TREE PROPERTY AT ALL REGULAR EVEN CARDINALS

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Abstract. Starting from a strong cardinal and a measurable cardinal above it, we construct a model of ZFC, in which, for every singular cardinal δ, δ is strong limit, $2^{\delta} = \delta^{+3}$ and the tree property holds at δ^{++}. It answers a question of Friedman, Honzik and Stejskalova. We also produce, relative to the existence of a strong cardinal and two measurable cardinals above it, a model of ZFC in which the tree property holds at all regular even cardinals. The result answers questions of Friedman, Halilovic and Honzik .

1. introduction

Assume κ is a regular cardinal. Recall that the tree property at κ is the assertion "there are no κ-Aronszajn trees". By a result of König, the tree property at $\aleph_0$ holds, and by a theorem of Aronszajn, the tree property at $\aleph_1$ fails (see [15]). However the problem of getting tree property at higher regular cardinals is different; as it has turned out, the tree property at κ for regular $\kappa > \aleph_1$, is independent of ZFC (modulo some large cardinal assumptions). The major problem, due to Magidor, is to prove the consistency of the tree property at all regular cardinals $\kappa > \aleph_1$.

In this paper, we are interested in the tree property at regular even cardinals. First we consider the problem of getting tree property at double successor of singular strong limit cardinals. The first result in this direction is due to Cummings and Foreman [3], who produced, starting from a supercompact cardinals κ and a weakly compact cardinal above it, a model of ZFC in which κ is a singular strong limit cardinal of countable cofinality and such that the tree property holds at κ^{++}. They also stated that their result can be improved to the case $\kappa = \aleph_\omega$. Later, Friedman and Halilovic [5] proved the same results from large cardinals close to being optimal. In [10], Gitik obtained a model of "$\aleph_\omega$ is strong limit + the tree property holds at $\aleph_{\omega+2}$" from optimal hypotheses. The papers [4], [7] and

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[8] have continued the work, where more results about the tree property at double successor of singular strong limit cardinals of countable cofinality are obtained.

In [13], singular cardinals of uncountable cofinality are considered, and in it, a model is constructed in which the tree property holds at double successor of a singular strong limit cardinal of any prescribed cofinality. Before we state the results of the paper, let us fix a notation.

Notation 1.1. *For a regular cardinal κ, we use $TP(\kappa)$ for the assertion "the tree property holds at κ".*

In [8], Friedman, Honzik and Stejskalova have produced a model of ZFC in which $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+3}$ and the tree property holds at $\aleph_{\omega+2}$. They asked if we can replace $\aleph_\omega$ with $\aleph_{\omega_1}$. We answer their question; in fact we prove the following global consistency result:

Theorem 1.2. *Assume κ is an $H(\lambda^{++})$-hypermeasurable cardinal where $\lambda > \kappa$ is measurable. Then there is a generic extension W of V in which the following hold:*

(a) *κ is inaccessible.*

(b) *For every singular cardinal $\delta < \kappa$, δ is strong limit and $2^\delta = \delta^{+3}$.*

(c) *For every singular cardinal $\delta < \kappa$, $TP(\delta^{++})$ holds.*

In particular the rank initial segment W_κ of W is a model of ZFC in which the tree property holds at double successor of every singular cardinal δ and $2^\delta = \delta^{+3}$.

Remark 1.3. *Given any finite $n \geq 2$, we can replace $2^\delta = \delta^{+3}$ with $2^\delta = \delta^{+n}$.*

Then we consider the problem of getting the tree property at even cardinals. In his paper [20], Mitchell observed that starting from two weakly compact cardinals one can get the tree property at both $\aleph_2$ and $\aleph_4$. His result can be easily extended to get the tree property at all $\aleph_{2n}$'s, $0 < n < \omega$, starting from infinitely many weakly compact cardinals. However if we want to get the tree property at $\aleph_{2n}$'s, $0 < n < \omega$ and at $\aleph_{\omega+2}$ and make $\aleph_\omega$ strong limit, then the above idea does not work. In fact, it turned out that this problem is more difficult. In [6], Friedman and Honzik produced a model in which $\aleph_\omega$ is strong limit with $2^{\aleph_\omega} = \aleph_{\omega+2}$ and such that in it, the tree property holds at all even cardinals below $\aleph_\omega$. Unger [24] has extended their result to get the tree property at all $\aleph_n$'s, $1 < n < \omega$. None of these papers obtain the tree property at $\aleph_{\omega+2}$. We address this question and prove the following, which in particular answers a question of [6]:

Theorem 1.4. *Assume $\eta > \lambda$ are measurable cardinals above κ and κ is $H(\eta)$-hypermeasurable. Then there is a generic extension W of V in which:*

(a) *$\kappa = \aleph_\omega$ and $\lambda = \aleph_{\omega+2}$.*

(b) *The tree property holds at all $\aleph_{2n}$'s, $0 < n < \omega$ and at $\aleph_{\omega+2}$.*

Then, we prove the following much stronger result, which is related to a question asked in [5].

Theorem 1.5. *Assume $\eta > \lambda$ are measurable cardinals above κ and κ is $H(\eta^+)$-hypermeasurable. Then there is a generic extension W of V in which:*

(a) *κ is inaccessible.*

(b) *The tree property holds at all regular even cardinals below κ.*

In particular the rank initial segment W_κ of W is a model of ZFC in which the tree property holds at all regular even cardinals.

The structure of the paper is as follows. In Section 3, we prove Theorem 1.2, in Section 4 we prove Theorem 1.4 and in Section 5 we prove Theorem 1.5. Section 2 is devoted to some preliminary results.

We assume familiarity with forcing and large cardinals. For a forcing notion $\mathbb{P}$ we use $p \leq q$ to mean p gives more information than q, i.e., $p \Vdash$"$q \in \dot{G}$", where $\dot{G}$ is the canonical $\mathbb{P}$-name for the generic filter.

2. Variants of Mitchell forcing and their properties

In this section we present two variants of Mitchell's forcing and discuss some of their properties.

2.1. **First variant of Mitchell's forcing.** The first version of the forcing, presented below, is essentially the same as in Mitchell's forcing, but it also allows us to blow up the power function. We will use this forcing in the proof of Theorem 1.2.

Definition 2.1. *Assume $\alpha < \beta$ are regular cardinals and $\gamma \geq \beta$ is an ordinal. Let $\mathbb{M}(\alpha, \beta, \gamma)$ be the following variant of Mitchell forcing for making $2^\alpha = \gamma$ and forcing the tree property at $\beta = \alpha^{++}$:*

(a) *A condition in $\mathbb{M}(\alpha, \beta, \gamma)$ is a pair (p, q), where*

  (1) $p \in \mathrm{Add}(\alpha, \gamma)$,

  (2) $\mathrm{dom}(q)$ *is a subset of β of size $\leq \alpha$,*

  (3) *For each $\xi \in \mathrm{dom}(q), 1_{\mathrm{Add}(\alpha,\xi)} \Vdash$ "$q(\xi) \in \dot{\mathrm{Add}}(\alpha^+, 1)$".*

(b) *For $(p, q), (p', q') \in \mathbb{M}(\alpha, \beta, \gamma)$, say $(p', q') \leq (p, q)$ iff*

  (1) $p' \leq_{\mathrm{Add}(\alpha,\gamma)} p$,

  (2) $\mathrm{dom}(q') \supseteq \mathrm{dom}(q)$,

  (3) *For all $\xi \in \mathrm{dom}(q), 1_{\mathrm{Add}(\alpha,\xi)} \Vdash$ "$q'(\xi) \leq_{\dot{\mathrm{Add}}(\alpha^+,1)} q(\xi)$".*

We refer to [8] for more discussion about the above forcing notion and just present its basic properties. Assume GCH holds and let $\alpha < \beta \leq \gamma$ be such that α is regular and β is a measurable cardinal. Also let $\mathbb{M}(\alpha, \beta, \gamma)$ be the above defined forcing notion. The next lemma is standard.

Lemma 2.2. (a) $\mathbb{M}(\alpha, \beta, \gamma)$ *is α-directed closed.*

(b) $\mathbb{M}(\alpha, \beta, \gamma)$ *is β-c.c.; in fact it is β-Knaster.*

(c) *In the generic extension by $\mathbb{M}(\alpha, \beta, \gamma)$, α^+ is preserved, $2^\alpha \geq \gamma$, $\beta = \alpha^{++}$, and $TP(\beta)$ holds.*

Let $\mathbb{T}(\alpha, \beta, \gamma)$ be the term forcing notion defined by

$$\mathbb{T}(\alpha, \beta, \gamma) = \{(\emptyset, q) : (\emptyset, q) \in \mathbb{M}(\alpha, \beta, \gamma)\}.$$

It is easily seen that $\mathbb{T}(\alpha, \beta, \gamma)$ is α^+-closed and that, there exists a projection from $\mathrm{Add}(\alpha, \gamma) \times \mathbb{T}(\alpha, \beta, \gamma)$ onto $\mathbb{M}(\alpha, \beta, \gamma)$. Also $\mathbb{M}(\alpha, \beta, \gamma) \simeq \mathrm{Add}(\alpha, \gamma) * \dot{\mathbb{Q}}$ for an $\mathrm{Add}(\alpha, \gamma)$-name $\dot{\mathbb{Q}}$ which is forced to be α^+-distributive.

2.2. **Second variant of Mitchell's forcing.** We now present another version of Mitchell's forcing, that we will use it in the proof of Theorems 1.4 and 1.5. We would like to thank Yair Hayut for introducing this version of forcing to us and for the proof of Lemma 2.5. In personal communication, he told us that this version of forcing is well-known, though we could not find a reference for it.

Definition 2.3. *Assume $\alpha < \beta$ are regular cardinals. Let $\widetilde{\mathbb{M}}(\alpha, \beta)$ be the two step iteration $\widetilde{\mathbb{M}}(\alpha, \beta) = \mathrm{Add}(\alpha, \beta) * \dot{\mathbb{C}}(\alpha, \beta)$, where $\mathbb{C}(\alpha, \beta)$ is defined as follows. Let $G \subseteq \mathrm{Add}(\alpha, \beta)$ be a generic filter. Then in $V[G]$:*

(a) *$p \in \mathbb{C}(\alpha, \beta)$ iff*

  (1) *p is a sequence of length β with at most α non-trivial coordinates,*

  (2) *$p(\zeta)$ belongs to the collection of names for elements in $\mathrm{Add}(\alpha^+, 1)^{V^{\mathrm{Add}(\alpha,\zeta)}}$.*

(b) *$p \leq q$ iff for every $\zeta < \beta$, $p(\zeta)^{V[G\restriction\zeta]} \leq q^{V[G\restriction\zeta]}$.*

The only difference between $M(\alpha, \beta)$ and Mitchell's forcing is that in this model we force at each step first with $\mathrm{Add}(\alpha^+, 1)$ and then with $\mathrm{Add}(\alpha, 1)$ and in Mitchell's forcing we use the opposite order. The next lemma can be proved as in Lemma 2.2.

Lemma 2.4. (a) *$\widetilde{\mathbb{M}}(\alpha, \beta)$ is α-directed closed.*

(b) *$\widetilde{\mathbb{M}}(\alpha, \beta)$ is β-c.c.; in fact it is β-Knaster.*

(c) *Assume β is a measurable cardinal. Then in the generic extension by $\widetilde{\mathbb{M}}(\alpha, \beta)$, α^+ is preserved, $2^\alpha = \beta = \alpha^{++}$, and $TP(\beta)$ holds.*

Here, we would like to prove something stronger than clause (c) above, which is the main reason of working with $\widetilde{\mathbb{M}}(\alpha, \beta)$ rather than $\mathbb{M}(\alpha, \beta, \beta)$ in the proof of theorems 1.4 and 1.5.

Lemma 2.5. *Assume $\alpha < \beta < \gamma$, where α is regular and β, γ are measurable cardinals and let $G * H$ be generic over V for the forcing notion $\widetilde{\mathbb{M}}(\alpha, \beta) * \dot{\widetilde{\mathbb{M}}}(\beta, \gamma)$. Then:*

(a) *$Card^{V[G*H]} \cap [\alpha, \gamma] = \{\alpha, \alpha^+, \beta, \beta^+, \gamma\}$.*

(b) *$V[G * H] \models$ "For each $2^\alpha = \beta = \alpha^{++}$ and $2^\beta = \gamma = \beta^{++}$".*

(c) *$V[G * H] \models$ "The tree property holds at both β and γ".*

Proof. [1] (a) and (b) can be proved easily using Lemma 2.4. Let us prove (c). Thus let $\alpha < \beta < \gamma$ be regular cardinals, with β and γ measurable. Let us show that the tree property holds at β in the generic extension (the proof for γ is easier).

[1]The proof presented here is suggested by Yair Hayut, which is based on ideas of Unger [23]

Let $\dot{T}$ be a $\widetilde{\mathbb{M}}(\alpha,\beta) * \dot{\widetilde{\mathbb{M}}}(\beta,\gamma)$-name for a β-tree. By the β^+-distributivity of the $\mathbb{C}$-part of $\widetilde{\mathbb{M}}(\beta,\gamma)$, $\dot{T}$ is added by $\widetilde{\mathbb{M}}(\alpha,\beta) * \dot{\mathrm{Add}}(\beta,\gamma)$. By the chain condition and the homogeneity of $\mathrm{Add}(\beta,\gamma)$, $\dot{T}$ is equivalent to an $\widetilde{\mathbb{M}}(\alpha,\beta) * \dot{\mathrm{Add}}(\beta,1)$-name. Let $\mathbb{Q} = \widetilde{\mathbb{M}}(\alpha,\beta) * \dot{\mathrm{Add}}(\beta,1)$.

Let $j\colon V \to M$ be an elementary embedding with critical point β. Let $H \subseteq \mathbb{Q}$ be a generic filter. Then

$$j(\mathbb{Q}) = \widetilde{\mathbb{M}}(\alpha,\beta) * \dot{\mathrm{Add}}(\alpha^+,1) * \dot{\mathbb{R}} * \dot{\mathrm{Add}}(j(\beta),1)$$

for some $\dot{\mathbb{R}}$.

Since $\mathrm{Add}(\alpha^+,1)$ is equivalent to $\mathrm{Col}(\alpha^+,2^\alpha)$ and after forcing with $\widetilde{\mathbb{M}}(\alpha,\beta)$, $2^\alpha=\beta$, it adds a generic filter for $\mathrm{Add}(\beta,1)$. Thus, we can represent $j(\mathbb{Q})$ in the following way:

$$j(\mathbb{Q}) \cong \widetilde{\mathbb{M}}(\alpha,\beta) * \dot{\mathrm{Add}}(\beta,1) * \dot{\mathrm{Col}}(\alpha^+,\beta) * \dot{\mathbb{R}} * \dot{\mathrm{Add}}(j(\beta),1).$$

Using the closure of $\mathrm{Add}(j(\beta),1)$, one can obtain a master condition and force a generic filter K over $M[H]$ such that $H*K$ is a generic filter for $j(\mathbb{Q})$ and $j''[H] \subseteq H*K$. Therefore, in $V[H*K]$ there is an elementary embedding $\tilde{j}\colon V[H] \to M[H*K]$ extending j. In particular, $\tilde{j}(\dot{T}^H)$ is a $j(\beta)$-tree and thus by taking any element from the β-th level of $\tilde{j}(\dot{T}^H)$ one can obtain a branch in $\dot{T}$.

Let us show that the forcing

$$\mathrm{Col}(\alpha^+,\beta) * \dot{\mathbb{R}} * \dot{\mathrm{Add}}(j(\beta),1)$$

(that introduced K) cannot add a branch to a β-tree in $V[H]$. Indeed, this forcing is a projection of the product $\mathrm{Add}(\alpha,j(\beta)) \times \mathrm{Col}^{V^{\mathbb{Q}}}(\alpha^+,j(\beta))$. By standard arguments, this forcing cannot add a branch to a β-tree in the model $V[H]$. $\square$

Similarly we can prove the following result:

Lemma 2.6. *Assume $\alpha_0 < \alpha_1 < \cdots < \alpha_n$, where α_0 is regular and $\alpha_1,\ldots,\alpha_n$ are measurable cardinals and let $G = G_0 * G_1 * \cdots * G_{n-1}$ be generic over V for the forcing notion*

$$\widetilde{\mathbb{M}}(\alpha_0,\alpha_1) * \dot{\widetilde{\mathbb{M}}}(\alpha_1,\alpha_2) * \cdots * \dot{\widetilde{\mathbb{M}}}(\alpha_{n-1},\alpha_n).$$

Then

(a) $Card^{V[G]} \cap [\alpha_0,\alpha_n] = \{\alpha_0,\alpha_0^+,\alpha_1,\alpha_1^+,\ldots,\alpha_{n-1},\alpha_{n-1}^+,\alpha_n\}$.

(b) $V[G] \models$ *"For each* $i < n, 2^{\alpha_i} = \alpha_{i+1} = \alpha_i^{++}$*"*.

(c) $V[G] \models$ *"The tree property holds at each* $\alpha_i, 1 \leq i \leq n$*"*.

Assume that $\eta > \lambda$ are measurable cardinals above κ and there is $j : V \to M$ with critical point κ such that $H(\eta) \subseteq M$ and j is generated by a (κ, η)-extender. Suppose there exists $\bar{g} \in V$ which is $i(\mathrm{Add}(\kappa, \lambda)_V)$-generic over N, where U is the normal measure derived from j and $i : V \to N \simeq \mathrm{Ult}(V, U)$ is the ultrapower embedding.

Let

$$\mathbb{P}_\kappa = \langle \langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle \rangle$$

be the reverse Easton iteration, where

(1) If $\alpha < \kappa$ is a measurable limit of measurable cardinals, then

$$\Vdash_\alpha \text{"}\dot{\mathbb{Q}}_\alpha = \dot{\widetilde{\mathbb{M}}}(\alpha, \alpha_*) * \dot{\widetilde{\mathbb{M}}}(\alpha_*, \alpha_{**})\text{"},$$

where for each $\alpha \leq \kappa$, α_* and α_{**} denote the first and the second measurable cardinals above α.

(2) Otherwise, $\Vdash_\alpha$"$\dot{\mathbb{Q}}_\alpha$ is the trivial forcing".

Let

$$G_k = \langle \langle G_\alpha : \alpha \leq \kappa \rangle, \langle G(\alpha) : \alpha < \kappa \rangle \rangle$$

be $\mathbb{P}_\kappa$-generic over V. Also let $\mathbb{M} = \widetilde{\mathbb{M}}(\aleph_0, \kappa)$.

We need the following Lemma.

Lemma 2.7. *Forcing with* $\mathbb{P} \times \mathbb{M}$ *forces "*$\kappa = \aleph_2 + TP(\kappa)$ *holds".*

Similarly, if we replace $\mathbb{P}$ *with* $\mathbb{P}_{(\lambda,\kappa]}$*, the tail iteration after* λ*, and* $\mathbb{M}$ *with* $\widetilde{\mathbb{M}}(\lambda, \kappa)$*, then the resulting product forces "*$\kappa = \lambda^{++} + TP(\kappa)$ *holds"*

Proof. Since $\mathbb{P}$ is κ-c.c. and $\mathbb{M}$ is κ-Knaster, their product is κ-c.c., so κ is preserved. On the other hand we can assume $\mathbb{P}$ is $\aleph_2$-closed, and so one can easily check that $\aleph_1$ is preserved. Thus κ becomes $\aleph_2$ in the extension by $\mathbb{P} \times \mathbb{M}$. Let us show that it forces the tree property at κ.

Let $G \times H$ be $\mathbb{P} \times \mathbb{M}$-generic over V and assume by contradiction that T is a κ-Aronszajn tree in $V[G \times H]$.

Note that we can take $j(\mathbb{P})(\kappa)$ to be $\widetilde{\mathbb{M}}(\kappa,\lambda)*\dot{\widetilde{\mathbb{M}}}(\lambda,\eta)$, and note that it is forcing isomorphic to a forcing notion of the form

$$\mathrm{Add}(\kappa,\lambda)*\dot{\mathbb{Q}},$$

where $\dot{\mathbb{Q}}$ is forced to be κ^+-distributive. So using arguments from [1] (see also Lemma 3.19), we can force with $j(\mathbb{P})/G\times\mathbb{M}/H$ and extend the embedding j to

$$\tilde{j}: V[G\times H]\to M[j(G)\times j(H)].$$

Then $T\in M[G\times H]$ and T has a cofinal branch in $M[j(G)\times j(H)]$. But note that

$$j(\mathbb{P})/G\simeq \mathrm{Add}(\kappa,\lambda)*\dot{\mathbb{B}},$$

where $\Vdash_{\mathrm{Add}(\kappa,\lambda)}$ "$\dot{\mathbb{B}}$ does not add new subsets to κ". Now we can use arguments similar to the proof of Lemma 2.5 to show that forcing with $j(\mathbb{P})/G\times\mathbb{M}/H$ can not add a branch through T, which leads to a contradiction. The lemma follows. $\square$

In fact one can say more regarding Lemma 2.7. Let U be a normal measure on κ. Then for any $\lambda<\kappa$, one can show that

$$\Lambda_\lambda=\{\gamma\in(\lambda,\kappa):\mathbb{P}_{(\lambda,\gamma]}\times\widetilde{\mathbb{M}}(\lambda,\gamma)\Vdash \text{ ``}\gamma=\lambda^{++}+TP(\gamma)\text{ ''}\}\in U.$$

and hence

$$\Lambda=\triangle_{\lambda<\kappa}\Lambda_\lambda=\{\gamma<\kappa:\forall\lambda<\gamma,\gamma\in\Lambda_\lambda\}\in U.$$

By combining the above proofs, it is possible to prove some other preservation lemmas. For example, one can show, if $\alpha<\beta<\kappa$ are such that α is regular, β is measurable and κ is supercompact, then an iteration of the form

$$\widetilde{\mathbb{M}}(\alpha,\beta)*(\dot{\mathbb{P}}_{(\beta,\kappa])}\times\dot{\widetilde{\mathbb{M}}}(\beta,\kappa))$$

forces the tree property at both β and κ.

3. Getting tree property at double successor of all singular cardinals

In this section we prove Theorem 1.2. In Subsection 3.1 we define the notion of measure sequences and then in Subsection 3.2, we assign to each measure sequence w a forcing notion $\mathbb{R}_w$, which is a version of Radin forcing which is needed for the proof of Theorem 1.2. In Subsection 3.3, we review some of the basic properties of the forcing notion $\mathbb{R}_w$. Then in Subsection 3.4 we define the required model and in Subsections 3.5 and 3.6 we complete the proof of Theorem 1.2.

3.1. measure sequences.

In this subsection we review the definition of measure sequences as we need them for the proof of Theorem 1.2. Our presentation follows [11], but we present the details for completeness. During the Subsections 3.1, 3.2 and 3.3, we assume that the following conditions are satisfied:

- κ is an $H(\kappa^{++})$-hypermeasurable cardinal and that $2^\kappa = 2^{\kappa^+} = 2^{\kappa^{++}} = \kappa^{+3}$.
- There is $j : V \to M$ with critical point κ such that $H(\kappa^{++}) \subseteq M$.
- j is generated by a (κ, κ^{+4})-extender.
- If U is the normal measure derived from j and if $i : V \to N \simeq \mathrm{Ult}(V, U)$ is the ultrapower embedding, then there exists $F \in V$ which is $\mathrm{Col}(\kappa^{+5}, < i(\kappa))_N$-generic over N.

Let $k : N \to M$ be the induced elementary embedding with $j = k \circ i$. Then $crit(k) = \kappa_N^{+4} < \kappa_M^{+4} = \kappa^{+4}$. Set

$$P^* = \{f : \kappa \to V_\kappa \mid \mathrm{dom}(f) \in U_j \text{ and } \forall \alpha, f(\alpha) \in \mathrm{Col}(\alpha^{+5}, < \kappa)\}.$$

$$F^* = \{f \in P^* \mid i(f)(\kappa) \in F\}.$$

Then U can be read off F^* as

$$U = \{X \subseteq \kappa \mid \exists f \in F^*, X = \mathrm{dom}(f)\}.$$

The following definitions are based on [1] and [11] with the modifications required for our purposes.

Definition 3.1. *A constructing pair, is a pair (j, F), where*

- *$j : V \to M$ is a non-trivial elementary embedding into a transitive inner model, and if $\kappa = \mathrm{crit}(j)$, then $M^\kappa \subseteq M$.*

- *F is $\mathrm{Col}(\kappa^{+5}, < i(\kappa))_N$-generic over N, where $i : V \to N \simeq \mathrm{Ult}(V, U)$ is the ultrapower embedding approximating j. Also factor j through i, say $j = k \circ i$.*
- *$F \in M$.*
- *F can be transferred along k to give a $\mathrm{Col}(\kappa^{+5}, < j(\kappa))_M$-generic over M.*

In particular note that the pair (j, F) constructed above is a constructing pair.

Definition 3.2. *If (j, F) is a constructing pair as above, then $F^* = \{f \in P^* \mid i(f)(\kappa) \in F\}$.*

Definition 3.3. *Suppose (j, F) is a constructing pair as above. A sequence w is constructed by (j, F) iff*

- *$w \in M$.*
- *$w(0) = \kappa = \mathrm{crit}(j)$.*
- *$w(1) = F^*$.*
- *For $1 < \beta < \mathrm{lh}(w), w(\beta) = \{X \subseteq V_\kappa \mid w \restriction \beta \in j(X)\}$.*
- *$M \models |\,\mathrm{lh}(w)| \leq w(0)^+$.*

If w is constructed by (j, F), then we set $\kappa_w = w(0)$, and if $\mathrm{lh}(w) \geq 2$, then we define

$$F^*_w = w(1).$$

$$\mu_w = \{X \subseteq \kappa_w \mid \exists f \in F^*_w, X = \mathrm{dom}(f)\}.$$

$$\bar{\mu}_w = \{X \subseteq V_{\kappa_w} \mid \{\alpha \mid \langle\alpha\rangle \in \mu_w\} \in \mu_w\}.$$

$$F_w = \{[f]_{\mu_w} \mid f \in F^*_w\}.$$

$$\mathcal{F}_w = \bar{\mu}_w \cap \bigcap\{w(\alpha) \mid 1 < \alpha < \mathrm{lh}(w)\}.$$

Definition 3.4. *Define inductively*

$$\mathcal{U}_0 = \{w \mid \exists (j, F) \text{ such that } (j, F) \text{ constructs } w\}.$$

$$\mathcal{U}_{n+1} = \{w \in \mathcal{U}_n \mid \mathcal{U}_n \cap V_{\kappa_w} \in \mathcal{F}_w\}.$$

$$\mathcal{U}_\infty = \bigcap_{n\in\omega} \mathcal{U}_n.$$

The elements of $\mathcal{U}_\infty$ are called measure sequences.

Now let u be the measure sequences constructed using (j, F) above. It is easily seen that for each $\alpha < \kappa^{++}, u \restriction \alpha$ exists and is in $\mathcal{U}_\infty$.

3.2. **Radin forcing with interleaved collapses.** In this subsection, to each measure sequence $w \in \mathcal{U}_\infty$ we assign a forcing notion $\mathbb{R}_w$. First we define the building blocks of the forcing.

Definition 3.5. *Assume $w \in \mathcal{U}_\infty$. Then $\mathbb{P}_w$ is the set of all tuples $p = (w, \lambda, A, H, h)$, where*

(1) *w is a measure sequence.*

(2) *$\lambda < \kappa_w$ is measurable.*

(3) *$A \in \mathcal{F}_w$.*

(4) *$H \in F^*_w$ with $\mathrm{dom}(H) = \{\kappa_v > \lambda \mid v \in A\}$.*

(5) *$h \in \mathrm{Col}(\lambda^{+5}, < \kappa_w)$.*

Note that if $\mathrm{lh}(w) = 1$, then the above tuple is of the form $(w, \lambda, \emptyset, \emptyset, h)$ (where $\lambda < \kappa_w$ and $h \in \mathrm{Col}(\lambda^{+5}, < \kappa_w)$).

Given $p \in \mathbb{P}_w$ as above, we denote it by

$$p = (w^p, \lambda^p, A^p, H^p, h^p).$$

The order on $\mathbb{P}_w$ is defined as follows.

Definition 3.6. *Assume $p, q \in \mathbb{P}_w$. Then $p \leq^* q$ iff:*

(1) *$w^p = w^q$.*

(2) *$\lambda^p = \lambda^q$.*

(3) *$A^p \subseteq A^q$.*

(4) *For all $v \in A^p$, $H^p(\kappa_v) \leq H^q(\kappa_v)$.*

(5) *$h^p \leq h^q$.*

Next we define the forcing notion $\mathbb{R}_w$.

Definition 3.7. *If w is a measure sequence, then $\mathbb{R}_w$ is the set of all finite sequences*

$$p = \langle p_k \mid k \leq n \rangle,$$

where

(1) *$p_k = (w_k, \lambda_k, A_k, H_k, h_k) \in \mathbb{P}_{w_k}$, for each $k \leq n$.*

(2) *$w_n = w$.*

(3) $\lambda_0 = \omega$.

(4) *If* $k < n$, *then* $\lambda_{k+1} = \kappa_{w_k}$.

Given $p \in \mathbb{R}_w$ as above, we denote it by

$$p = \langle p_k \mid k \leq n^p \rangle$$

and call n^p the length of p. We also use w^p_k for w^{p_k} (for $k \leq n^p$). The direct extension relation $\leq^*$ is defined on $\mathbb{R}$ in the natural way:

Definition 3.8. *Assume* $p, q \in \mathbb{R}_w$. *Then* $p \leq^* q$ *iff*

(1) $n^p = n^q$.

(2) *For all* $k \leq n^p, p_k \leq^* q_k$ *in* $\mathbb{P}_{w^p_k}$.

The following definition is the key step towards defining the order relation $\leq$ on $\mathbb{R}_w$

Definition 3.9. *Assume* $p = (w, \lambda, A, H, h) \in \mathbb{P}_w$ *and* $w' \in A$. *Then* $\mathrm{Add}(p, w')$ *is the condition* $\langle p_0, p_1 \rangle \in \mathbb{R}_w$ *defined by*

(1) $p_0 = (w', \lambda, A \cap V_{\kappa_{w'}}, H \restriction V_{\kappa_{w'}}, h)$.

(2) $p_1 = (w, \kappa_{w'}, A \setminus V_\eta, H \restriction \mathrm{dom}(H) \setminus V_\eta, H(\kappa_{w'}))$, *where* $\eta = \sup \mathrm{range}(H(\kappa_{w'}))$.

In the case that this does not yield a member of $\mathbb{R}_w$, *then* $\mathrm{Add}(s, w')$ *is undefined.*

If $p = \langle p_0, \dots, p_n \rangle \in \mathbb{R}_w$ *and* $u \in A_k$ *for some* $k \leq n$ *then* $\mathrm{Add}(p, u)$ *is the member of* $\mathbb{R}_w$ *obtained by replacing* p_k *with the two members of* $\mathrm{Add}(p_k, u)$, *That is,*

- $\mathrm{Add}(p, u) \restriction i = p \restriction i$.
- $\mathrm{Add}(p, u)_i = \mathrm{Add}(p_i, u)_0$.
- $\mathrm{Add}(p, u)_{i+1} = \mathrm{Add}(p_i, u)_1$.
- $\mathrm{Add}(p, u) \restriction [i + 2, n + 1] = p \restriction [i + 1, n]$.

Definition 3.10. *The forcing order* $\leq$ *on* $\mathbb{R}_w$ *is the smallest transitive relation containing the direct order* $\leq^*$ *and all pairs of the form* $(p, \mathrm{Add}(p, u))$.

3.3. **Basic properties of the forcing notion $\mathbb{R}_w$.** We now state the main properties of the forcing notion $\mathbb{R}_w$.

Lemma 3.11. $(\mathbb{R}_w, \leq)$ *satisfies the* κ^+_w*-c.c.*

Proof. Assume on the contrary that $A \subseteq \mathbb{R}_w$ is an antichain of size κ_w^+. We can assume that all $p \in A$ have the same length n. Write each $p \in A$ as $p = d_p^\frown p_n$, where $d_p \in V_{\kappa_w}$ and $p_n = (w, \lambda^p, A^p, H^p, h^p)$. By shrinking A, if necessary, we can assume that there are fixed $d \in V_{\kappa_w}$ and $\lambda < \kappa_w$ such that for all $p \in A$, $d_p = d$ and $\lambda_p = \lambda$.

Note that for $p \neq q$ in A, as p and q are incompatible, we must have h^p is incompatible with h^q. But $\mathrm{Col}(\lambda^{+5}, < \kappa_w)$ satisfies the κ_w-c.c., and we get a contradiction. □

The following factorization lemma can be proved easily.

Lemma 3.12. *(The factorization lemma) Assume that* $p = \langle p_0, \dots, p_n \rangle \in \mathbb{R}_w$*, where* $p_i = (w_i, \lambda_i, A_i, H_i, h_i)$ *and* $m < n$*. Set* $p^{\leq m} = \langle p_0, \dots, p_m \rangle$ *and* $p^{>m} = \langle p_{m+1}, \dots, p_n \rangle$*. Then*

(*a*) $p^{\leq m} \in \mathbb{R}_{w_m}$, $p^{>m} \in \mathbb{R}_w$ *and there exists*

$$i : \mathbb{R}_w/p \longrightarrow \mathbb{R}_{w_m}/p^{\leq m} \times \mathbb{R}_w/p^{>m}$$

which is an isomorphism with respect to both $\leq^*$ *and* $\leq$.

(*b*) *If* $m + 1 < n$*, then there exists*

$$i : \mathbb{R}_w/p \longrightarrow \mathbb{R}_{w_m}/p^{\leq m} \times \mathrm{Col}(\kappa_{w_m}^{+5}, < \kappa_{w_{m+1}}) \times \mathbb{R}_w/p^{>m+1}$$

which is an isomorphism with respect to both $\leq^*$ *and* $\leq$. □

Lemma 3.13. $(\mathbb{R}_w, \leq, \leq^*)$ *satisfies the Prikry property, i.e., given* $p \in \mathbb{R}_w$ *and any statement* σ *in the forcing language of* $(\mathbb{R}_w, \leq)$*, there exists* $q \leq^* p$ *which decides* σ.

Proof. We follow the argument given in [14]. We prove the lemma by induction on κ_w. Thus, assuming it is true for $\mathbb{R}_u$ with $\kappa_u < \kappa_w$; we prove it for $\mathbb{R}_w$. Thus suppose $p \in \mathbb{R}_w$ and σ is a statement in the forcing language of $(\mathbb{R}_w, \leq)$. First, we assume that $\mathrm{lh}(p) = 1$. So let us write it as $p = (w, \lambda, A, H, h) \in \mathbb{P}_w$.

Given $q \in \mathbb{R}_w$, we can write it as $q = d_q^\frown \langle q_{\mathrm{lh}(q)} \rangle$, where $d_q \in V_{\kappa_w}$ and then we set $stem(q) = d_q$, the stem of q. Let $\mathbf{L}$ be the set of stems of conditions in $\mathbb{R}_w$ which extend p.

Suppose $v \in A$ and $s = \langle q_k = (w_k, \lambda_k, A_k, H_k, h_k) : k \leq n \rangle \in \mathbf{L}$ are such that

$$q = s^\frown \langle v, \lambda, A_v, H_v, h_v \rangle^\frown \langle w, \kappa_v, A', H', h' \rangle \in \mathbb{R}_w,$$

and $q \leq p$. Then we have $\kappa_{w_n} = \lambda$, in particular there are less than λ^{+5}-many such stems s. For each $v \in A$ and each stem $s \in \mathbf{L}$ define the sets $D^{\mathrm{top}}(0, s, v)$ and $D^{\mathrm{top}}(1, s, v)$, as follows:

- $D^{\mathrm{top}}(0,s,v)$ is the set of all $g \leq H(\kappa_v)$ for which there exist A_v, H_v, h_v, A' and H' such that $s^\frown\langle v,\lambda,A_v,H_v,h_v\rangle^\frown\langle w,\kappa_v,A',H',g\rangle \leq \mathrm{Add}(p,v)$ and it decides σ.
- $D^{\mathrm{top}}(1,s,v)$ is the set of all $g \leq H(\kappa_v)$ such that for all A_v, H_v, h_v, A', H' and $g' \leq g$, $s^\frown\langle v,\lambda,A_v,H_v,h_v\rangle^\frown\langle w,\kappa_v,A',H',g'\rangle$ does not decide σ.

Clearly, $D^{\mathrm{top}}(0,s,v)\cup D^{\mathrm{top}}(1,s,v)$ is dense in $\mathrm{Col}(\kappa_v^{+5},<\kappa_w)/H(\kappa_v)$, and so by the distributivity of $\mathrm{Col}(\kappa_v^{+5},<\kappa_w)$, the intersection

$$D_v^{\mathrm{top}} = \bigcap_{s\in\mathbf{L}\cap V_{\kappa_v}} D^{\mathrm{top}}(0,s,v)\cup D^{\mathrm{top}}(1,s,v)$$

is also dense in $\mathrm{Col}(\kappa_v^{+5},<\kappa_w)/H(\kappa_v)$. Take $\tilde{H}\in F_w^*$ such that

$$\tilde{A} = \{v\in A : \tilde{H}(v)\in D_v^{\mathrm{top}}\}\in\mathcal{F}_w.$$

Let $H^*\in F_w^*$ extends both of H and $\tilde{H}$. Now define the sets $D^{\mathrm{low}}(0,s,v)$ and $D^{\mathrm{low}}(0,s,v)$ as follows:

- $D^{\mathrm{low}}(0,s,v)$ is the set of all $g\leq h$ for which there exist A_v, H_v, A' and H' such that $s^\frown\langle v,\lambda,A_v,H_v,g\rangle^\frown\langle w,\kappa_v,A',H',H^*(v)\rangle \leq \mathrm{Add}(p,v)$ and it decides σ.
- $D^{\mathrm{low}}(1,s,v)$ is the set of all $g\leq h$ such that for all A_v,H_v,A',H' and $g'\leq H^*(v)$, $s^\frown\langle v,\lambda,A_v,H_v,g\rangle^\frown\langle w,\kappa_v,A',H',H^*(v)\rangle$ does not decide σ.

Again, $D^{\mathrm{top}}(0,s,v)\cup D^{\mathrm{top}}(1,s,v)$ is dense in $\mathrm{Col}(\lambda^{+5},<\kappa_v)/h$, and so by the distributivity of $\mathrm{Col}(\lambda^{+5},<\kappa_v)$ and the remarks above, the intersection

$$D_v^{\mathrm{low}} = \bigcap_{s\in\mathbf{L}\cap V_{\kappa_v}} D^{\mathrm{low}}(0,s,v)\cup D^{\mathrm{low}}(1,s,v)$$

is also dense in $\mathrm{Col}(\lambda^{+5},<\kappa_v)/h$. Take $\tilde{h}_v\in D_v^{\mathrm{low}}$.

Now consider

$$p' = (w,\lambda,\tilde{A},H^*,h)\leq p.$$

For any stem s of a condition in $\mathbb{R}_w$ extending p' and every $\alpha<\mathrm{lh}(w)$, let $A(s,\alpha)\in\mathcal{F}_w$ be such that one of the following three possibilities holds for it:

$(1_{s,\alpha})$: For every $v\in A(s,\alpha)$ there exists $q'\leq p'$ such that q' forces σ and q' is of the form

$$q' = s^\frown\langle v,\lambda,A'_{s,v},H'_{s,v},h'_{s,v}\rangle^\frown\langle w,\kappa_v,A_{s,v},H_{s,v},h_{s,v}\rangle,$$

for some $A'_{s,v},H'_{s,v},h'_{s,v}\leq\tilde{h}_v, A_{s,v},H_{s,v}$ and $h_{s,v}\leq H^*(v)$.

$(2_{s,\alpha})$: For every $v \in A(s,\alpha)$ there exists $q' \leq p'$ such that q' forces $\neg\sigma$ and q' is of the above form.

$(3_{s,\alpha})$: For every $v \in A(s,\alpha)$ there does not exist $q' \leq p'$ of the above form such that q' decides σ.

For every v, we may suppose that $H_{s,v}, h_{s,v}, H'_{s,v}$ and $h'_{s,v}$'s depend only on v, and so we denote them by H_v, h_v, H'_v and h'_v respectively. For each α let $A(\alpha) = \triangle_s A(s,\alpha)$ be the diagonal intersection of the $A(s,\alpha)$'s and set

$$A^* = A' \cap \bigcup_{\alpha < \mathrm{lh}(w)} A(\alpha) \in \mathcal{F}_w.$$

Also let

$$p^* = (w, \lambda, A^*, H^*, h).$$

Note that if $v \in A' \cap A(s,\alpha)$ and if one of the $(1_{s,\alpha})$ or $(2_{s,\alpha})$ happen, then we may take $h_v = H^*(v)$ and $h'_v = \tilde{h}_v$. This is because if one of these possibilities happen, then $H^*(v) \in D(0,s,v)$, so there are $\tilde{A}_v, \tilde{H}_v, \tilde{A}'$ and $\tilde{H}'$ such that

$$q = s^\frown\langle v, \lambda, \tilde{A}_v, \tilde{H}_v, \tilde{h}_v\rangle^\frown\langle w, \kappa_v, \tilde{A}', \tilde{H}', H^*(\kappa_v)\rangle \leq \mathrm{Add}(p,v)$$

and it decides σ. On the other hand, there exists

$$q' = s^\frown\langle v, \lambda, A'_{s,v}, H'_s, h'_s\rangle^\frown\langle w, \kappa_v, A_{s,v}, H_v, h_v\rangle \leq p'$$

which also decides σ. But the conditions q and q' are compatible and they decide the same truth value; hence we can take $h_v = H^*(v)$ and $h'_v = \tilde{h}_v$.

We show that there exists a direct extension of p^* which decides σ. Assume not and let $r \leq p^*$ be of minimal length which decides σ, say it forces σ. Let us write

$$stem(r) = s^\frown\langle u, \lambda, A^r, H^r, h^r\rangle,$$

where $s \in V_{\kappa_u}$. By our assumption, there exists $\alpha < \mathrm{lh}(w)$ such that $A(s,\alpha) \in w(\alpha)$ satisfies $(1_{s,\alpha})$, so for every $v \in A(s,\alpha)$, there exists $q'_v \leq p'$ such that q'_v forces σ and q'_v is of the form

$$q'_v = s^\frown\langle v, \lambda, A'_v, H'_v, \tilde{h}_v\rangle^\frown\langle w, \kappa_v, A_v, H_v, H^*(v)\rangle,$$

for some $A'_v, H'_v, , A_v$ and H_v.

We show that there exists $q^* \leq p^*$ with $stem(q^*) = s$ such that every extension of q^* is compatible with q'_v, for some $v \in A(s, \alpha)$. This property implies that q^* forces σ, contradicting the minimal choice of $\mathrm{lh}(r)$. We note that by the definition of extension in the forcing $\mathbb{R}_w$, we may assume from this point on that s is empty.

Consider the map $\phi : A(\langle\rangle, \alpha) \to V$ which is defined by

$$\phi : v \mapsto (\phi_0(v), \phi_1(v)) = (A_\nu, H_\nu).$$

As $A(\langle\rangle, \alpha) \in w(\alpha)$, we have $w \restriction \alpha \in j(A(\langle\rangle, \alpha))$ (where j is the constructing embedding for w). Let

$$(A^{<\alpha}, H^{<\alpha}) = j(\phi)(w \restriction \alpha).$$

Also let

$$A^{\alpha} = \{v \in A(\langle\rangle, \alpha) : A^{<\alpha} \cap V_{\kappa_v} = A_v \text{ and } H^{<\alpha} \restriction V_{\kappa_v} = H_v\}$$

and

$$A^{>\alpha} = \bigcup_{\alpha < \beta < \mathrm{lh}(w)} \{v \in A^* : A^{\alpha} \cap V_{\kappa_v} \in w(\beta)\}.$$

Then $A^{**} = A^{<\alpha} \cup A^{\alpha} \cup A^{>\alpha} \in \mathcal{F}_w$. Set $H^{**} = H^{<\alpha} \wedge H^*$ and finally set

$$q^* = \langle w, \lambda, A^{**}, H^{**}, h\rangle \leq p^*.$$

We show that q^* is as required. Thus let

$$q = \langle (w_k, \lambda_k, A_k, H_k, h_k) : k \leq n\rangle$$

be an extension of q^*. There are various cases:

(1) There is no index k such that $\mathrm{lh}(u_k) > 0$ and $(A^{\alpha} \cup A^{>\alpha}) \cap V_{\kappa_{w_k}} \in \bigcup_{\beta < \mathrm{lh}(w_k)} w_k(\beta)$. Then pick some non-trivial measure sequence $v \in A^{\alpha} \cap A_n$, and note that for all $k < n$, $A^{<\alpha} \cap A_k \in \bigcap_{\beta < \mathrm{lh}(w_k)} w_k(\beta)$. Then one can easily show that q is compatible with q'_v.

(2) There is an index k with $\mathrm{lh}(u_k) > 0$ and $(A^{\alpha} \cup A^{>\alpha}) \cap V_{\kappa_{w_k}} \in \bigcup_{\beta < \mathrm{lh}(w_k)} w_k(\beta)$ and $A^{\alpha} \in w_k(\beta)$ for some $\beta < \mathrm{lh}(w_k)$. Let us pick k to be the least such an index. Let $v \in A_k$ be such that $A^{<\alpha} \cap A_k \in \bigcap_{\beta < \mathrm{lh}(v)} v(\beta)$. Then q is compatible with q'_v.

(3) There is an index k with $\mathrm{lh}(u_k) > 0$ and $(A^\alpha \cup A^{>\alpha}) \cap V_{\kappa_{w_k}} \in \bigcup_{\beta<\mathrm{lh}(w_k)} w_k(\beta)$ and $A^{>\alpha} \in w_k(\beta)$ for some $\beta < \mathrm{lh}(w_k)$. then by our choice of $A^{>\alpha}$, there is some $v \in A_k$ that can be added to q such that we reduce to the case (2).

This completes the proof for the case $\mathrm{lh}(p) = 1$. We now prove the lemma for an arbitrary condition p, by induction on $\mathrm{lh}(p)$. Thus suppose that $\mathrm{lh}(p) \geq 2$; say

$$p = s^\frown\langle (u, \lambda', A', H', h')\rangle \frown \langle (w, \lambda, A', H', h)\rangle.$$

By the factorization Lemma 3.12, we have

$$\mathbb{R}_w/p \simeq (\mathbb{R}_u/s^\frown\langle (u, \lambda', A', H', h')\rangle) \times (\mathbb{R}_w/\langle (w, \lambda, A', H', h)\rangle).$$

Let $\langle s_i : i < \kappa_u\rangle$ enumerate $\mathbf{L} \cap V_{\kappa_u}$, and define by recursion on i a $\leq^*$-decreasing chain $\langle p_i : i \leq \kappa_u\rangle$ of conditions in $\mathbb{R}_w/\langle (w, \lambda, A', H', h)\rangle$ as follows:

Set $p_o = \langle (w, \lambda, A', H', h)\rangle$. Given p_i, let $p_{i+1} \leq^* p_i$ decide whether there is a condition in $\mathbb{R}_u/s^\frown\langle (u, \lambda', A', H', h')\rangle$ with stem s_i which decides σ and if so, then it forces one of σ or $\neg\sigma$. At limit ordinals $i \leq \kappa_u$, use the fact that $(\mathbb{R}_w/\langle (w, \lambda, A', H', h)\rangle, \leq^*)$ is κ_w-closed to find an p_i which $\leq^*$-extends all $p_j, j < i$.

By our construction,

$$\Vdash_{\mathbb{R}_u/s^\frown\langle (u,\lambda',A',H',h')\rangle} \text{“}p_{\kappa_u} \text{ decides } \sigma\text{”}.$$

By the induction hypothesis, there exists $q \leq^* s^\frown\langle (u, \lambda', A', H', h')\rangle$ which decides which way p_{κ_u} decides σ, and then $q^\frown p_{\kappa_u} \leq^* p$ decides σ.

The lemma follows. □

Now suppose that $w = u \restriction \kappa^+$, where u is the measure sequence constructed by (j, F) and let $K \subseteq \mathbb{R}_w$ be generic over V. Set

$$C = \{\kappa_u \mid \exists p \in K, \exists \xi < \mathrm{lh}(p), p_\xi = (u, \lambda, A, H, h)\}.$$

By standard arguments, C is a club of κ, also we can suppose that $\min(C) = \aleph_0$. Let $\langle \kappa_\xi : \xi < \kappa\rangle$ be the increasing enumeration of the club C and let $\vec{u} = \langle u_\xi \mid \xi < \kappa\rangle$ be the enumeration of

$$\{u \mid \exists p \in K, \exists \xi < \mathrm{lh}(p), p_\xi = (u, \lambda, A, H, h)\}$$

such that for $\xi < \zeta < \kappa, \kappa_{u_\xi} = \kappa_\xi < \kappa_\zeta = \kappa_{u_\zeta}$. Also let $\vec{F} = \langle F_\xi \mid \xi < \kappa \rangle$ be such that each F_ξ is $\mathrm{Col}(\kappa_\xi^{+5}, < \kappa_{\xi+1})$-generic over V produced by K.

Lemma 3.14. (a) $V[K] = V[\vec{u}, \vec{F}]$.

(b) *For every limit ordinal* $\xi < \kappa$, $\langle \vec{u} \restriction \xi, \vec{F} \restriction \xi \rangle$ *is* $\mathbb{R}_{u_\xi}$*-generic over* V, *and* $\langle \vec{u} \restriction [\xi, \kappa), \vec{F} \restriction [\xi, \kappa) \rangle$ *is* $\mathbb{R}_w$*-generic over* $V[\vec{u} \restriction \xi, \vec{F} \restriction \xi]$.

(c) *For every* $\gamma < \kappa$ *and every* $A \subseteq \gamma$ *with* $A \in V[\vec{u}, \vec{F}]$, *we have* $A \in V[\vec{u} \restriction \xi, \vec{F} \restriction \xi]$, *where* ξ *is the least ordinal such that* $\gamma < \kappa_\xi$.

Proof. The proof is similar to the proof of Lemma 2.13 from [12].

(a) It suffices to show that K is definable from $\vec{u}$ and $\vec{F}$. Let K' be the set of all conditions $p \in \mathbb{R}_w$ such that

- For all measure sequences $u \in V_\kappa$, if u appears in p, then $u = u_\xi$, for some $\xi < \kappa$,
- For all $\xi < \kappa$, there exists $q \leq p$ such that u_ξ appears in q,
- If $f \in V_\kappa$ appears in p, then $f \subset F_\xi$, for some $\xi < \kappa$,
- For all $\xi < \kappa$ and all $f \in \mathcal{P}(F_\xi) \cap \mathrm{Col}(\kappa_\xi^{+5}, < \kappa_{\xi+1})$, there exists $q \leq p$ such that f appears in q.

It is clear that $K' \in V[\vec{u}, \vec{F}]$. It is also easily seen that K' is a filter which includes K. It follows from the genericity of K that $K = K'$. So $K \in V[\vec{u}, \vec{F}]$, as required.

(b) Follows from (a) and the factorization lemma 3.12.

(c) First note that ν is not a limit ordinal, so assume $\nu = \xi + 1$ is a successor ordinal (if $\nu = 0$, then the proof is similar). Let $p \in K$ be such that p mentions both u_ξ and $u_{\xi+1}$, say $u_\xi = u^{p_m}$ and $u_{\xi+1} = u^{p_{m+1}}$. By the Factorization Lemma 3.12,

$$\mathbb{R}_w / p \simeq \mathbb{R}_{u_\xi} / p^{\leq m} \times \mathrm{Col}(\kappa_\xi^{+5}, < \kappa_{\xi+1}) \times \mathbb{R}_w / p^{> m+1}.$$

Let $\dot{A}$ be an $\mathbb{R}_w$-name for A such that $\Vdash_{\mathbb{R}_w} \dot{A} \subseteq \gamma$. Let $\dot{B}$ be an $\mathbb{R}_w / p^{>m+1}$-name for a subset of $\mathbb{R}_{u_\xi} / p^{\leq m} \times \mathrm{Col}(\kappa_\xi^{+5}, < \kappa_{\xi+1}) \times \gamma$ such that

$$\Vdash_{\mathbb{R}_w / p^{>m+1}} \forall \eta < \gamma, ((r, f, \eta) \in \dot{B} \iff (r, f) \Vdash_{\mathbb{R}_{u_\xi}/p^{\leq m} \times \mathrm{Col}(\kappa_\xi^{+5}, < \kappa_{\xi+1})} \eta \in \dot{A}).$$

Let $\langle y_\alpha : \alpha < \kappa_{\xi+1} \rangle$ be an enumeration of $\mathbb{R}_{u_\xi} / p^{\leq m} \times \mathrm{Col}(\kappa_\xi^{+5}, \kappa_{\xi+1}) \times \gamma$. Define a $\leq^*$-decreasing sequence $\langle q_\alpha \mid \alpha < \kappa_{\xi+1} \rangle$ of conditions in $\mathbb{R}_w / p^{>m+1}$ such that for all α, q_α

decides "$y_\alpha \in \dot{B}$". This is possible as $(\mathbb{R}_w/p^{>m+1}, \leq^*)$ is $\kappa^+_{\xi+1}$-closed and satisfies the Prikry property, Lemma 3.13. Let $q \leq^* q_\alpha$ for all $\alpha < \kappa_{\xi+1}$. Then q decides each "$y_\alpha \in \dot{B}$". It follows that $A \in V[\vec{u} \restriction \nu, \vec{F} \restriction \nu]$ $\square$

We now state a geometric characterization of generic filters for $\mathbb{R}_w$. Such a characterization was first given by Mitchell [21] for Radin forcing. The characterization given bellow is essentially due to Cummings [1].

Lemma 3.15. *(Geometric characterization) The pair $(\vec{u}, \vec{F})$ is $\mathbb{R}_w$-generic over V if and only if it satisfies the following conditions:*

(1) *If $\xi < \kappa$ and $\mathrm{lh}(u_\xi) > 1$, then the pair $(\vec{u} \restriction \xi, \vec{F} \restriction \xi)$ is $\mathbb{R}_{u_\xi}$-generic over V.*
(2) *For all $A \in V_{\kappa+1}$ $(A \in \mathcal{F}_w \iff \exists \alpha < \kappa\ \forall \xi > \alpha,\ u_\xi \in A)$.*
(3) *For all $f \in w(1)$ there exists $\alpha < \kappa$ such that $\forall \xi > \alpha, f(\kappa_\xi) \in F_\xi$.*

As $\mathrm{lh}(w) = \kappa^+$, it follows from Mitchell [21] (see also [9]) that

Lemma 3.16. *κ remains strongly inaccessible in $V[K]$.*

Proof. We follow Cummings [1]. Suppose not and let $p \in \mathbb{R}_w, \delta < \kappa$ and $\dot{f}$ be such that

$$p \Vdash \text{“}\dot{f} : \delta \to \kappa \text{ is cofinal”}.$$

Let $\theta > \kappa$ be large enough regular such that $p, \dot{f}, w, \mathbb{R}_w \in H(\theta)$ and let $\mathbf{X} \prec H(\theta)$ be such that

(1) $p, \kappa^+, \dot{f}, w, \mathbb{R}_w \in \mathbf{X}$.
(2) $V_\kappa \subseteq \mathbf{X}$.
(3) ${}^{<\kappa}\mathbf{X} \subseteq \mathbf{X}$.
(4) $|\mathbf{X}| = \kappa$.

Let $\pi : \mathbf{X} \to \mathbf{N}$ be the Mostowski collapse of $\mathbf{X}$ onto a transitive model $\mathbf{N}$. Note that $\pi \restriction \mathbf{X} \cap V_{\kappa+1} = id \restriction \mathbf{X} \cap V_{\kappa+1}$.

Let $v = \pi(w)$ and $\beta = \pi(\kappa^+)$. Then

$$\pi(\mathcal{F}_w) = \mathcal{F}_w \cap \mathbf{X} = \mathcal{F}_w \cap \mathbf{N}$$

and

$$\forall \alpha \in \mathbf{X} \cap \kappa^+, \ \pi(v(\alpha)) = v(\pi(\alpha)) = v(\alpha) \cap \mathbf{X} = w(\alpha) \cap \mathbf{N}.$$

Let $\bar{\beta} = \sup(\mathbf{X} \cap \kappa^+) < \kappa^+$. Using Lemma 3.15, if K is $\mathbb{R}_{w\restriction\bar{\gamma}}$-generic over V, where $\bar{\beta} \leq \gamma < \kappa^+$, then G is $\pi(\mathbb{R}_w)$-generic over V.

We get a contradiction and the lemma follows. □

It follows that

$$CARD^{V[K]} \cap \kappa = \bigcup_{\alpha \in C} \{\alpha, \alpha^+, \alpha^{++}, \alpha^{+3}, \alpha^{+4}, \alpha^{+5}\}.$$

As every limit point of C is singular in $V[K]$, it follows that κ is the least inaccessible cardinal. Also note that $\lim(C)$ is exactly the set of all singular cardinals below κ in $V[K]$.

3.4. **The final model.** Here we define the final model we are going to work with. Suppose that the GCH holds, κ is an $H(\lambda^{++})$-hypermeasurable cardinal, where λ is the least measurable cardinal above κ. Also let $f : \kappa \to \kappa$ be defined by

$$f(\alpha) = (\min\{\beta > \alpha : \beta \text{ is a measurable cardinal }\})^+.$$

Then $j(f)(\kappa) = \lambda^+$. Let $j : V \to M$ witness the $H(\lambda^{++})$-hypermeasurability of κ and suppose j is generated by a (κ, λ^{++})-extender, i.e.,

$$M = \{j(f)(\alpha) : f : \kappa \to V, \alpha < \lambda^{++}\}.$$

Also let U be the normal measure derived from j; $U = \{X \subseteq \kappa : \kappa \in j(X)\}$ and let $i : V \to N \simeq \mathrm{Ult}(V, U)$ be the ultrapower embedding. From now on, we fix the following notation.

Notation 3.17. (a) *For each infinite cardinal $\alpha \leq \kappa$ let α_* denote the least measurable cardinal above α. Note that $\kappa_* = \lambda$.*

(b) *For an infinite cardinal $\alpha \leq \kappa$ let $\mathbb{M}_\alpha = \mathbb{M}(\alpha, \alpha_*, \alpha_*^+)$.*

We start with a simple lemma.

Lemma 3.18. *Then there exists a cofinality preserving generic extension V^1 of V satisfying the following conditions:*

(a) *$V^1 \models$ "GCH".*

(b) *There is $j^1 : V^1 \to M^1$ with critical point κ such that $H(\lambda^{++}) \subseteq M^1$ and $j^1 \upharpoonright V = j$.*

(c) *j^1 is generated by a (κ, λ^{++})-extender.*

(d) *If U^1 is the normal measure derived from j^1 and if $i^1 : V^1 \to N^1 \simeq \mathrm{Ult}(V^1, U^1)$ is the ultrapower embedding, then there exists $\bar{g} \in V^1$ which is $i^1(\mathrm{Add}(\kappa, \lambda^+)_{V^1})$-generic over N^1. Further $i^1 \upharpoonright V = i$.*

Proof. The lemma is proved in [8]. We just sketch what the generic extension V^1 is. Work in V. Let

$$C = \{\alpha < \kappa : \forall \beta < \alpha,\ f(\beta) < \alpha\}.$$

Then C is the club consisting of the closure points of f. Let

$$\mathbb{P}^1_{\kappa+1} = \langle \langle \mathbb{P}^1_\alpha : \alpha \leq \kappa + 1 \rangle, \langle \dot{\mathbb{Q}}^1_\alpha : \alpha \leq \kappa \rangle \rangle$$

be a reverse Easton iteration of forcing notions, such that if

$$G^1_{\kappa+1} = \langle \langle G^1_\alpha : \alpha \leq \kappa + 1 \rangle, \langle g^1_\alpha : \alpha \leq \kappa \rangle \rangle$$

is the $\mathbb{P}^1_{\kappa+1}$-generic filter, then for each $\alpha \leq \kappa$,

(1) In $V[G^1_\alpha], \mathbb{Q}^1_\alpha$ is the lottery sum of all forcing notions $\mathbb{R}$ which satisfy the following:
   (a) If $\alpha \in C$ is a measurable cardinal in $V[G^1_\alpha]$, choose some normal measure U_α on α such that the derived ultrapower embedding satisfies $i : V[G^1_\alpha] \to N_\alpha[i(G^1_\alpha)]$, for some N_α.
   (b) $R = i(\mathrm{Add}(\alpha, f(\alpha))_{V[G^1_\alpha]})$.

   If the above is not possible, then $\mathbb{Q}^1_\alpha = \{1_{\mathbb{Q}^1_\alpha}\}$ is the trivial forcing

(2) Suppose i lifts in $V[G^1_\kappa]$ to $i : V[G^1_\kappa] \to N[i(G^1_\kappa)]$. Then set $\mathbb{Q}_\kappa = i(\mathrm{Add}(\kappa, \lambda^+)_{V[G^1_\kappa]})$.
   If i does not lift to $V[G^1_\kappa]$, set $\mathbb{Q}^1_\kappa = \{1_{\mathbb{Q}^1_\kappa}\}$ be the trivial forcing notion.

Then $V^1 = V[G^1_{\kappa+1}] = V[G^1_\kappa * g^1_\kappa]$ is as required. □

Let V^1 be the model constructed above.

Lemma 3.19. *Work in V^1. There exists a forcing iteration $\mathbb{P}_\kappa$ of length κ such that if $G_\kappa * g$ is $\mathbb{P}_\kappa * \dot{\mathbb{M}}(\kappa, \lambda, \lambda^+)$-generic over V^1, then in $V^2 = V^1[G_\kappa * g]$, the following holds:*

(a) *There is* $j^2 : V^2 \to M^2$ *with critical point* κ *such that* $H(\kappa^{++}) \subseteq M^2$ *and* $j^2 \restriction V^1 = j^1$.

(b) j^2 *is generated by a* (κ, μ^+)*-extender.*

(c) $V^2 \models$ *"*$\lambda = \kappa^{++} + 2^\kappa = \lambda^+ = \kappa^{+3} + TP(\lambda)$*".*

(d) *If* U^2 *is the normal measure derived from* j^2 *and if* $i^2 : V^2 \to N^2 \simeq \mathrm{Ult}(V^2, U^2)$ *is the ultrapower embedding, then there exists* $F \in V^2$ *which is* $\mathrm{Col}(\kappa^{+5}, < i(\kappa))_{N^2}$*-generic over* N^2.

Proof. We follow [1]. Work in V^1. Factor j^1 in two steps through the models

$$N' = \text{ the transitive collapse of } \{j^1(f)(\kappa) : f : \kappa \to V^1\}$$

$$\bar{N}' = \text{ the transitive collapse of } \{j^1(f)(\alpha) : f : \kappa \to V^1, \alpha < \lambda^+\}.$$

N' is the familiar ultrapower approximating M^1, while $\bar{N}'$ corresponds to the extender of length λ^+. We have maps

$$i' : V^1 \to N',$$
$$k' : N' \to M^1,$$
$$\bar{i}' : N' \to \bar{N}',$$
$$\bar{k}' : \bar{N}' \to M^1$$

such that

$$k' \circ i' = j^1 \ \& \ \bar{k}' \circ \bar{i}' = k'.$$

Let

$$\mathbb{P}_\kappa = \langle \langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle \rangle$$

be the reverse Easton iteration, such that

(1) If $\alpha < \kappa$ is a measurable limit of measurable cardinals, then $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{M}}(\alpha, \alpha_*, \alpha_*^+)$".

(2) Otherwise, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ is the trivial forcing".

Let $G_k * g$ be be $\mathbb{P}_\kappa * \dot{\mathbb{M}}(\kappa, \lambda, \lambda^+)$-generic over V^1.

Note that we can factor $\mathbb{M}(\kappa, \lambda, \lambda^+)$ as

$$\mathbb{M}(\kappa, \lambda, \lambda^+) = \mathrm{Add}(\kappa, \lambda^+) * \dot{\dot{\mathbb{Q}}},$$

where $\dot{\dot{\mathbb{Q}}}$ is forced to be κ^+-distributive. Let us factor g as $g = g(0) * g(1)$.

By standard arguments, we can lift the maps $\bar{i}', k', \bar{k}'$ to get

$$k' : N'[G_\kappa] \to M^1[G_\kappa],$$
$$\bar{i}' : N'[G_\kappa] \to \bar{N}'[G_\kappa],$$
$$\bar{k}' : \bar{N}'[G_\kappa] \to M^1[G_\kappa],$$

where $\bar{k}' \circ \bar{i}' = k'$. The models $\bar{N}'[G_\kappa]$ and $M^1[G_\kappa]$ are closed under κ-sequences in $V^1[G_\kappa]$ and they compute the cardinals up to λ^+ in the correct way, in particular, the least measurable above κ in these models is λ, and so if we set $\mathbb{Q}_\kappa = \mathrm{Add}(\kappa, \lambda^+)_{V^1[G_\kappa]}$, then it is computed in the same way in the models $\bar{N}'[G_\kappa]$ and $M^1[G_\kappa]$, i.e.,

$$\mathbb{Q}_\kappa = (\mathbb{Q}_\kappa)_{\bar{N}'[G_\kappa]} = (\mathbb{Q}_\kappa)_{M^1[G_\kappa]}.$$

On the other hand

$$(\mathbb{Q}_\kappa)_{N'[G_\kappa]} = \mathrm{Add}(\kappa, \bar{\lambda})_{V^1[G_\kappa]},$$

where $\bar{\lambda} = (i'(f)(\kappa)^+)_{N'}$. Note that $\kappa^+ < \bar{\lambda} < \kappa^{++}$. Factor $g(0)$ as $g(0)_1 \times g(0)_2$, which corresponds to

$$\mathrm{Add}(\kappa, \lambda^+)_{V^1[G_\kappa]} = \mathrm{Add}(\kappa, \bar{\lambda})_{V^1[G_\kappa]} \times \mathrm{Add}(\kappa, \lambda^+ \setminus \bar{\lambda})_{V^1[G_\kappa]}.$$

We build further extensions

$$k' : N'[G_\kappa][g(0)_1] \to M^1[G_\kappa][g(0)],$$
$$\bar{i}' : N'[G_\kappa][g(0)_1] \to \bar{N}'[G_\kappa][g(0)],$$
$$\bar{k}' : \bar{N}'[G_\kappa][g(0)] \to M^1[G_\kappa][g(0)],$$

still preserving the relation $\bar{k}' \circ \bar{i}' = k'$. By standard arguments, we can find $H \in V^1[G_\kappa][g(0)_1]$, which is $i'(\mathbb{P}_\kappa)_{(\kappa+1, i'(\kappa))}$-generic over $N'[G_\kappa][g(0)_1]$. We transfer H along $\bar{i}', k'$ to get

$$i' : V^1[G_\kappa] \to N'[i'(G_\kappa)],$$
$$k' : N'[i(G_\kappa)] \to M^1[j^1(G_\kappa)],$$
$$\bar{i}' : N'[i(G_\kappa)] \to \bar{N}'[\bar{i}' \circ i'(G_\kappa)],$$
$$\bar{k}' : \bar{N}'[\bar{i}' \circ i'(G_\kappa)] \to M^1[j^1(G_\kappa)],$$

where all the maps are defined in $V^1[G_\kappa][g(0)]$. Let $l' = \bar{i}' \circ i'$. Since $\mathbb{P}_\kappa$ has size κ and is κ-c.c., so the term forcing

$$\mathrm{Add}(\kappa, \lambda^+)_{V^1[G_\kappa]}/\mathbb{P}^2_\kappa$$

is forcing isomorphic to $\mathrm{Add}(\kappa, \lambda^+)_{V^1}$ (see [1] Fact 2, §1.2.6). By our assumption, we have $\bar{g} \in V^1$, which is $i'(\mathrm{Add}(\kappa, \mu)_{V^1})$-generic over N', and using it we can define g_a which is

$$\mathrm{Add}(i(\kappa), i(\lambda^+))_{N'[i'(G_\kappa)]}$$

generic over $N'[i'(G_\kappa)]$. Using the fact that

$$V^1[G_\kappa][g(0)_1] \models \text{“} {}^\kappa N'[i'(G_\kappa)] \subseteq N'[i'(G_\kappa)]\text{”}$$

we also build F, which is $\mathrm{Col}(\kappa^{+5}, < i'(\kappa))_{N'[i'(G_\kappa)]}$ generic over $N'[i'(G_\kappa)]$. Note that g_a and F are mutually generic.

Transfer g_a and F along $\bar{i}'$ to get new generics $\bar{g}_a$ and $\bar{F}$. Now using Woodin's surgery argument, we can alter the filter $\bar{g}_a$ to find a generic filter h_a with the additional property $l''[g(0)] \subseteq h_a$. Also h_a is easily seen to be mutually generic with $\bar{F}$.

We now transfer h_a along $\bar{k}'$ to get H_a which is $j^1(\mathbb{Q}_\kappa)$-generic over M^1. Further, ${j^1}''[g(0)] \subseteq H_a$, so we can build maps

$$\bar{j} : V^1[G_\kappa * g(0)] \to M^1[j^1(G_\kappa * g(0))],$$
$$\bar{k}' : \bar{N}'[l(G_\kappa * g(0))] \to M^1[j^1(G_\kappa * g(0))],$$
$$l : V^1[G_\kappa * g(0)] \to \bar{N}'[l(G_\kappa * g)].$$

where $j^2 = \bar{k} \circ l$. Now let us look at $\dot{\bar{\mathbb{Q}}}[G_\kappa * g(0)]$. It is κ^+-distributive in $V^1[G_\kappa * g(0)]$, so by standard arguments, we can further extend the above embeddings and get

$$\bar{j} : V^1[G_\kappa * g(0) * g(1)] \to M^1[j^1(G_\kappa * g(0) * g(1))],$$
$$\bar{k}' : \bar{N}'[l(G_\kappa * g(0) * g(1))] \to M^1[j^1(G_\kappa * g(0) * g(1))],$$
$$l : V^1[G_\kappa * g(0) * g(1)] \to \bar{N}'[l(G_\kappa * g(0) * g(1))].$$

Let

$$V^2 = V^1[G_\kappa * g(0) * g(1)],$$

$$M^2 = M^1[j^1(G_\kappa * g(0) * g(1))]$$

and

$$N^2 = \bar{N}'[l(G_\kappa * g(0) * g(1))].$$

Also let $j^2 = \bar{j}$. We argue

$$\mathrm{Ult}(V[G_\kappa * g(0) * g(1)], U^2) \simeq N^2,$$

where U^2 is the normal measure derived from j^2. To see this, factor l through $l^\dagger : V^2 \to N^\dagger \simeq \mathrm{Ult}(V^2, U')$, where U' is the normal measure derived from l. Also let $k^\dagger : N^\dagger \to N^2$. Then $P(\kappa)_{V^2} \subseteq N^\dagger$ and $N^\dagger \models$"$2^\kappa = \lambda^+$". So $\mathrm{crit}(k^\dagger) > \lambda^+$. Since $\lambda^+ \subseteq \mathrm{range}(k^\dagger)$ and N^2 is generated by a (κ, λ^+)-extender, we have $N^2 = N^\dagger$ and we are done.

So if we let $i^2 = l$, then

$$i^2 : V^2 \to N^2$$

is the ultrapower embedding. Finally note that F is generic for the appropriate collapse ordering. The lemma follows. $\square$

Note that in the model $V^2 = V^1[G_\kappa * g]$, the following conditions are satisfied:

- $V^2 \models$"$\lambda = \kappa^{++} + 2^\kappa = \lambda^+ = \kappa^{+3}$".
- There is $j^2 : V^2 \to M^2$ with critical point κ such that $H(\kappa^{++}) \subseteq M^2$ and $j^2 \restriction V^1 = j^1$.
- j^2 is generated by a (κ, μ^+)-extender.
- If U^2 is the normal measure derived from j^2 and if $i^2 : V^2 \to N^2 \simeq \mathrm{Ult}(V^2, U^2)$ is the ultrapower embedding, then there exists $F \in V^2$ which is $\mathrm{Col}(\kappa^{+5}, < i(\kappa))_{N^2}$-generic over N^2.

Thus the hypotheses made at the beginning of Subsection 3.1 are satisfied, and so, working in V^2, we can construct the pair (j, F). Let u be the measure sequence constructed from it. Set $w = u \restriction \kappa^+$ and let $\mathbb{R}_w$ be the corresponding forcing notion as in Definition 3.7. Also let K be $\mathbb{R}_w$-generic over V^2. Build the sequences $\vec{\kappa} = \langle \kappa_\xi : \xi < \kappa \rangle, \vec{u} = \langle u_\xi : \xi < \kappa \rangle$ and $\vec{F} = \langle F_\xi : \xi < \kappa \rangle$ from K, as in Subsection 3.3.

3.5. $TP(\kappa^{++})$ **holds in** $V^1[G_\kappa * g * K]$**.** In this subsection we show that $TP(\kappa^{++})$ holds in $V^1[G_\kappa * g * K]$, and then in the next subsection, we complete the proof of Theorem 1.2 by showing that

$$V^1[G_\kappa * g * K] \models \text{“}TP(\alpha^{++}) \text{ holds for all singular cardinals } \alpha < \kappa\text{”}.$$

As $V^1[G_\kappa * g * K] \models$"$\kappa^{++} = \lambda$", thus it suffices to prove the following:

Theorem 3.20. *$V^1[G_\kappa * g * K] \models$"$TP(\lambda)$".*

The rest of this subsection is devoted to the proof of the above theorem. The proof we present follows ideas of [8].

Lemma 3.21. *The forcing* $\mathbb{P}_\kappa * \dot{\mathbb{M}} * \dot{\mathbb{R}}_w$ *satisfies the* λ*-c.c.*

Proof. The forcing $\mathbb{P}_\kappa$ is κ-c.c. Now the result follows from the facts that $\mathbb{P}_\kappa$ forces "$\dot{\mathbb{M}}$ is λ-c.c." (by Lemma 2.2) and $\mathbb{P}_\kappa * \dot{\mathbb{M}}$ forces "$\dot{\mathbb{R}}_w$ is κ^+-c.c." (by Lemma 3.11). $\square$

Assume towards contradiction that $TP(\lambda)$ fails in $V^1[G_\kappa * g][\vec{u}, \vec{F}]$ and let $\dot{T} \in V^1[G_\kappa]$ be an $\mathbb{M} * \dot{\mathbb{R}}_w$-name for a λ-Aronszajn tree in $V^1[G_\kappa * g][\vec{u}, \vec{F}]$. Suppose for simplicity that the trivial condition forces that $\dot{T}$ is a λ-Aronszajn tree and let us view it as a nice name for a subset of λ; so that $\dot{T} = \bigcup_{\xi<\lambda} \{\check{\xi}\} \times A_\xi$, where each A_ξ is a maximal antichain in $\mathbb{M} * \dot{\mathbb{R}}_w$. Note that by Lemma 3.21, each A_ξ has size less than λ.

Recall from the remarks after Lemma 2.2 that the forcing $\mathbb{M}$ is forcing isomorphic to $\mathrm{Add}(\kappa, \lambda^+) * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is some $\mathrm{Add}(\kappa, \lambda^+)$-name for a forcing notion which is forced to be κ^+-distributive.

Lemma 3.22. *Work in* $V^1[G_\kappa]$*. The set*

$$\{r = ((p, \dot{q}), \check{d}^\frown\langle w, \lambda, \dot{A}, \dot{H}, \check{h}\rangle) \in \mathbb{M} * \dot{\mathbb{R}}_w : d, h \in V^1[G_\kappa] \textit{ and } \dot{A}, \dot{H} \textit{ are } \mathrm{Add}(\kappa, \lambda^+)-\textit{names}\ \}$$

is dense in $\mathbb{M} * \dot{\mathbb{R}}_w$.

Proof. Recall that a condition in $\mathbb{R}_w$ is of the form $p = d^\frown\langle w, \lambda, A, H, h\rangle$ where

(1) $d \in V_\kappa$.

(2) $\langle w, \lambda, A, H, h\rangle \in \mathbb{P}_w$.

(3) $h \in \mathrm{Col}(\lambda^{+5}, <\kappa)$.

As $\mathbb{M}$ does not add bounded subsets to κ, so any condition $((p, \dot{q}), \dot{d}^\frown\langle w, \lambda, \dot{A}, \dot{H}, \dot{h}\rangle)$ has an extension of the form $((p', \dot{q}'), \check{d}'^\frown\langle \kappa, \lambda, \dot{A}', \dot{H}', \check{h}'\rangle)$, where $d', h' \in V^1[G_\kappa]$.

Also note that all conditions in $\mathbb{P}_w$ and hence in $\mathbb{R}_w$ exist already in the extension by $\mathrm{Add}(\kappa, \lambda^+)$, the Cohen part of $\mathbb{M}$ (though the definition of $\mathbb{R}_w$ may require the whole $\mathbb{M}$). Thus we can further extend $((p', \dot{q}'), \check{d}'^\frown\langle \kappa, \lambda, \dot{A}', \dot{H}', \check{h}'\rangle)$ to another condition

$$((p'', \dot{q}''), \check{d}''^\frown\langle w, \lambda, \dot{A}'', \dot{H}'', \check{h}''\rangle)$$

where $\dot{A}''$ and $\dot{H}''$ are forced to be $\mathrm{Add}(\kappa, \lambda^+)$-names (over $V^1[G_\kappa]$). The result follows immediately. $\square$

From now on, we assume that all the conditions in $\mathbb{M} * \dot{\mathbb{R}}_w$ are of the above form. This is useful in some of the arguments below (see for example Lemma 3.23(a)). Let us define

$$\mathbb{C} = \{((p, \emptyset), r) : ((p, \emptyset), r) \in \mathbb{M} * \dot{\mathbb{R}}_w\}$$

and

$$\mathbb{T} = \{(\emptyset, q) : (\emptyset, q) \in \mathbb{M}\}.$$

Let $\tau : \mathbb{C} \times \mathbb{T} \to \mathbb{M} * \dot{\mathbb{R}}_w$ be defined by

$$\tau(\langle((p, \emptyset), r), (\emptyset, q)\rangle) = ((p, q), r).$$

Lemma 3.23. (a) *τ is a projection from $\mathbb{C} \times \mathbb{T}$ onto $\mathbb{M} * \dot{\mathbb{R}}_w$.*

(b) *$\mathbb{T}$ is κ^+-closed in $V^1[G_\kappa]$.*

(c) *$\mathbb{C}$ is κ^+-c.c. in $V^1[G_\kappa]$.*

Proof. The proof is similar to the proof of Lemma 1.5 from [8]. We present it for completeness.

(a) It is clear that τ is order preserving. Suppose that

$$(p', \dot{q}'), \dot{r}') \leq_{\mathbb{M} * \dot{\mathbb{R}}_w} \tau(\langle((p, \emptyset), \dot{r}), (\emptyset, \dot{q})\rangle) = ((p, \dot{q}), \dot{r}).$$

We are going to find $p^*, \dot{q}^*$ and $\dot{r}^*$ such that $\langle((p^*, \emptyset), \dot{r}^*), (\emptyset, \dot{q}^*)\rangle \leq_{\mathbb{C} \times \mathbb{T}} \langle((p, \emptyset), \dot{r}), (\emptyset, \dot{q})\rangle$ and

$$\tau(\langle((p^*, \emptyset), \dot{r}^*), (\emptyset, \dot{q}^*)\rangle) = (p^*, \dot{q}^*), \dot{r}^*) \leq_{\mathbb{M} * \dot{\mathbb{R}}_w} (p', \dot{q}'), \dot{r}').$$

Let $p^* = p'$. Let $\dot{q}^*$ be a name such that

- $p^* \Vdash$ "$\dot{q}^* = \dot{q}'$".
- If $\tilde{p}$ is incompatible with p', then $\tilde{p} \Vdash$ "$\dot{q}^* = \dot{q}$".

Also set $\dot{r}^* = \dot{r}'$. Then $p^*, \dot{q}^*$ and $\dot{r}^*$ are as required.

(b) follows from the fact that $1_{\mathrm{Add}(\kappa, \lambda^+)} \Vdash$ "$\mathrm{Add}(\kappa^+, 1)$ is κ^+-closed".

(c) follows from the fact that $\mathrm{Add}(\kappa, \lambda^+)$ is κ^+-c.c. and Lemma 3.11. $\square$

Let $k : V^1 \to N^1$ witness the measurability of λ in V^1. As $|\mathbb{P}_\kappa| = \kappa < \lambda$, so by the Levy-Solovay's theorem [17], we can lift k to $k : V^1[G_\kappa] \to N^1[G_\kappa]$.

Let $\mathbb{M}^* * \dot{\mathbb{R}}^*_w = k(\mathbb{M} * \dot{\mathbb{R}}_w)$. The next lemma follows from Lemma 3.21.

Lemma 3.24. *(in $V^1[G_\kappa]$) $k \restriction \mathbb{M} * \dot{\mathbb{R}}_w : \mathbb{M} * \dot{\mathbb{R}}_w \to \mathbb{M}^* * \dot{\mathbb{R}}^*_w$ is a regular embedding.*

Proof. It is clear that k is order preserving and if $p \perp_{\mathbb{M} * \dot{\mathbb{R}}_w} q$ (p is incompatible with q in $\mathbb{M} * \dot{\mathbb{R}}_w$), then $k(p) \perp_{\mathbb{M}^* * \dot{\mathbb{R}}^*_w} k(q)$ ($k(p)$ and $k(q)$ are incompatible in $\mathbb{M}^* * \dot{\mathbb{R}}^*_w$). Now suppose that $A \subseteq \mathbb{M} * \dot{\mathbb{R}}_w$ is a maximal antichain in $\mathbb{M} * \dot{\mathbb{R}}_w$. By Lemma 3.21, $|A| < \lambda$ and so by elementarity of k, $k''[A] = k(A)$ is a maximal antichain in $\mathbb{M}^* * \dot{\mathbb{R}}^*_w$. $\square$

Thus let $g^* * K^*$ be $\mathbb{M}^* * \dot{\mathbb{R}}^*_w$-generic over $V^1[G_\kappa]$ such that $k''[g * K] \subseteq g^* * K^*$. It follows that we can lift k to

$$k : V^1[G_\kappa * g * K] \to N^1[G_\kappa * g^* * K^*].$$

Hence, in $V^1[G_\kappa]$, by Lemma 3.24, there is a projection

$$\pi : \mathbb{M}^* * \dot{\mathbb{R}}^*_w \to RO(\mathbb{M} * \dot{\mathbb{R}}_w),$$

where $RO(\mathbb{M} * \dot{\mathbb{R}}_w)$ denotes the Boolean completion of $\mathbb{M} * \dot{\mathbb{R}}_w$.

As in [8], given a condition $(((p, q), r) \in \mathbb{M}^* * \dot{\mathbb{R}}^*_w$, let us identify $\pi(p) = \pi((p, \emptyset), 1_{\mathbb{R}^*_w})$ with

$$(k^{-1})''[p] = p \restriction (\kappa \times \lambda) \cup \{((\gamma, \alpha), i) : \gamma < \kappa, \alpha \geq \lambda, i \in \{0, 1\}, ((\gamma, k(\alpha)), i) \in p\}.$$

Let $\mathbb{Q}_\pi$ be the quotient forcing determined by π:

$$\mathbb{Q}_\pi = \{((p, \dot{q}), \dot{r}) \in \mathbb{M}^* * \dot{\mathbb{R}}^*_w : \pi((p, \dot{q}), \dot{r}) \in g * K\}.$$

Let us define

$$\mathbb{C}_\pi = \{((p, \emptyset), r) : ((p, \emptyset), r) \in \mathbb{Q}_\pi\}$$

where the ordering is the one inherited from $\mathbb{Q}_\pi$, and let

$$\mathbb{T}_\pi = \{(\emptyset, q) \in \mathbb{M}^* : (\emptyset, q) \in g\},$$

with the ordering inherited from $\mathbb{M}^*$. Also define $\tau_\pi : \mathbb{C}_\pi \times \mathbb{T}_\pi \to \mathbb{Q}_\pi$ by

$$\tau_\pi(\langle ((p, \emptyset), r), (\emptyset, q) \rangle) = ((p, q), r).$$

This is well-defined.

Lemma 3.25. *τ_π is a projection from $\mathbb{C}_\pi \times \mathbb{T}_\pi$ onto $\mathbb{Q}_\pi$.*

Proof. The proof is similar to the proof of Lemma 3.23(a). Clearly τ_π is order preserving. Suppose that

$$(p', \dot{q}'), \dot{r}') \leq_{\mathbb{Q}_\pi} \tau(\langle((p, \emptyset), \dot{r}), (\emptyset, \dot{q})\rangle) = ((p, \dot{q}), \dot{r}).$$

We are going to find $p^*, \dot{q}^*$ and $\dot{r}^*$ such that $\langle((p^*, \emptyset), \dot{r}^*), (\emptyset, \dot{q}^*)\rangle \leq_{\mathbb{C}_\pi \times \mathbb{T}_\pi} \langle((p, \emptyset), \dot{r}), (\emptyset, \dot{q})\rangle$ and

$$\tau(\langle((p^*, \emptyset), \dot{r}^*), (\emptyset, \dot{q}^*)\rangle) = (p^*, \dot{q}^*), \dot{r}^*) \leq_{\mathbb{Q}_\pi} (p', \dot{q}'), \dot{r}').$$

Let $p^* = p'$. Let $\dot{q}^*$ be a name such that

- $p^* \Vdash$ "$\dot{q}^* = \dot{q}'$".
- If $\tilde{p}$ is incompatible with p', then $\tilde{p} \Vdash$ "$\dot{q}^* = \dot{q}$".

Also set $\dot{r}^* = \dot{r}'$. Then $p^*, \dot{q}^*$ and $\dot{r}^*$ are as required. □

Lemma 3.26. *$\mathbb{T}_\pi$ is κ^+-closed in $N^1[G_\kappa * g]$.*

Proof. Similar to the proof of Lemma 3.23(b). □

Also, as in the proof of 3.23(c), one can show that $\mathbb{C}_\pi$ is κ^+-c.c. in $N^1[G_\kappa * g * K]$. Here we prove something stronger, which is needed for the proof of Theorem 3.20.

Lemma 3.27. *$\mathbb{C}_\pi \times \mathbb{C}_\pi$ is κ^+-c.c. in $N^1[G_\kappa * g * K]$.*

Proof. The proof is similar to the proof of Lemma 3.8 from [8], but we need more work as we are using the more complicated Radin forcing $\mathbb{R}_w$ than the Prikry collapse forcing.

Assume towards contradiction that $A \in N^1[G_\kappa * g * K]$ is an antichain in $\mathbb{C}_\pi \times \mathbb{C}_\pi$ of size κ^+. Let $\langle(a_i^1, a_i^2) : i < \kappa^+\rangle$ be an enumeration of A, and for $i < \kappa^+$ and $k \in \{1, 2\}$ let us write a_i^k as

$$a_i^k = ((p_i^k, \emptyset), \check{d}_i^k{}^\frown\langle w, \lambda_i^k, \dot{A}_i^k, \dot{H}_i^k, \check{h}_i^k\rangle).$$

By shrinking A in necessary, we assume that there is a condition $((p, \dot{q}), \check{d}^\frown\langle w, \lambda, \dot{A}, \dot{H}, \check{h}\rangle) \in g * K$ which forces the following:

(1) $\dot{A}$ is an antichain.

(2) There exists t_1 such that all $i < \kappa^+, d_i^1 = t_1$.

(3) There exists t_2 such that all $i < \kappa^+, d_i^2 = t_2$.

(4) For some fixed $\eta_1 < \kappa$ and all $i < \kappa^+, \lambda_i^1 = \eta_1$.

(5) For some fixed $\eta_2 < \kappa$ and all $i < \kappa^+, \lambda_i^2 = \eta_2$.

(6) For some $f_1 \in \mathrm{Col}(\eta_1^{+5}, < \kappa)$ and all $i < \kappa^+, h_i^1 = f_1$.

(7) For some $f_2 \in \mathrm{Col}(\eta_2^{+5}, < \kappa)$ and all $i < \kappa^+, h_i^2 = f_2$.

For each i, choose $((p_i, \dot{q}_i), \check{d_i}^\frown \langle w, \lambda, \dot{A}_i, \dot{H}_i, \check{h}_i\rangle) \in g{*}K$ which extends $((p, \dot{q}), \check{d}^\frown \langle w, \lambda, \dot{A}, \dot{H}, \check{h}\rangle)$ and decides both a_i^1 and a_i^2, say it forces (for $k \in \{1, 2\}$)

$$a_i^k = ((p_i^k, \emptyset), \check{t_1}^\frown \langle w, \eta_k, \dot{A}_i^k, \dot{H}_i^k, \check{f}_k\rangle).$$

By further shrinking and extending the conditions, we may assume that for some s and all $i < \kappa^+, d = d_i = s$.

Let

$$((p_i^{*1}, \dot{q}_i^{*1}), \check{s}^\frown \langle w, \lambda, \dot{A}_i^{*1}, \dot{H}_i^{*1}, \check{h}_i^{*1}\rangle)$$

extends a_i^1, $((p_i, \dot{q}_i), \check{d_i}^\frown \langle w, \lambda, \dot{A}_i, \dot{H}_i, \check{h}_i\rangle)$ and $((p, \dot{q}), \check{d}^\frown \langle w, \lambda, \dot{A}, \dot{H}, \check{h}\rangle)$ and such that $\pi(p_i^{*1}) = (k^{-1})''[p_i^{*1}]$ is in the Cohen part of $g * K$. Similarly let

$$((p_i^{*2}, \dot{q}_i^{*2}), \check{s}^\frown \langle w, \lambda, \dot{A}_i^{*2}, \dot{H}_i^{*2}, \check{h}_i^{*2}\rangle)$$

extends a_i^2, $((p_i, \dot{q}_i), \check{d_i}^\frown \langle w, \lambda, \dot{A}_i, \dot{H}_i, \check{h}_i\rangle)$ and $((p, \dot{q}), \check{d}^\frown \langle w, \lambda, \dot{A}, \dot{H}, \check{h}\rangle)$ and such that $\pi(p_i^{*2}) = (k^{-1})''[p_i^{*2}]$ is in the Cohen part of $g * K$. Note that in particular $\pi(p_i^{*1}) \| \pi(p_i^{*2})$ ($\pi(p_i^{*1})$ is compatible with $\pi(p_i^{*2})$).

By Δ-system arguments, we can find $i < j$ such that $p_i^{*1} \| p_j^{*1}$ and $p_i^{*2} \| p_j^{*2}$. Let

$$g^1 = ((p_i^{*1}, \dot{q}_i^{*1}), \check{s}^\frown \langle w, \lambda, \dot{A}_i^{*1}, \dot{H}_i^{*1}, \check{h}_i^{*1}, \rangle) \wedge ((p_j^{*1}, \dot{q}_j^{*1}), \check{s}^\frown \langle w, \lambda, \dot{A}_j^{*1}, \dot{H}_j^{*1}, \check{h}_j^{*1}\rangle)$$

be the greatest lower bound of

$$((p_i^{*1}, \dot{q}_i^{*1}), \check{s}^\frown \langle w, \lambda, \dot{A}_i^{*1}, \dot{H}_i^{*1}, \check{h}_i^{*1}\rangle)$$

and

$$((p_j^{*1}, \dot{q}_j^{*1}), \check{s}^\frown \langle w, \lambda, \dot{A}_j^{*1}, \dot{H}_j^{*1}, \check{h}_j^{*1}\rangle).$$

Similarly let

$$g^2 = ((p_i^{*2}, \dot{q}_i^{*2}), \check{s}^\frown\langle w, \lambda, \dot{A}_i^{*2}, \dot{H}_i^{*2}, \check{h}_i^{*2}\rangle) \wedge ((p_j^{*2}, \dot{q}_j^{*2}), \check{s}^\frown\langle w, \lambda, \dot{A}_j^{*2}, \dot{H}_j^{*2}, \check{h}_j^{*2}\rangle).$$

Let

$$p' = \pi(p_i^{*1}) \cup \pi(p_j^{*1}) \cup \pi(p_i^{*2}) \cup \pi(p_j^{*2}),$$

which is well-defined. Let

$$g = ((p', \emptyset), \emptyset) \wedge ((p, \dot{q}), \check{d}^\frown\langle w, \lambda, \dot{A}, \dot{H}, \check{h}\rangle) \bigwedge_{l=i,j} ((p_l, \dot{q}_l), \check{d_l}^\frown\langle w, \lambda, \dot{A}_l, \dot{H}_l, \check{h}_l\rangle),$$

the greatest lower bound of the conditions considered. To continue, we need the following two claims:

Claim 3.28. *Assume that*

$$r = ((p, \dot{q}), \check{d}^\frown\langle w, \lambda, \dot{A}, \dot{H}, \check{h}\rangle) \in \mathbb{M} * \dot{\mathbb{R}}_w$$

and

$$r^* = ((p^*, \emptyset), \check{d^*}^\frown\langle w, \lambda, \dot{A}^*, \dot{H}^*, \check{h}^*\rangle) \in \mathbb{M}^* * \dot{\mathbb{R}}^*_w$$

and the following conditions are satisfies:

(1) $r \leq \pi(r^*)$.

(2) *Suppose that* $d = \langle d_0, \ldots, d_{n-1}\rangle$ *and* $d^* = \langle d^*_0, \ldots, d^*_{m-1}\rangle$. *Then* $n = m$ *and for all* $k < n, \kappa^{d_k} = \kappa^{d^*_k}$ *and* $\lambda^{d_k} = \lambda^{d^*_k}$.

(3) *For all* $k < n, h^{d_k} \leq h^{d^*_k}$.

(4) $h \leq h^*$.

Then r *does not force* r^* *out the quotient* $\mathbb{C}_\pi$.

Proof. Consider the conditions r and r^*. The above conditions imply that they are compatible, so let $r \wedge r^*$ be a common extension of them. Let $\bar{g} \times \bar{K}$ be $\mathbb{M}^* * \dot{\mathbb{R}}^*_w$-generic over $V^1[G_\kappa]$ such that $r \wedge r^* \in \bar{g} \times \bar{K}$. But then $\pi(r) \in \langle \pi''[\bar{g} \times \bar{K}]\rangle$, the filter on $\mathbb{M} * \dot{\mathbb{R}}_w$ generated by $\pi''[\bar{g} \times \bar{K}]$. The result follows immediately. □

Claim 3.29. *Assume* r *and* r^* *are as in Claim 3.28. Then there exists* $\bar{r} \leq^* r$ *such that* $\bar{r}$ *forces "*$r^* \in \mathbb{C}_\pi$*".*

Proof. By Lemma 3.13, there exists $\bar{r} \leq^* r$ which decides "$r^* \in \mathbb{C}_\pi$". By Claim 3.28, $\bar{r}$ cannot force "$r^* \notin \mathbb{C}_\pi$". So $\bar{r} \Vdash$"$r^* \in \mathbb{C}_\pi$". $\square$

Note that conditions g and g^1 satisfy the conditions in Claim 3.28, hence by Claim 3.29, there exists $\bar{g}_1 \leq^* g$ which forces "$g^1 \in \mathbb{C}_\pi$". Then $\bar{g}_1$ and g^2 satisfy the conditions in Claim 3.28, so again by Claim 3.29, there exists $\bar{g}_2 \leq^* \bar{g}_1$ which forces "$g^2 \in \mathbb{C}_\pi$". It follows that

$$\bar{g}_2 \Vdash \text{“} g^1, g^2 \in \mathbb{C}_\pi\text{”}.$$

But then

- $\bar{g}_2 \Vdash$"$(g^1, g^2) \in \mathbb{C}_\pi \times \mathbb{C}_\pi$".
- $\bar{g}_2 \Vdash$"$(g^1, g^2) \leq (a^1_i, a^2_i), (a^1_j, a^2_j)$".
- $\bar{g}_2 \leq ((p, \dot{q}), \check{d}^\frown\langle \kappa, \lambda, \dot{A}, \check{f}, \dot{F}\rangle)$.

It follows that $\bar{g}_2 \Vdash$"$\dot{A}$ is an antichain", and from the above, we get a contradiction. $\square$

To complete the argument, we need the following lemma.

Lemma 3.30. *(Unger)*

(a) *Assume κ is a regular cardinal and $\mathbb{P}$ is a forcing notion such that $\mathbb{P} \times \mathbb{P}$ is κ-c.c. Then forcing with $\mathbb{P}$ adds no new branches to κ-trees.*

(b) *Suppose $\mathbb{P}$ is κ^+-c.c. and preserves $\kappa, \mathbb{Q}$ is κ^+-closed and $2^\kappa > \kappa^+$. Also assume that $G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over V. If T is a κ^{++}-tree in $V[G]$, then in $V[G \times H], T$ has no new branches.*

Proof. For (a), see Fact 3.12 in [8]. For (b), see [23]. $\square$

We are now ready to complete the proof of Theorem 3.20. Note that by our assumption $\dot{T} \in V^1[G_\kappa]$ is an $\mathbb{M} * \dot{\mathbb{R}}_w$ name such that $T = \dot{T}[g * K]$ is a λ-Aronszajn tree in $V^1[G_\kappa * g * K]$. Also note that $\dot{T} \in N^1[G_\kappa]$.

By standard arguments, $k(T)_{<\lambda} = T$ and so T has a cofinal branch in $N^1[G_\kappa * g^* * K^*] \subseteq V^1[G_\kappa * g^* * K^*]$.

Suppose that $X \times Y$ is $\mathbb{C}_\pi \times \mathbb{Q}_\pi$-generic over $V^1[G_\kappa * g * K]$ so that $N^1[G_\kappa * g^* * K^*] \subseteq N^1[G_\kappa * g * K][X \times Y]$, which is possible by Lemma 3.25. It follows that T has a cofinal branch in $N[G * H * K][X \times Y]$. Now lemmas 3.26, 3.27 and 3.30 can be used to show that forcing with $\mathbb{C}_\pi \times \mathbb{Q}_\pi$ over $N^1[G_\kappa * g * K]$ does not add cofinal branches to T (see [8] for details). We get a contradiction and Theorem 3.20 follows.

3.6. **Completing the proof of Theorem 1.2.** In this subsection we complete the proof of Theorem 1.2, by showing that in the model $V^1[G_\kappa * g * K]$, $TP(\alpha^{++})$ holds for all singular cardinals $\alpha < \kappa$.

Recall that $C = \{\kappa_i : i < \kappa\}$ is the Radin club added by K and $\min(C) = \aleph_0$. Recall that G_κ is assumed to be $\mathbb{P}_\kappa$-generic over V^1. Let us write it as

$$G_\kappa = \langle\langle G_\alpha : \alpha \leq \kappa\rangle, \langle G(\alpha) : \alpha < \kappa\rangle\rangle,$$

which corresponds to the iteration

$$\mathbb{P}_\kappa = \langle\langle \mathbb{P}_\alpha : \alpha \leq \kappa\rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa\rangle\rangle.$$

By simple reflecting arguments, we have the following lemma.

Lemma 3.31. *The set $X \in \mathcal{F}_w$, where X consists of all those $u \in \mathcal{U}_\infty$ such that $\alpha = \kappa_u$ satisfies the following conditions:*

- *α is a measurable cardinal.*
- *$\mathbb{P}_\alpha$ is α-c.c. and of size α.*
- *α remains measurable after forcing with $\mathbb{P}_\alpha$ and $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{M}}(\alpha, \alpha_*, \alpha_*^+)$.*
- *Some elementary embedding $j : V^1 \to M^1$ with $crit(j) = \alpha$ can be extended to*

$$j : V^1[G_\alpha] \to M^1[j(G_\alpha)]$$

  and then to

$$j : V^1[G_\alpha * G(\alpha)] \to M^1[j(G_\alpha * G(\alpha))].$$

- *$\mathbb{P}_{\alpha+1} \Vdash$ "$\dot{\mathbb{P}}_{(\alpha+1,\kappa)}$ does not add any new subsets to α_*".*

Proof. It suffices to show that $\forall \alpha < \kappa^+, w \restriction \alpha \in j(X)$, which can be easily checked. □

Thus we can assume that

$$\aleph_0 < \alpha \in C \implies \alpha \in X.$$

On the other hand, if $\alpha < \kappa$ is a limit cardinal in $V^1[G_\kappa * g * K]$, then $\alpha \in \lim(C)$, the set of limit points of C, and $2^\alpha = \alpha^{+3}$. Thus the following completes the proof:

Theorem 3.32. *Assume $\alpha \in \lim(C)$. Then $V^1[G_\kappa * g * K] \models$"$TP(\alpha^{++})$".*

Proof. Fix $\alpha \in \lim(C)$, and let $\xi < \kappa$ be such that $\alpha = \kappa_\xi$. Note that ξ is a limit ordinal. We have

$$V^1[G_\kappa * g * K] = V^1[G_{\alpha+1}][G_{(\alpha+1,\kappa]}][g][[\vec{u} \restriction \xi, \vec{F} \restriction \xi][\vec{u} \restriction [\xi,\kappa), \vec{F} \restriction [\xi,\kappa)].$$

and the following hold:

(1) $V^1[G_\kappa * g * K]$ is a generic extension of $V^1[G_\kappa * g][\vec{u} \restriction \xi, \vec{F} \restriction \xi]$ by a forcing notion which does not add any new subsets to α_*[2].

(2) Forcing with $\mathbb{P}_{(\alpha+1,\kappa]} * \dot{\mathbb{M}}$ does not add any subsets to α_*; in particular, the forcing notion $\mathbb{R}_{u_\xi}$ is defined in the same way in the models $V^1[G_\kappa * g]$ and $V^1[G_{\alpha+1}]$.

It follows that $V^1[G_\kappa * g * K]$ is a generic extension of $V^1[G_{\alpha+1}][\vec{u} \restriction \xi, \vec{F} \restriction \xi]$, by a forcing notion which does not add any new subsets to α_*. Also note that

(3) $V^1[G_{\alpha+1}][\vec{u} \restriction \xi, \vec{F} \restriction \xi] \models$"$\alpha^{++} = \alpha_*$".

Thus it suffices to prove the following:

Lemma 3.33. *Tree property at α_* holds in the generic extension $V^1[G_{\alpha+1}][\vec{u} \restriction \xi, \vec{F} \restriction \xi]$, which is obtained using the forcing notion*

$$\mathbb{P}_{\alpha+1} * \dot{\mathbb{R}}_{u_\xi} = \mathbb{P}_\alpha * \dot{\mathbb{M}}_\alpha * \dot{\mathbb{R}}_{u_\xi}.$$

Proof. The proof is very similar to the proof of Theorem 3.20. □

This completes the proof of Theorem 3.32. □

Theorem 1.2 follows.

4. Tree property at $\aleph_{2n}$'s and $\aleph_{\omega+2}$

In this section we prove Theorem 1.4. Thus assume that GCH holds, $\eta > \lambda$ are measurable cardinals above κ. We assume that they are the least such cardinals. Suppose κ is an $H(\eta)$-hypermeasurable cardinal. Let $j : V \to M \supseteq H(\eta)$ witness this. We may assume that it is generated by a (κ,η)-extender. Let $i : V \to N$ be the ultrapower embedding derived from j and let $k : N \to M$ be such that $j = k \circ i$.

[2] Recall that α_* is the least measurable cardinal above α.

The next lemma can be proved as in Lemma 3.18

Lemma 4.1. *Then there exists a cofinality preserving generic extension V^1 of V satisfying the following conditions:*

(a) $V^1 \models$ *"GCH".*

(b) *There is $j^1 : V^1 \to M^1$ with critical point κ such that $H(\eta) \subseteq M^1$ and $j^1 \restriction V = j$.*

(c) *j^1 is generated by a (κ, η)-extender.*

(d) *If U^1 is the normal measure derived from j^1 and if $i^1 : V^1 \to N^1 \simeq \mathrm{Ult}(V^1, U^1)$ is the ultrapower embedding, then there exists $\bar{g} \in V^1$ which is $i^1(\mathrm{Add}(\kappa, \lambda)_{V^1})$-generic over N^1. Further $i^1 \restriction V = i$.*

Let V^1 be the model constructed above. We need the following lemma which is an analogue of Lemma 3.19.

Lemma 4.2. *Work in V^1. There exists a forcing iteration $\mathbb{P}_\kappa$ of length κ such that if $G_\kappa * g * h$ is $\mathbb{P}_\kappa * \dot{\tilde{\mathbb{M}}}(\kappa, \lambda) * \dot{\tilde{\mathbb{M}}}(\lambda, \eta)$-generic over V^1, then in $V^2 = V^1[G_\kappa * g * h]$, the following holds:*

(a) *There is $j^2 : V^2 \to M^2$ with critical point κ such that $j^2 \restriction V^1 = j^1$.*

(b) *j^2 is generated by a (κ, η)-extender.*

(c) $V^2 \models$ *"$\lambda = \kappa^{++} + \eta = \kappa^{+4} + TP(\lambda) + TP(\eta)$".*

(d) *If U^2 is the normal measure derived from j^2 and if $i^2 : V^2 \to N^2 \simeq \mathrm{Ult}(V^2, U^2)$ is the ultrapower embedding, then there exists $F \in V^2$ which is $\dot{\tilde{\mathbb{M}}}(\kappa^{+4}, i^2(\kappa))_{N^2}$-generic over N^2.*

Proof. We follow the proof of Lemma 3.19. For an ordinal $\alpha \leq \kappa$ let α_* and α_{**} denote the first and second measurable cardinals above α. Note that $\kappa_* = \lambda$ and $\kappa_{**} = \eta$.

Work in V^1. Factor j^1 in two steps through the models

$$N' = \text{ the transitive collapse of } \{j^1(f)(\kappa) : f : \kappa \to V^1\}$$

$$\bar{N}' = \text{ the transitive collapse of } \{j^1(f)(\alpha) : f : \kappa \to V^1, \alpha < \lambda\}.$$

Again, note that N' is the familiar ultrapower approximating M^1, while $\bar{N}'$ corresponds to the extender of length λ. We have maps

$$i' : V^1 \to N',$$

$$k' : N' \to M^1,$$

$$\bar{i}' : N' \to \bar{N}',$$

$$\bar{k}' : \bar{N}' \to M^1$$

such that

$$k' \circ i' = j^1 \ \& \ \bar{k}' \circ \bar{i}' = k'.$$

Let

$$\mathbb{P}_\kappa = \langle \langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle \rangle$$

be the reverse Easton iteration, where

(1) If $\alpha < \kappa$ is a measurable limit of measurable cardinals, then $\Vdash_\alpha$"$\dot{\mathbb{Q}}_\alpha = \dot{\tilde{\mathbb{M}}}(\alpha, \alpha_*) * \dot{\tilde{\mathbb{M}}}(\alpha_*, \alpha_{**})$".

(2) Otherwise, $\Vdash_\alpha$"$\dot{\mathbb{Q}}_\alpha$ is the trivial forcing".

Let

$$G_k = \langle \langle G_\alpha : \alpha \leq \kappa \rangle, \langle G(\alpha) : \alpha < \kappa \rangle \rangle$$

be be $\mathbb{P}_\kappa$-generic over V^1.

Note that we can factor $\mathbb{P}_\kappa * \dot{\tilde{\mathbb{M}}}(\kappa, \lambda) * \dot{\tilde{\mathbb{M}}}(\lambda, \eta)$ as

$$\mathbb{P}_\kappa * \dot{\tilde{\mathbb{M}}}(\kappa, \lambda) * \dot{\tilde{\mathbb{M}}}(\lambda, \eta) = \mathbb{P}_\kappa * \mathrm{Add}(\kappa, \lambda) * \dot{\mathbb{Q}},$$

where $\dot{\mathbb{Q}}$ is forced to be κ^+-distributive. So the arguments of the proof of Lemma 3.19 can be used to get the embeddings

$$\bar{j} : V^1[G_\kappa * g * h] \to M^1[j^1(G_\kappa * g * h)],$$

$$\bar{k}' : \bar{N}'[l(G_\kappa * g * h)] \to M^1[j^1(G_\kappa * g * h)],$$

$$l : V^1[G_\kappa * g * h] \to \bar{N}'[l(G_\kappa * g * h)],$$

together a filter F which is $\widetilde{\mathbb{M}}(\kappa^{+4}, i^2(\kappa))_{N'[i'(G_\kappa)]}$ generic over $N'[i'(G_\kappa)]$.

Let

$$V^2 = V^1[G_\kappa * g * h],$$

$$M^2 = M^1[j^1(G_\kappa * g * h)]$$

and

$$N^2 = \bar{N}'[l(G_\kappa * g * h)].$$

Also let $j^2 = \bar{j}$. We argue

$$\mathrm{Ult}(V[G_\kappa * g * h], U^2) \simeq N^2,$$

where U^2 is the normal measure derived from j^2. To see this, factor l through $l^\dagger : V^2 \to N^\dagger \simeq \mathrm{Ult}(V^2, U')$, where U' is the normal measure derived from l. Also let $k^\dagger : N^\dagger \to N^2$. Then $P(\kappa)_{V^2} \subseteq N^\dagger$ and $N^\dagger \models$"$2^\kappa = \lambda$". So $\mathrm{crit}(k^\dagger) > \lambda$. Since $\lambda \subseteq \mathrm{range}(k^\dagger)$ and N^2 is generated by a (κ, λ)-extender, we have $N^2 = N^\dagger$ and we are done. So if we let $i^2 = l$, then

$$i^2 : V^2 \to N^2$$

is the ultrapower embedding. Finally note that F is generic for the appropriate ordering. The lemma follows. $\square$

Also note that $F \in M^2$. Now, working in $V^2 = V^1[G_\kappa * g * h]$, we would like to define a version of Prikry forcing. Set

$$P^* = \{f : \kappa \to V^2_\kappa \mid \mathrm{dom}(f) \in U^2 \text{ and } \forall \alpha, f(\alpha) \in \widetilde{\mathbb{M}}(\alpha^{+4}, \kappa)\}.$$
$$F^* = \{f \in P^* \mid i(f)(\kappa) \in F\}.$$

Now define the notion of a constructing pair as in Definition 3.2, where the forcing notions $\mathrm{Col}(\kappa^{+5}, < i(\kappa))_N$ and $\mathrm{Col}(\kappa^{+5}, < j(\kappa))_M$ are replaced by $\widetilde{\mathbb{M}}(\kappa^{+4}, i(\kappa))_N$ and $\widetilde{\mathbb{M}}(\kappa^{+4}, j(\kappa))_M$ respectively. Then definitions 3.3-3.5 go in the same way.

We are going to define our desired notion forcing in a similar way to what we defined in Section 3, but using the different guiding generic filters we obtained above. For this aim, and as before, we define forcing notions $\mathbb{P}_w, w \in \mathcal{U}_\infty$, which are the building blocks of our main forcing notion.

Definition 4.3. *If $w \in \mathcal{U}_\infty$, then $\mathbb{P}_w$ is the set of tuples $p = \langle w, \lambda, A, H, h \rangle$ such that*

1. *w is a measure sequence.*
2. *$\lambda < \kappa_w$ is measurable.*
3. *$A \in \mathcal{F}_w$.*
4. *$H \in F^*_w$ with $\mathrm{dom}(H) = \{\kappa_v > \lambda \mid v \in A\}$.*
5. *$h \in \widetilde{\mathbb{M}}(\lambda^{+4}, \kappa_w)$.*

Note that if $\mathrm{lh}(w) = 1$, then the above tuple is of the form $(w, \lambda, \emptyset, \emptyset, h)$ (where $\lambda < \kappa_w$ and $h \in \widetilde{\mathbb{M}}(\lambda^{+4}, \kappa_w)$).

The forcing notion $\mathbb{R}_w$ is defined in the same way as before:

Definition 4.4. *If w is a measure sequence, then $\mathbb{R}_w$ is the set of finite sequences*

$$p = \langle p_k \mid k \leq n \rangle,$$

where

(1) $p_k = (w_k, \lambda_k, A_k, H_k, h_k) \in \mathbb{P}_w$, *for each* $k \leq n$.

(2) $w_n = w$.

(3) $\lambda_0 = \omega$.

(4) *If* $k < n$, *then* $\lambda_{k+1} = \kappa_{w_k}$.

Given $p \in \mathbb{R}_w$ as above, we call n, the length of p and denote it by $\mathrm{lh}(p)$. The order relations $\leq^*$ and $\leq$ are defined as before.

Now assume that u is the measure sequence constructed using (j^2, F) and set $w = u \restriction 2$. Let $\mathbb{R} = \mathbb{R}_w$. Let K be $\mathbb{R}$-generic over $V^1[G_\kappa * g * h]$. Let C be the ω-sequence added by K and let $\vec{\kappa} = \langle \kappa_n : n < \omega \rangle$ enumerate C in the increasing order. Then $\kappa_0 = \aleph_0$ and $\sup_{n<\omega} \kappa_n = \kappa$. Also let $\vec{F} = \langle F_n : n < \omega \rangle$ be the ω-sequence added by K, where each F_n is $\widetilde{\mathbb{M}}(\kappa_n^{+4}, \kappa_{n+1})$-generic over $V^1[G_\kappa * g * h]$. The following lemma summarizes the basic properties of $\mathbb{R}$.

Lemma 4.5. (a) $(\mathbb{R}, \leq)$ *satisfies the* κ^+*-c.c.*

(b) *Assume* $p \in \mathbb{R}$ *and* $m < n^p$. *Then*

$$\mathbb{R}/p \simeq \prod_{i \leq m} \widetilde{\mathbb{M}}(\kappa_i^{+4}, \kappa_{i+1}) \times \mathbb{R}/p^{>m},$$

where $p^{>m} = \langle p_{m+1}, \ldots, p_{\mathrm{lh}(p)} \rangle$.

(c) $(\mathbb{R}, \leq, \leq^*)$ *satisfies the Prikry property.*

(d) $V^1[G_\kappa * g * h * K] = V^1[G_\kappa * g * h][\vec{F}]$.

(e) *In* $V^1[G_\kappa * g * h * K], \kappa = \aleph_\omega,\ \lambda = \aleph_{\omega+2}$ *and* $\eta = \aleph_{\omega+4}$.

(f) $Card^{V^1[G_\kappa * g * h * K]} \cap \kappa = \{\kappa_n, \kappa_n^+, \kappa_n^{++}, \kappa_n^{+3}, \kappa_n^{+4}, \kappa_n^{+5} : n < \omega\}$.

Recall that, given a cardinal $\alpha \leq \kappa$, we are using α_* to denote the least measurable cardinal above α and α_{**} to denote the second measurable cardinal above α; so that $\alpha_{**} =$

$(\alpha_*)_*$. The next lemma can be proved by the same arguments as in [8] (and using Lemma 2.5); see also Theorem 3.20:

Lemma 4.6. *In $V^1[G_\kappa * g * h * K]$, the tree property holds at $\aleph_{\omega+2}$.*

We now show that the tree property holds at all $\aleph_{2n}$'s, $0 < n < \omega$. The next lemma can be proved by simple reflection arguments.

Lemma 4.7. *The set $X \in \mathcal{F}_w$, where X consists of cardinals $\alpha < \kappa$ such that*

(1) $\mathbb{P}_\alpha$ *is α-c.c. and of size α.*
(2) $\mathbb{P}_\alpha \Vdash$ " $\mathbb{P}(\alpha) = \widetilde{\mathbb{M}}(\alpha^{+4}, \alpha_*) * \dot{\widetilde{\mathbb{M}}}(\alpha_*^{+4}, \alpha_{**})$ ".
(3) *α remains measurable after forcing with $\mathbb{P}_\alpha$ and $\mathbb{P}_{\alpha+1}$.*
(4) *Some elementary embedding $j : V^1 \to M^1$ with $crit(j) = \alpha$ can be extended to*

$$j : V^1[G_\alpha] \to M^1[j(G_\alpha)]$$

and then to

$$j : V^1[G_\alpha * G(\alpha)] \to M^1[j(G_\alpha * G(\alpha))].$$

(5) $\mathbb{P}_{\alpha+1} \Vdash$ "$\dot{\mathbb{P}}_{(\alpha+1,\kappa)}$ *does not add any new subsets to* α_{**}^{+4}".
(6) $\forall \gamma < \alpha, \mathbb{P}_{(\gamma,\alpha]} \times \widetilde{\mathbb{M}}(\gamma, \alpha) \Vdash$ " $\alpha = \gamma^{++} + TP(\alpha)$ ".

So we assume that each $\kappa_n \in X$.

The next lemma follows from Lemma 4.5(f).

Lemma 4.8. *In $V^1[G_\kappa * g * h * K]$,*

$$\{\aleph_{2n} : n < \omega\} = \{\kappa_m : m < \omega\} \cup \{\kappa_m^{++} : m < \omega\} \cup \{\kappa_m^{+4} : m < \omega\}.$$

We now prove, in a sequence of lemmas that the tree property holds at all $\aleph_{2n}$'s, $n < \omega$. The case $\kappa_0 = \aleph_0$ follows from König's theorem stated in the introduction. We start with the simple case of the tree property at κ_{m+1}.

Lemma 4.9. *For each m, $V^1[G_\kappa * g * h * K] \models$ "$TP(\kappa_{m+1})$".*

Proof. We can write

$$V^1[G_\kappa * g * h * K] = V^1[G_\kappa * g * h][\langle F_i : i < m\rangle][F_m][\langle F_i : m < i < \omega\rangle].$$

Working in $V^1[G_\kappa * g * h][\langle F_i : i < m\rangle]$, the filter F_m is generic filter for the forcing notion $\widetilde{\mathbb{M}}(\kappa_m^{+4}, \kappa_{m+1})$, so by Lemma 2.4($c$).

$$V^1[G_\kappa * g * h][\langle F_i : i < m\rangle][F_m] \models \text{ `` } TP(\kappa_{m+1}) \text{ ''}.$$

But $V^1[G_\kappa * g * h * K]$ is a generic extension of $V^1[G_\kappa * g * h][\langle F_i : i < m\rangle][F_m]$ by a forcing notion which does not add new subsets to κ_{m+1}, and so

$$V^1[G_\kappa * g * h * K] \models \text{ `` } TP(\kappa_{m+1}) \text{ ''}.$$

□

Next we consider cardinals κ_m^{++}.

Lemma 4.10. *For each m, $V^1[G_\kappa * g * h * K] \models$ "$TP(\kappa_m^{++})$".*

Proof. We have

$$V^1[G_\kappa * g * h * K] = V^1[G_{\kappa_m}][G(\kappa_m)][G_{(\kappa_m+1,\kappa])}][g * h][\langle F_i : i < m\rangle][\langle F_i : m \le i < \omega\rangle].$$

Since the forcing notion $\prod_{i<m} \widetilde{\mathbb{M}}(\kappa_i^{+4}, \kappa_{i+1})$ is defined in the same way in the models $V^1[G_{\kappa_m}]$ and $V^1[G_\kappa * g * h]$, so

$$V^1[G_\kappa * g * h * K] = V^1[G_{\kappa_m}][\langle F_i : i < m\rangle][G(\kappa_m)][G_{(\kappa_m+1,\kappa]}][g * h][\langle F_i : m \le i < \omega\rangle].$$

By our convention $\kappa_m \in X$, $G(\kappa_m)$ is generic for $\widetilde{\mathbb{M}}(\kappa_m^{+4}, (\kappa_m)_*) * \dot{\widetilde{\mathbb{M}}}((\kappa_m)_*^{+4}, (\kappa_m)_{**})$, so by Lemma 2.5,

$$V^1[G_{\kappa_m}][\langle F_i : i < m\rangle][G(\kappa_m)] \Vdash \text{ ``}\kappa_m^{++} = (\kappa_m)_* \ + \ TP(\kappa_m^{++}) \text{ ''}.$$

But $V^1[G_\kappa * g * h * K]$ is a generic extension of $V^1[G_{\kappa_m}][\langle F_i : i < m\rangle][G(\kappa_m)]$ by a forcing notion which does not add any new subsets to $(\kappa_m)_*$, and so $V^1[G_\kappa * g * h * K] \models$"$TP(\kappa_m^{++})$".

□

Now we consider the cardinals κ_m^{+4}.

Lemma 4.11. *For each m, $V^1[G_\kappa * g * h * K] \models$ "$TP(\kappa_m^{+4})$".*

Proof. As above,

$$V^1[G_\kappa * g * h * K] = V[G_{\kappa_m}][\langle F_i : i < m\rangle][G(\kappa_m)][G_{(\kappa_m+1,\kappa]}][g * h][F_m][\langle F_i : m < i < \omega\rangle].$$

But $\mathbb{M}(\kappa_m^{+4}, \kappa_{m+1})$ is defined in the same way in the models $V^1[G_{\kappa_m}][\langle F_i : i < m\rangle][G(\kappa_m)]$ and $V^1[G_\kappa * g * h * K]$, so $V^1[G_\kappa * g * h * K]$ is equal to

$$V^1[G_{\kappa_m}][\langle F_i : i < m\rangle][G(\kappa_m)][G_{(\kappa_m+1,\kappa_{m+1})} \times F_m][G_{(\kappa_{m+1}+1,\kappa]}][g * h][\langle F_i : m < i < \omega\rangle].$$

Since $V^1[G_\kappa * g * h * K]$ is obtained from $V^1[G_{\kappa_m}][\langle F_i : i < m\rangle][G(\kappa_m)][G_{(\kappa_m+1,\kappa_{m+1})} \times F_m]$ by a forcing which does not add new subsets to $(\kappa_m)_{**}(= ((\kappa_m)^{+4})^{V^1[G*g*h*K]})$, it suffices to show $TP((\kappa_m)_{**})$ holds in $V^1[G_{\kappa_m}][\langle F_i : i < m\rangle][G(\kappa_m)][G_{(\kappa_m+1,\kappa_{m+1})} \times F_m]$. Now, the model $V^1[G_{\kappa_m}][\langle F_i : i < m\rangle][G(\kappa_m)][G_{(\kappa_m+1,\kappa_{m+1})} \times F_m]$ is a generic extension of $V^1[G_{\kappa_m}][\langle F_i : i < m\rangle]$ by the forcing notion

$$\mathbb{P}(\kappa_m) * (\dot{\mathbb{P}}_{(\kappa_m+1,\kappa_{m+1})} \times \dot{\tilde{\mathbb{M}}}(\kappa_m^{+4}, \kappa_{m+1})),$$

and by Lemmas 2.6 and 2.7,

$$V^1[G_{\kappa_m}][\langle F_i : i < m\rangle][G(\kappa_m)][G_{(\kappa_m+1,\kappa_{m+1})} \times F_m] \models \text{“ } TP((\kappa_m)_{**}) \text{ ”}.$$

The lemma follows. □

Putting the above lemmas together, give us a proof of Theorem 1.4.

5. Tree property at all regular even cardinals

In this section we prove Theorem 1.5. In Subsection 5.1, we present some of the basic properties of the new version of Radin forcing we defined in Section 4. Then in Subsection 5.2, we define the forcing notion needed which is used for the proof of our main theorem. Finally in Subsection 5.3, we complete the proof of Theorem 1.5.

5.1. **A new variant of Radin forcing.** Through this subsection, we assume that the following conditions are satisfied:

- κ is an $H(\kappa^{++})$-hypermeasurable cardinal, $2^\kappa = \kappa^{++}$ and $2^{\kappa^{++}} = \kappa^{+4}$.
- There is $j : V \to M$ with critical point κ such that $H(\kappa^{++}) \subseteq M$.
- j is generated by a (κ, κ^{+5})-extender.

- If U is the normal measure derived from j and if $i : V \to N \simeq \mathrm{Ult}(V, U)$ is the ultrapower embedding, then there exists $F \in V$ which is $\widetilde{\mathbb{M}}(\kappa^{+4}, i(\kappa))_N$-generic over N.

Let $\mathbb{R}_w$ be the modified version of Radin forcing that we defined in Section 4, and let us review it basic properties in the general context. The next lemma can be proved as in Lemma 3.11

Lemma 5.1. $\mathbb{R}_w$ *satisfies the* κ_w^+*-chain condition.*

The following is an analogue of the factorization lemma 3.12.

Lemma 5.2. *(The factorization lemma) Suppose that* $p = \langle p_0, \dots, p_n \rangle \in \mathbb{R}_w$ *where* $p_i = \langle \kappa_i, \lambda_i, A_i, f_i, F_i \rangle$ *and* $m < n$. *Set* $p^{\leq m} = \langle p_0, \dots, p_m \rangle$ *and* $p^{>m} = \langle p_{m+1}, \dots, p_n \rangle$.

(a) $p^{\leq m} \in \mathbb{R}_{w \restriction \kappa_m + 1}$, $p^{>m} \in \mathbb{R}_w$ *and there exists*

$$i : \mathbb{R}_w / p \to \mathbb{R}_{w \restriction \kappa_m + 1} / p^{\leq m} \times \mathbb{R}_w / p^{>m}$$

which is an isomorphism with respect to both $\leq^*$ *and* $\leq$.

(b) *If* $m + 1 < n$, *then there exists*

$$i : \mathbb{R}_w / p \to \mathbb{R}_{w \restriction \kappa_m + 1} / p^{\leq m} \times \widetilde{\mathbb{M}}(\kappa_m^{+4}, \kappa_{m+1}) \times \mathbb{R}_w / p^{>m+1}$$

which is an isomorphism with respect to both $\leq^*$ *and* $\leq$. □

The following can be proved as before:

Lemma 5.3. (a) $(\mathbb{R}_w, \leq, \leq^*)$ *satisfies the Prikry property.*

(b) *Assume* $\mathrm{lh}(w) = \kappa_w^+$ *Then forcing with* $\mathbb{R}_w$ *preserves the inaccessibility of* κ_w.

From now on assume that $\mathrm{lh}(w) = \kappa^+$. Suppose $K \subseteq \mathbb{R}_w$ is generic over V and define the club C and the sequence $\vec{u} = \langle u_\xi : \xi < \kappa \rangle$ and $\vec{\kappa} = \langle \kappa_\xi : \xi < \kappa \rangle$ as before. Let the sequence $\vec{F} = \langle F_\xi : \xi < \kappa \rangle$ be such that each F_ξ is $\widetilde{\mathbb{M}}(\kappa_\xi^{+4}, \kappa_{\xi+1})$-generic over V, which is produced by K. The next lemma can be proved as in Lemma 3.14.

Lemma 5.4. (a) $V[K] = V[\vec{u}, \vec{F}]$.

(b) *For every limit ordinal* $\xi < \kappa$, $\langle \vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi \rangle$ *is* $\mathbb{R}_{u_\xi}$*-generic over* V, *and* $\langle \vec{u} \upharpoonright [\xi, \kappa), \vec{F} \upharpoonright [\xi, \kappa) \rangle$ *is* $\mathbb{R}_w$*-generic over* $V[\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi]$.

(c) *For every* $\gamma < \kappa$ *and every* $A \subseteq \gamma$ *with* $A \in V[\vec{u}, \vec{F}]$, *we have* $A \in V[\vec{u} \upharpoonright \xi, \vec{F} \upharpoonright \xi]$, *where* ξ *is the least ordinal such that* $\gamma < \kappa_\xi$.

5.2. **The final model.** In this subsection we define the final model we are going to work with. Thus assume that GCH holds, $\eta > \lambda$ are measurable cardinals above κ. We assume that they are the least such cardinals. Suppose κ is an $H(\eta^+)$-hypermeasurable cardinal. Let $j : V \to M \supseteq H(\eta^+)$ witness this. We may assume that it is generated by a (κ, η^+)-extender. Let $i : V \to N$ be the ultrapower embedding derived from j and let $k : N \to M$ be such that $j = k \circ i$.

The next lemma can be proved as in Lemma 3.18

Lemma 5.5. *Then there exists a cofinality preserving generic extension* V^1 *of* V *satisfying the following conditions:*

(a) $V^1 \models$ *"GCH"*.

(b) *There is* $j^1 : V^1 \to M^1$ *with critical point* κ *such that* $H(\eta^+) \subseteq M^1$ *and* $j^1 \upharpoonright V = j$.

(c) j^1 *is generated by a* (κ, η^+)*-extender.*

(d) *If* U^1 *is the normal measure derived from* j^1 *and if* $i^1 : V^1 \to N^1 \simeq \mathrm{Ult}(V^1, U^1)$ *is the ultrapower embedding, then there exists* $\bar{g} \in V^1$ *which is* $i^1(\mathrm{Add}(\kappa, \lambda)_{V^1})$*-generic over* N^1. *Further* $i^1 \upharpoonright V = i$.

Lemma 5.6. *Work in* V^1. *There exists a forcing iteration* $\mathbb{P}_\kappa$ *of length* κ *such that if* $G_\kappa * g * h$ *is* $\mathbb{P}_\kappa * \dot{\tilde{\mathbb{M}}}(\kappa, \lambda) * \dot{\tilde{\mathbb{M}}}(\lambda, \eta)$*-generic over* V^1, *then in* $V^2 = V^1[G_\kappa * g * h]$, *the following holds:*

(a) $V^2 \models$ *"*$\lambda = \kappa^{++} + \eta = \kappa^{+4} + TP(\lambda) + TP(\eta)$*"*.

(b) *There is* $j^2 : V^2 \to M^2$ *with critical point* κ *and* $H(\kappa^{++}) \subseteq M^2$ *such that* $j^2 \upharpoonright V^1 = j^1$.

(c) j^2 *is generated by a* (κ, η^+)*-extender.*

(d) *If* U^2 *is the normal measure derived from* j^2 *and if* $i^2 : V^2 \to N^2 \simeq \mathrm{Ult}(V^2, U^2)$ *is the ultrapower embedding, then there exists* $F \in V^2$ *which is* $\dot{\tilde{\mathbb{M}}}(\kappa^{+4}, i^2(\kappa))_{N^2}$*-generic over* N^2.

Proof. The model $V^2 = V^1[G_\kappa * g * h]$ constructed in the proof of Lemma 4.2 does the job. The additional assumption of κ being $H(\eta^+)$-hypermeasurable guarantees that $H(\kappa^{++}) \subseteq M^2$. □

In particular, note that in the model V^2, the hypotheses at the beginning of Subsection 5.1 are satisfied; so we can consider the forcing notion $\mathbb{R}_w$, where $w = u \restriction \kappa^+$ and u is constructed using the pair (j^2, F). Let K be $\mathbb{R}_w$-generic over V^2. Build the sequences $\vec{\kappa} = \langle \kappa_\xi : \xi < \kappa \rangle, \vec{u} = \langle u_\xi : \xi < \kappa \rangle$ and $\vec{F} = \langle F_\xi : \xi < \kappa \rangle$ from K, as before.

5.3. **In $V^1[G_\kappa * g * h * K]$, the tree property holds at all regular even cardinals below κ.** Here we complete the proof of Theorem 1.5. As before, given a cardinal $\alpha \leq \kappa$, let α_* denote the least measurable cardinal above α and let α_{**} denote the second measurable cardinal above α. Now note that

$$Card^{V^1[G_\kappa * g * h * K]} \cap \kappa = \{\kappa_\xi, \kappa_\xi^+ : \xi < \kappa\} \cup \{(\kappa_\xi)_*, (\kappa_\xi)_*^+ : \xi < \kappa\} \cup \{(\kappa_\xi)_{**}, (\kappa_\xi)_{**}^+ : \xi < \kappa\},$$

Also note that if $\alpha < \kappa$ is a singular cardinal in $V^1[G_\kappa * g * h * K]$, then $\alpha \in \lim(C)$, i.e., $\alpha = \kappa_\xi$ for some limit ordinal $\xi < \kappa$. The following lemma is immediate:

Lemma 5.7. *In $V^1[G_\kappa * g * h * K]$, the set of uncountable regular even cardinals below κ is equal to*

$$\{\kappa_\xi^{++} : \xi < \kappa\} \cup \{\kappa_\xi^{+4} : \xi < \kappa\} \cup \{\kappa_{\xi+1} : \xi < \kappa\}.$$

Before we continue, let us show that we can choose the cardinals κ_ξ in a suitable way, which is guaranteed by the following lemma, which is an analogue of Lemma 3.31.

Lemma 5.8. *The set $X \in \mathcal{F}_w$, where X consists of all those $u \in \mathcal{U}_\infty$ such that $\alpha = \kappa_u$ satisfies the following conditions:*

(1) *$\mathbb{P}_\alpha$ is α-c.c. and of size α.*

(2) *$\mathbb{P}_\alpha \Vdash$ " $\mathbb{P}(\alpha) = \widetilde{\mathbb{M}}(\alpha, \alpha_*) * \dot{\widetilde{\mathbb{M}}}(\alpha_*, \alpha_{**})$ ".*

(3) *α remains measurable after forcing with $\mathbb{P}_\alpha$ and $\mathbb{P}_{\alpha+1}$.*

(4) *Some elementary embedding $j : V^1 \to M^1$ with $crit(j) = \alpha$ can be extended to*

$$j : V^1[G_\alpha] \to M^1[j(G_\alpha)]$$

and then to

$$j : V^1[G_\alpha * G(\alpha)] \to M^1[j(G_\alpha * G(\alpha))].$$

(5) $\mathbb{P}_{\alpha+1} \Vdash$ " $\dot{\mathbb{P}}_{(\alpha+1,\kappa)}$ *does not add any new subsets to* α_{**}^{+4} ".

(6) $\forall\gamma < \alpha, \mathbb{P}_{(\gamma,\alpha]} \times \widetilde{\mathbb{M}}(\gamma, \alpha) \Vdash$ " $\alpha = \gamma^{++} + TP(\alpha)$ ".

The next lemma can be proved as in Theorems 3.20 and 3.32, combined with ides of the proof of Lemma 4.10.

Lemma 5.9. $V^1[G_\kappa * g * h * K] \models$ *"For all limit ordinals* $\xi < \kappa, \kappa_\xi^{++} = (\kappa_\xi)_*$ *and* $TP((\kappa_\xi)_*)$ *holds".*

Before we continue, let us make a simple remark. Assume $\bar{\xi}$ is a limit ordinal. Then we can write $V^1[G_\kappa * g * h * K]$ as

$$V[G_\kappa * g * h][\vec{u} \restriction \bar{\xi}, \vec{F} \restriction \bar{\xi}][\vec{u} \restriction [\bar{\xi}, \bar{\xi} + \omega), \vec{F} \restriction [\bar{\xi}, \bar{\xi} + \omega)][\vec{u} \restriction [\bar{\xi} + \omega, \kappa), \vec{F} \restriction [\bar{\xi} + \omega, \kappa)].$$

On the other hand:

(1) $V^1[G_\kappa * g * h * K]$ is a generic extension of $V^1[G_\kappa * g * h][\vec{u} \restriction \bar{\xi}, \vec{F} \restriction \bar{\xi}][\vec{u} \restriction [\bar{\xi}, \bar{\xi} + \omega), \vec{F} \restriction [\bar{\xi}, \bar{\xi} + \omega)]$ by a forcing notion which does not add any new subsets to $\kappa_{\bar{\xi}+\omega}$.

(2) By standard arguments, $V^1[G_\kappa * g * h][\vec{u} \restriction \bar{\xi}, \vec{F} \restriction \bar{\xi}][\vec{u} \restriction [\bar{\xi}, \bar{\xi} + \omega), \vec{F} \restriction [\bar{\xi}, \bar{\xi} + \omega)]$ is generic extension of $V^1[G_\kappa * g * h][\vec{u} \restriction \bar{\xi}, \vec{F} \restriction \bar{\xi}]$ by a forcing notion which is forcing equivalent to the forcing notion $\mathbb{R}$ of Section 4 (for suitable choices of the normal measure and guiding generic filters).

So, given any limit ordinal $\bar{\xi}$, we can use the arguments of Section 4 to conclude that the model $V^1[G_\kappa * g * h][\vec{u} \restriction \bar{\xi}, \vec{F} \restriction \bar{\xi}][\vec{u} \restriction [\bar{\xi}, \bar{\xi} + \omega), \vec{F} \restriction [\bar{\xi}, \bar{\xi} + \omega)]$ satisfies:

$$\text{“ } \forall n < \omega,\ TP(\kappa_{\bar{\xi}+n}^{++}) + TP(\kappa_{\bar{\xi}+n}^{+4}) + TP(\kappa_{\bar{\xi}+\omega+2}) \text{ ”}.$$

Using the above remark, and by the same arguments as in Sections 3 and 4 one can prove Theorem 1.5. Below we present more details for completeness. First we prove an analogue of Lemma 5.9 for successor ordinals.

Lemma 5.10. $V^1[G_\kappa * g * h * K] \models$ *"For all successor ordinals* $\xi < \kappa, \kappa_\xi^{++} = (\kappa_\xi)_*$ *and the tree property at* $(\kappa_\xi)_*$ *holds".*

Proof. Suppose $\xi = \zeta + 1$. The model $V^1[G_\kappa * g * h * K]$ is an extension of $V^1[G_\kappa * g * h][\vec{u} \restriction \xi, \vec{F} \restriction \xi]$ by a forcing notion which does not add new subsets to $\kappa_\xi^{++} = (\kappa_\xi)_*$; so it suffices to show that $TP((\kappa_\xi)_*)$ holds in $V^1[G_\kappa * g * h][\vec{u} \restriction \xi, \vec{F} \restriction \xi]$. But this last model is equal to

$$V^1[G_{\kappa_\xi}][G(\kappa_\xi)][G_{(\kappa_\xi+1,\kappa]}][g * h][\vec{u} \restriction \zeta, \vec{F} \restriction \zeta][F_\zeta].$$

Since the forcing notion $\mathbb{R}_{u_\zeta}$ is defined in the same way in the models $V^1[G_\kappa * g * h]$ and $V^1[G_{\kappa_\xi}]$, so $V^1[G_\kappa * g * h][\vec{u} \restriction \xi, \vec{F} \restriction \xi]$ equals

$$V^1[G_{\kappa_\xi}][\vec{u} \restriction \zeta, \vec{F} \restriction \zeta][G(\kappa_\xi)][G_{(\kappa_\xi+1,\kappa]}][g{*}h][F_\zeta] = V[G_{\kappa_\xi}][\vec{u} \restriction \zeta, \vec{F} \restriction \zeta][F_\zeta][G(\kappa_\xi)][G_{(\kappa_\xi+1,\kappa]}][g{*}h].$$

It follows that $V^1[G_\kappa{*}g{*}h][\vec{u} \restriction \xi, \vec{F} \restriction \xi]$ is an extension of $V^1[G_{\kappa_\xi}][\vec{u} \restriction \zeta, \vec{F} \restriction \zeta][F_\zeta][G(\kappa_\xi)]$ by a forcing which does not add new subsets to $(\kappa_\xi)_*$, so we just need to show that $TP((\kappa_\xi)_*)$ holds in $V^1[G_{\kappa_\xi}][\vec{u} \restriction \zeta, \vec{F} \restriction \zeta][F_\zeta][G(\kappa_\xi)]$.

But $G(\kappa_\xi)$ is generic for $\widetilde{\mathbb{M}}(\kappa_\xi, (\kappa_\xi)_*) * \dot{\widetilde{\mathbb{M}}}((\kappa_\xi)_*, (\kappa_\xi)_{**})$ and so by Lemma 2.5,

$$V^1[G_{\kappa_\xi}][\vec{u} \restriction \zeta, \vec{F} \restriction \zeta][F_\zeta][G(\kappa_\xi)] \models \text{“} TP((\kappa_\xi)_*) \text{”}.$$

The lemma follows. □

Next we prove the following:

Lemma 5.11. $V^1[G_\kappa * g * h * K] \models$*“For all ordinals $\xi < \kappa$, $TP(\kappa_{\xi+1})$ holds”.*

Proof. We have

$$V^1[G_\kappa * g * h * K] = V^1[G_\kappa * g * h][\vec{u} \restriction \xi, \vec{F} \restriction \xi][F_\xi][\vec{u} \restriction (\xi+1, \kappa), \vec{F} \restriction (\xi+1, \kappa))]$$

Now F_ξ is a generic filter for $\widetilde{\mathbb{M}}(\kappa_\xi^{+4}, \kappa_{\xi+1})$ and by Lemma 2.4(c), we have

$$V^1[G_\kappa * g * h][\vec{u} \restriction \xi, \vec{F} \restriction \xi][F_\xi] \models \text{“}TP(\kappa_{\xi+1})\text{”}.$$

On the other hand, the models $V^1[G_\kappa * g * h * K]$ and $V^1[G_\kappa * g * h][\vec{u} \restriction \xi, \vec{F} \restriction \xi][F_\xi]$ have the same subsets of $\kappa_{\xi+1}$, and hence $V^1[G_\kappa * g * h * K] \models$“$TP(\kappa_{\xi+1})$”. □

Theorem 5.12. $V^1[G_\kappa * g * h * K] \models$*“For all ordinals $\xi < \kappa, \kappa_\xi^{+4} = (\kappa_\xi)_{**}$ and the tree property at $(\kappa_\xi)_{**}$ holds”.*

Proof. By similar analysis as above, it suffices to show that $TP((\kappa_\xi)_{**})$ holds in the model

$$V^1[G_{\kappa_\xi}][\vec{u}\restriction\xi,\vec{F}\restriction\xi][G(\kappa_\xi)][G_{(\kappa_\xi+1,\kappa_{\xi+1}]}][F_\xi],$$

which is an extension of $V^1[G_{\kappa_\xi}][\vec{u}\restriction\xi,\vec{F}\restriction\xi]$ by the forcing notion

$$\widetilde{\mathbb{M}}(\kappa_\xi,(\kappa_\xi)_*) * \dot{\widetilde{\mathbb{M}}}((\kappa_\xi)_*,(\kappa_\xi)_{**}) * (\dot{\mathbb{L}}_{(\kappa_\xi+1,\kappa_{\xi+1}]} \times \dot{\widetilde{\mathbb{M}}}((\kappa_\xi)_{**},\kappa_{\xi+1})).$$

So, by lemmas by Lemmas 2.6 and 2.7,

$$V^1[G_{\kappa_\xi}][\vec{u}\restriction\xi,\vec{F}\restriction\xi][G(\kappa_\xi)][G_{(\kappa_\xi+1,\kappa_{\xi+1}]}][F_\xi] \models \text{ `` } TP((\kappa_\xi)_{**}) \text{ ''}.$$

The lemma follows. □

The above lemmas give us a proof of Theorem 1.5.

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