langle F_i : i < m \rangle][F_m][\langle F_i : m < i < \omega \rangle].$$

Working in  $V^1[G_\kappa * g * h][\langle F_i : i < m \rangle]$ , the filter  $F_m$  is generic filter for the forcing notion  $\mathbb{M}(\kappa_m^{+4}, \kappa_{m+1})$ , so by Lemma 3.3(c).

$$V^1[G_\kappa * g * h][\langle F_i : i < m \rangle][F_m] \models \text{“ TP}(\kappa_{m+1}) \text{”}.$$

But  $V^1[G_\kappa * g * h * K]$  is a generic extension of  $V^1[G_\kappa * g * h][\langle F_i : i < m \rangle][F_m]$  by a forcing notion which does not add new subsets to  $\kappa_{m+1}$ , and so

$$V^1[G_\kappa * g * h * K] \models \text{“ TP}(\kappa_{m+1}) \text{”}.$$

□

Next we consider cardinals  $\kappa_m^{++}$ .

**Lemma 5.10.** *For each  $m$ ,  $V^1[G_\kappa * g * h * K] \models \text{“TP}(\kappa_m^{++})\text{”}$ .*

*Proof.* We have

$$V^1[G_\kappa * g * h * K] = V^1[G_{\kappa_m}][G(\kappa_m)][G(\kappa_{m+1}, \kappa)][g * h][\langle F_i : i < m \rangle][\langle F_i : m \leq i < \omega \rangle].$$

Since the forcing notion  $\prod_{i < m} \mathbb{M}(\kappa_i^{+4}, \kappa_{i+1})$  is defined in the same way in the models  $V^1[G_{\kappa_m}]$  and  $V^1[G_\kappa * g * h]$ , so

$$V^1[G_\kappa * g * h * K] = V^1[G_{\kappa_m}][\langle F_i : i < m \rangle][G(\kappa_m)][G(\kappa_{m+1}, \kappa)][g * h][\langle F_i : m \leq i < \omega \rangle].$$

By our convention  $\kappa_m \in X$ ,  $G(\kappa_m)$  is generic for  $\mathbb{M}(\kappa_m^{+4}, (\kappa_m)_*) * \dot{\mathbb{M}}((\kappa_m)_*^{+4}, (\kappa_m)_{**})$ , so by Lemma 3.5,

$$V^1[G_{\kappa_m}][\langle F_i : i < m \rangle][G(\kappa_m)] \Vdash \text{“}\kappa_m^{++} = (\kappa_m)_* + \text{TP}(\kappa_m^{++})\text{”}.$$

But  $V^1[G_\kappa * g * h * K]$  is a generic extension of  $V^1[G_{\kappa_m}][\langle F_i : i < m \rangle][G(\kappa_m)]$  by a forcing notion which does not add any new subsets to  $(\kappa_m)_*$ , and so  $V^1[G_\kappa * g * h * K] \models \text{“TP}(\kappa_m^{++})\text{”}$ .

□

Now we consider the cardinals  $\kappa_m^{+4}$ .

**Lemma 5.11.** *For each  $m$ ,  $V^1[G_\kappa * g * h * K] \models \text{“TP}(\kappa_m^{+4})\text{”}$ .*

*Proof.* As above,

$$V^1[G_\kappa * g * h * K] = V[G_{\kappa_m}][\langle F_i : i < m \rangle][G(\kappa_m)][G_{(\kappa_m+1, \kappa)}][g * h][F_m][\langle F_i : m < i < \omega \rangle].$$

But  $\mathbb{M}(\kappa_m^{+4}, \kappa_{m+1})$  is defined in the same way in the models  $V^1[G_{\kappa_m}][\langle F_i : i < m \rangle][G(\kappa_m)]$  and  $V^1[G_\kappa * g * h * K]$ , so  $V^1[G_\kappa * g * h * K]$  is equal to

$$V^1[G_{\kappa_m}][\langle F_i : i < m \rangle][G(\kappa_m)][G_{(\kappa_m+1, \kappa_{m+1})} \times F_m][G_{(\kappa_{m+1}+1, \kappa)}][g * h][\langle F_i : m < i < \omega \rangle].$$

Since  $V^1[G_\kappa * g * h * K]$  is obtained from  $V^1[G_{\kappa_m}][\langle F_i : i < m \rangle][G(\kappa_m)][G_{(\kappa_m+1, \kappa_{m+1})} \times F_m]$  by a forcing which does not add new subsets to  $(\kappa_m)_{**} (= ((\kappa_m)^{+4})^{V^1[G * g * h * K]})$ , it suffices to show  $\text{TP}((\kappa_m)_{**})$  holds in  $V^1[G_{\kappa_m}][\langle F_i : i < m \rangle][G(\kappa_m)][G_{(\kappa_m+1, \kappa_{m+1})} \times F_m]$ . Now, the model  $V^1[G_{\kappa_m}][\langle F_i : i < m \rangle][G(\kappa_m)][G_{(\kappa_m+1, \kappa_{m+1})} \times F_m]$  is a generic extension of  $V^1[G_{\kappa_m}][\langle F_i : i < m \rangle]$  by the forcing notion

$$\mathbb{P}(\kappa_m) * (\dot{\mathbb{P}}_{(\kappa_m+1, \kappa_{m+1})} \times \dot{\mathbb{M}}(\kappa_m^{+4}, \kappa_{m+1})),$$

and by Lemmas 3.7 and 3.8,

$$V^1[G_{\kappa_m}][\langle F_i : i < m \rangle][G(\kappa_m)][G_{(\kappa_m+1, \kappa_{m+1})} \times F_m] \models \text{“ TP}((\kappa_m)_{**})\text{”}.$$

The lemma follows. □

Theorem 1.3 follows.

## 6. TREE PROPERTY AT ALL REGULAR EVEN CARDINALS

In this section we prove Theorem 1.4. In Subsection 6.1, we present some of the basic properties of the new version of Radin forcing we defined in Section 5. Then in Subsection 6.2, we define the forcing notion needed which is used for the proof of our main theorem. Finally in Subsection 6.3, we complete the proof of Theorem 1.4.

**6.1. A new variant of Radin forcing.** Through this subsection, we assume that the following conditions are satisfied:

- $\kappa$  is an  $H(\kappa^{++})$ -hypermeasurable cardinal,  $2^\kappa = \kappa^{++}$  and  $2^{\kappa^{++}} = \kappa^{+4}$ .
- There is  $j : V \rightarrow M$  with critical point  $\kappa$  such that  $H(\kappa^{++}) \subseteq M$ .
- $j$  is generated by a  $(\kappa, \kappa^{+5})$ -extender.
- If  $U$  is the normal measure derived from  $j$  and if  $i : V \rightarrow N \simeq \text{Ult}(V, U)$  is the ultrapower embedding, then there exists  $F \in V$  which is  $\mathbb{M}(\kappa^{+4}, i(\kappa))_N$ -generic over  $N$ .

Let  $\mathbb{R}_w$  be the modified version of Radin forcing that we defined in Section 5, and let us review its basic properties in the general context. The next lemma can be proved as in Lemma 4.13

**Lemma 6.1.**  $\mathbb{R}_w$  satisfies the  $\kappa_w^+$ -chain condition.

The following is an analogue of the factorization lemma 4.14.

**Lemma 6.2.** (The factorization lemma) Suppose that  $p = \langle p_0, \dots, p_n \rangle \in \mathbb{R}_w$  where  $p_i = \langle \kappa_i, \lambda_i, A_i, f_i, F_i \rangle$  and  $m < n$ . Set  $p^{\leq m} = \langle p_0, \dots, p_m \rangle$  and  $p^{> m} = \langle p_{m+1}, \dots, p_n \rangle$ .

- (a)  $p^{\leq m} \in \mathbb{R}_{w \upharpoonright \kappa_{m+1}}$ ,  $p^{> m} \in \mathbb{R}_w$  and there exists

$$i : \mathbb{R}_w/p \rightarrow \mathbb{R}_{w \upharpoonright \kappa_{m+1}}/p^{\leq m} \times \mathbb{R}_w/p^{> m}$$

which is an isomorphism with respect to both  $\leq^*$  and  $\leq$ .

- (b) If  $m + 1 < n$ , then there exists

$$i : \mathbb{R}_w/p \rightarrow \mathbb{R}_{w \upharpoonright \kappa_{m+1}}/p^{\leq m} \times \mathbb{M}(\kappa_m^{+4}, \kappa_{m+1}) \times \mathbb{R}_w/p^{> m+1}$$

which is an isomorphism with respect to both  $\leq^*$  and  $\leq$ . □

The following can be proved as before:

**Lemma 6.3.** (a)  $(\mathbb{R}_w, \leq, \leq^*)$  satisfies the Prikry property.

- (b) Assume  $\text{lh}(w) = \kappa_w^+$ . Then forcing with  $\mathbb{R}_w$  preserves the inaccessibility of  $\kappa_w$ .

From now on assume that  $\text{lh}(w) = \kappa^+$ . Suppose  $K \subseteq \mathbb{R}_w$  is generic over  $V$  and define the club  $C$  and the sequence  $\vec{u} = \langle u_\xi : \xi < \kappa \rangle$  and  $\vec{\kappa} = \langle \kappa_\xi : \xi < \kappa \rangle$  as before. Let the sequence  $\vec{F} = \langle F_\xi : \xi < \kappa \rangle$  be such that each  $F_\xi$  is  $\mathbb{M}(\kappa_\xi^{+4}, \kappa_{\xi+1})$ -generic over  $V$ , which is produced by  $K$ . The next lemma can be proved as in Lemma 4.16.

- Lemma 6.4.** (a)  $V[K] = V[\vec{u}, \vec{F}]$ .
- (b) For every limit ordinal  $\xi < \kappa$ ,  $\langle \vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi \rangle$  is  $\mathbb{R}_{u_\xi}$ -generic over  $V$ , and  $\langle \vec{u} \upharpoonright [\xi, \kappa), \vec{F} \upharpoonright [\xi, \kappa) \rangle$  is  $\mathbb{R}_w$ -generic over  $V[\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi]$ .
- (c) For every  $\gamma < \kappa$  and every  $A \subseteq \gamma$  with  $A \in V[\vec{u}, \vec{F}]$ , we have  $A \in V[\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi]$ , where  $\xi$  is the least ordinal such that  $\gamma < \kappa_\xi$ .

**6.2. The final model.** In this subsection we define the final model we are going to work with. Thus assume that  $GCH$  holds,  $\eta > \lambda$  are measurable cardinals above  $\kappa$ . We assume that they are the least such cardinals. Suppose  $\kappa$  is an  $H(\eta^+)$ -hypermeasurable cardinal. Let  $j : V \rightarrow M \supseteq H(\eta^+)$  witness this. We may assume that it is generated by a  $(\kappa, \eta^+)$ -extender. Let  $i : V \rightarrow N$  be the ultrapower embedding derived from  $j$  and let  $k : N \rightarrow M$  be such that  $j = k \circ i$ .

The next lemma can be proved as in Lemma 4.20

**Lemma 6.5.** *Then there exists a cofinality preserving generic extension  $V^1$  of  $V$  satisfying the following conditions:*

- (a)  $V^1 \models \text{“}GCH\text{”}$ .
- (b) There is  $j^1 : V^1 \rightarrow M^1$  with critical point  $\kappa$  such that  $H(\eta^+) \subseteq M^1$  and  $j^1 \upharpoonright V = j$ .
- (c)  $j^1$  is generated by a  $(\kappa, \eta^+)$ -extender.
- (d) If  $U^1$  is the normal measure derived from  $j^1$  and if  $i^1 : V^1 \rightarrow N^1 \simeq \text{Ult}(V^1, U^1)$  is the ultrapower embedding, then there exists  $\bar{g} \in V^1$  which is  $i^1(\text{Add}(\kappa, \lambda)_{V^1})$ -generic over  $N^1$ . Further  $i^1 \upharpoonright V = i$ .

**Lemma 6.6.** *Work in  $V^1$ . There exists a forcing iteration  $\mathbb{P}_\kappa$  of length  $\kappa$  such that if  $G_\kappa * g * h$  is  $\mathbb{P}_\kappa * \dot{\mathbb{M}}(\kappa, \lambda) * \dot{\mathbb{M}}(\lambda, \eta)$ -generic over  $V^1$ , then in  $V^2 = V^1[G_\kappa * g * h]$ , the following holds:*

- (a)  $V^2 \models \text{“}\lambda = \kappa^{++} + \eta = \kappa^{+4} + TP(\lambda) + TP(\eta)\text{”}$ .



- (b) There is  $j^2 : V^2 \rightarrow M^2$  with critical point  $\kappa$  and  $H(\kappa^{++}) \subseteq M^2$  such that  $j^2 \upharpoonright V^1 = j^1$ .
- (c)  $j^2$  is generated by a  $(\kappa, \eta^+)$ -extender.
- (d) If  $U^2$  is the normal measure derived from  $j^2$  and if  $i^2 : V^2 \rightarrow N^2 \simeq \text{Ult}(V^2, U^2)$  is the ultrapower embedding, then there exists  $F \in V^2$  which is  $\dot{\mathbb{M}}(\kappa^{+4}, i^2(\kappa))_{N^2}$ -generic over  $N^2$ .

*Proof.* The model  $V^2 = V^1[G_\kappa * g * h]$  constructed in the proof of Lemma 5.2 does the job. The additional assumption of  $\kappa$  being  $H(\eta^+)$ -hypermeasurable guarantees that  $H(\kappa^{++}) \subseteq M^2$ .  $\square$

In particular, note that in the model  $V^2$ , the hypotheses at the beginning of Subsection 6.1 are satisfied; so we can consider the forcing notion  $\mathbb{R}_w$ , where  $w = u \upharpoonright \kappa^+$  and  $u$  is constructed using the pair  $(j^2, F)$ . Let  $K$  be  $\mathbb{R}_w$ -generic over  $V^2$ . Build the sequences  $\vec{\kappa} = \langle \kappa_\xi : \xi < \kappa \rangle$ ,  $\vec{u} = \langle u_\xi : \xi < \kappa \rangle$  and  $\vec{F} = \langle F_\xi : \xi < \kappa \rangle$  from  $K$ , as before.

**6.3. In  $V^1[G_\kappa * g * h * K]$ , the tree property holds at all regular even cardinals below  $\kappa$ .** Here we complete the proof of Theorem 1.4. As before, given a cardinal  $\alpha \leq \kappa$ , let  $\alpha_*$  denote the least measurable cardinal above  $\alpha$  and let  $\alpha_{**}$  denote the second measurable cardinal above  $\alpha$ . Now note that

$$\text{Card}^{V^1[G_\kappa * g * h * K]} \cap \kappa = \{\kappa_\xi, \kappa_\xi^+ : \xi < \kappa\} \cup \{(\kappa_\xi)_*, (\kappa_\xi)_*^+ : \xi < \kappa\} \cup \{(\kappa_\xi)_{**}, (\kappa_\xi)_{**}^+ : \xi < \kappa\},$$

Also note that if  $\alpha < \kappa$  is a singular cardinal in  $V^1[G_\kappa * g * h * K]$ , then  $\alpha \in \text{lim}(C)$ , i.e.,  $\alpha = \kappa_\xi$  for some limit ordinal  $\xi < \kappa$ . The following lemma is immediate:

**Lemma 6.7.** *In  $V^1[G_\kappa * g * h * K]$ , the set of uncountable regular even cardinals below  $\kappa$  is equal to*

$$\{\kappa_\xi^{++} : \xi < \kappa\} \cup \{\kappa_\xi^{+4} : \xi < \kappa\} \cup \{\kappa_{\xi+1} : \xi < \kappa\}.$$

Before we continue, let us show that we can choose the cardinals  $\kappa_\xi$  in a suitable way, which is guaranteed by the following lemma, which is an analogue of Lemma 4.32.

**Lemma 6.8.** *The set  $X \in \mathcal{F}_w$ , where  $X$  consists of all those  $u \in \mathcal{U}_\infty$  such that  $\alpha = \kappa_u$  satisfies the following conditions:*

- (1)  $\mathbb{P}_\alpha$  is  $\alpha$ -c.c. and of size  $\alpha$ .
- (2)  $\mathbb{P}_\alpha \Vdash$  “  $\mathbb{P}(\alpha) = \mathbb{M}(\alpha, \alpha_*) * \dot{\mathbb{M}}(\alpha_*, \alpha_{**})$  ”.
- (3)  $\alpha$  remains measurable after forcing with  $\mathbb{P}_\alpha$  and  $\mathbb{P}_{\alpha+1}$ .
- (4) Some elementary embedding  $j : V^1 \rightarrow M^1$  with  $\text{crit}(j) = \alpha$  can be extended to

$$j : V^1[G_\alpha] \rightarrow M^1[j(G_\alpha)]$$

and then to

$$j : V^1[G_\alpha * G(\alpha)] \rightarrow M^1[j(G_\alpha * G(\alpha))].$$

- (5)  $\mathbb{P}_{\alpha+1} \Vdash$  “  $\dot{\mathbb{P}}_{(\alpha+1, \kappa)}$  does not add any new subsets to  $\alpha_{**}^{+4}$  ”.
- (6)  $\forall \gamma < \alpha, \mathbb{P}_{(\gamma, \alpha)} \times \mathbb{M}(\gamma, \alpha) \Vdash$  “  $\alpha = \gamma^{++} + TP(\alpha)$  ”.

The next lemma can be proved as in Theorems 4.22 and 4.33, combined with ideas of the proof of Lemma 5.10.

**Lemma 6.9.**  $V^1[G_\kappa * g * h * K] \Vdash$  “For all limit ordinals  $\xi < \kappa, \kappa_\xi^{++} = (\kappa_\xi)_*$  and  $TP((\kappa_\xi)_*)$  holds”.

Before we continue, let us make a simple remark. Assume  $\bar{\xi}$  is a limit ordinal. Then we can write  $V^1[G_\kappa * g * h * K]$  as

$$V[G_\kappa * g * h][\bar{u} \upharpoonright \bar{\xi}, \bar{F} \upharpoonright \bar{\xi}][\bar{u} \upharpoonright [\bar{\xi}, \bar{\xi} + \omega), \bar{F} \upharpoonright [\bar{\xi}, \bar{\xi} + \omega)][\bar{u} \upharpoonright [\bar{\xi} + \omega, \kappa), \bar{F} \upharpoonright [\bar{\xi} + \omega, \kappa)].$$

On the other hand:

- (1)  $V^1[G_\kappa * g * h * K]$  is a generic extension of  $V^1[G_\kappa * g * h][\bar{u} \upharpoonright \bar{\xi}, \bar{F} \upharpoonright \bar{\xi}][\bar{u} \upharpoonright [\bar{\xi}, \bar{\xi} + \omega), \bar{F} \upharpoonright [\bar{\xi}, \bar{\xi} + \omega)]$  by a forcing notion which does not add any new subsets to  $\kappa_{\bar{\xi} + \omega}$ .
- (2) By standard arguments,  $V^1[G_\kappa * g * h][\bar{u} \upharpoonright \bar{\xi}, \bar{F} \upharpoonright \bar{\xi}][\bar{u} \upharpoonright [\bar{\xi}, \bar{\xi} + \omega), \bar{F} \upharpoonright [\bar{\xi}, \bar{\xi} + \omega)]$  is generic extension of  $V^1[G_\kappa * g * h][\bar{u} \upharpoonright \bar{\xi}, \bar{F} \upharpoonright \bar{\xi}]$  by a forcing notion which is forcing equivalent to the forcing notion  $\mathbb{R}$  of Section 5 (for suitable choices of the normal measure and guiding generic filters).

So, given any limit ordinal  $\bar{\xi}$ , we can use the arguments of Section 5 to conclude that the model  $V^1[G_\kappa * g * h][\bar{u} \upharpoonright \bar{\xi}, \bar{F} \upharpoonright \bar{\xi}][\bar{u} \upharpoonright [\bar{\xi}, \bar{\xi} + \omega), \bar{F} \upharpoonright [\bar{\xi}, \bar{\xi} + \omega)]$  satisfies:

$$“ \forall n < \omega, TP(\kappa_{\bar{\xi} + n}^{++}) + TP(\kappa_{\bar{\xi} + n}^{+4}) + TP(\kappa_{\bar{\xi} + \omega + 2}) ”.$$

Using the above remark, and by the same arguments as in Sections 4 and 5 one can prove Theorem 1.4. Below we present more details for completeness. First we prove an analogue of Lemma 6.9 for successor ordinals.

**Lemma 6.10.**  $V^1[G_\kappa * g * h * K] \models$  “For all successor ordinals  $\xi < \kappa, \kappa_\xi^{++} = (\kappa_\xi)_*$  and the tree property at  $(\kappa_\xi)_*$  holds”.

*Proof.* Suppose  $\xi = \zeta + 1$ . The model  $V^1[G_\kappa * g * h * K]$  is an extension of  $V^1[G_\kappa * g * h][\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi]$  by a forcing notion which does not add new subsets to  $\kappa_\xi^{++} = (\kappa_\xi)_*$ ; so it suffices to show that  $\text{TP}((\kappa_\xi)_*)$  holds in  $V^1[G_\kappa * g * h][\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi]$ . But this last model is equal to

$$V^1[G_{\kappa_\xi}][G(\kappa_\xi)][G_{(\kappa_\xi+1, \kappa)}][g * h][\vec{u} \upharpoonright \zeta, \vec{F} \upharpoonright \zeta][F_\zeta].$$

Since the forcing notion  $\mathbb{R}_{u_\zeta}$  is defined in the same way in the models  $V^1[G_\kappa * g * h]$  and  $V^1[G_{\kappa_\xi}]$ , so  $V^1[G_\kappa * g * h][\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi]$  equals

$$V^1[G_{\kappa_\xi}][\vec{u} \upharpoonright \zeta, \vec{F} \upharpoonright \zeta][G(\kappa_\xi)][G_{(\kappa_\xi+1, \kappa)}][g * h][F_\zeta] = V[G_{\kappa_\xi}][\vec{u} \upharpoonright \zeta, \vec{F} \upharpoonright \zeta][F_\zeta][G(\kappa_\xi)][G_{(\kappa_\xi+1, \kappa)}][g * h].$$

It follows that  $V^1[G_\kappa * g * h][\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi]$  is an extension of  $V^1[G_{\kappa_\xi}][\vec{u} \upharpoonright \zeta, \vec{F} \upharpoonright \zeta][F_\zeta][G(\kappa_\xi)]$  by a forcing which does not add new subsets to  $(\kappa_\xi)_*$ , so we just need to show that  $\text{TP}((\kappa_\xi)_*)$  holds in  $V^1[G_{\kappa_\xi}][\vec{u} \upharpoonright \zeta, \vec{F} \upharpoonright \zeta][F_\zeta][G(\kappa_\xi)]$ .

But  $G(\kappa_\xi)$  is generic for  $\mathbb{M}(\kappa_\xi, (\kappa_\xi)_*) * \dot{\mathbb{M}}((\kappa_\xi)_*, (\kappa_\xi)_{**})$  and so by Lemma 3.5,

$$V^1[G_{\kappa_\xi}][\vec{u} \upharpoonright \zeta, \vec{F} \upharpoonright \zeta][F_\zeta][G(\kappa_\xi)] \models \text{“TP}((\kappa_\xi)_*)\text{”}.$$

The lemma follows. □

Next we prove the following:

**Lemma 6.11.**  $V^1[G_\kappa * g * h * K] \models$  “For all ordinals  $\xi < \kappa$ ,  $\text{TP}(\kappa_{\xi+1})$  holds”.

*Proof.* We have

$$V^1[G_\kappa * g * h * K] = V^1[G_\kappa * g * h][\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi][F_\xi][\vec{u} \upharpoonright (\xi + 1, \kappa), \vec{F} \upharpoonright (\xi + 1, \kappa)]$$

Now  $F_\xi$  is a generic filter for  $\mathbb{M}(\kappa_\xi^{+4}, \kappa_{\xi+1})$  and by Lemma 3.3(c), we have

$$V^1[G_\kappa * g * h][\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi][F_\xi] \models \text{“TP}(\kappa_{\xi+1})\text{”}.$$

On the other hand, the models  $V^1[G_\kappa * g * h * K]$  and  $V^1[G_\kappa * g * h][\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi][F_\xi]$  have the same subsets of  $\kappa_{\xi+1}$ , and hence  $V^1[G_\kappa * g * h * K] \models \text{“TP}(\kappa_{\xi+1}\text{)”}$ .  $\square$

**Theorem 6.12.**  $V^1[G_\kappa * g * h * K] \models \text{“For all ordinals } \xi < \kappa, \kappa_\xi^{+4} = (\kappa_\xi)_{**} \text{ and the tree property at } (\kappa_\xi)_{**} \text{ holds”}$ .

*Proof.* By similar analysis as above, it suffices to show that  $\text{TP}((\kappa_\xi)_{**})$  holds in the model

$$V^1[G_{\kappa_\xi}][\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi][G(\kappa_\xi)][G_{(\kappa_\xi+1, \kappa_{\xi+1})}][F_\xi],$$

which is an extension of  $V^1[G_{\kappa_\xi}][\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi]$  by the forcing notion

$$\dot{\mathbb{M}}(\kappa_\xi, (\kappa_\xi)_*) * \dot{\mathbb{M}}((\kappa_\xi)_*, (\kappa_\xi)_{**}) * (\dot{\mathbb{L}}_{(\kappa_\xi+1, \kappa_{\xi+1})} \times \dot{\mathbb{M}}((\kappa_\xi)_{**}, \kappa_{\xi+1})).$$

So, by lemmas by Lemmas 3.7 and 3.8,

$$V^1[G_{\kappa_\xi}][\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi][G(\kappa_\xi)][G_{(\kappa_\xi+1, \kappa_{\xi+1})}][F_\xi] \models \text{“ TP}((\kappa_\xi)_{**}) \text{”}.$$

The lemma follows.  $\square$

Theorem 1.4 follows.

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