# WOODIN'S SURGERY METHOD

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Abstract. In this short paper we give an overview of Woodin's surgery method.

## 1. Surgery method for strong cardinals

In this section we present an abstract version of Woodin's surgery method for strong cardinals.

**Theorem 1.1.** *([3]) Let $j : M \to N$ be an elementary embedding with $\kappa = crit(j)$, where $\kappa$ is inaccessible in $M$, and $N = \{j(F)(a) : F \in M, F : [\kappa]^{<\omega} \to M$ and $a \in [\lambda]^{<\omega}\}$. Let $\mathbb{P} = Add(\kappa, \nu)_M$, where $\nu$ is a cardinal in $M$ and suppose that $j \restriction \nu \in M$. Let $G$ be $\mathbb{P}-$generic over $M$, and suppose that there exists $H$ such that:*

(1) $N \subseteq M[G]$,

(2) $M[G] \models N^\kappa \subseteq N$,

(3) *$H$ is $j(\mathbb{P})-$generic over $M[G]$.*

*Then there exists $H^* \in M[G][H]$ such that $H^*$ is $j(\mathbb{P})-$generic over $N$ and $j[G] \subseteq H^*$.*

We will present two proofs of the above theorem. The first one, essentially due to Woodin, is taken from [1].

**First proof.** Let

$$g = \bigcup G : \nu \to 2,$$
$$h = \bigcup H : j(\nu) \to 2.$$

Define $h^* : j(\nu) \to 2$ by

$$h^*(\beta) = \begin{cases} g(\alpha) & \text{if } \beta = j(\alpha), \\ h(\beta) & Otherwise. \end{cases}$$

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Let $H^*$ be the filter generated by $h^*$. Note that $H^* = \{p^* : p \in H\}$, where for each $p \in j(\mathbb{P})$, $p^*$ is defined by

- $dom(p^*) = dom(p)$,
- $p^*$ is defined by

$$p^*(\beta) = \begin{cases} g(\alpha) & \text{if } \beta = j(\alpha), \\ p(\beta) & Otherwise. \end{cases}$$

Let's first show that $H^*$ is well-defined.

**Lemma 1.2.** $p \in j(\mathbb{P}) \Rightarrow p^* \in j(\mathbb{P})$.

*Proof.* It suffices to show that $p^* \in N$. But clearly $p^* \in M[G]$, so by clause (2) of the theorem, it suffices to show that $X(p, p^*) = \{z : p(z) \neq p^*(z)\}$ has size $\leq \kappa$. We have $X(p, p^*) \subseteq dom(p) \cap j[\nu]$, so it suffices to show that the later set has size at most $\kappa$. Let $p = j(F)(a)$, where $a \in [\lambda]^{<\omega}, F \in M, F : [\kappa]^{|a|} \to M$. We may further suppose that $\forall x \in [\kappa]^{|a|}, F(x) \in \mathbb{P}$. Then

$$\alpha < \nu, j(\alpha) \in dom(p) \Rightarrow \exists x, \alpha \in dom(F(x)).$$

So if $X = \bigcup\{dom(F(x)) : x \in [\kappa]^{|a|}\}$, then $X \in M$ and $M[G] \models$" $|X| \leq \kappa$ and $dom(p) \cap j[\nu] \subseteq j[X]$". The result follows. □

It is easily seen that $H^*$ is a filter on $j(\mathbb{P})$.

**Lemma 1.3.** *$H^*$ is $j(\mathbb{P})-$generic over $N$.*

*Proof.* Let $D \in N$ be dense open in $j(\mathbb{P})$. Define an equivalence relation on $j(\mathbb{P})$ by

$$p \sim q \Leftrightarrow dom(p) = dom(q) \text{ and } |\{z : p(z) \neq q(z)\}| \leq \kappa.$$

Let $E = \{q \in j(\mathbb{P}) : \forall p, p \sim q \Rightarrow p \in D\}$. We show that $E$ is dense in $j(\mathbb{P})$. First we prove the following.

**Lemma 1.4.** *If $p \in j(\mathbb{P})$, then there is $q \leq p$ such that*

$$\forall r, r \sim q \Rightarrow r \cup (q \setminus p) \in D.$$

*Proof.* Let $\langle X_\alpha : \alpha < \mu \rangle, \mu < j(\kappa)$ be an enumeration of $\{X \subseteq dom(p) : |X| \leq \kappa\}$. Define by induction a decreasing sequence $\langle p_\alpha : \alpha \leq \mu \rangle$ of conditions as follows:

- $\alpha = 0$**:** Let $p_0 = p$,
- $\alpha = \beta + 1$**:** Suppose $p_\beta$ is defined. Let
$$q(z) = \begin{cases} p_\beta(z) & \text{if } z \in dom(p_\alpha) \setminus X_\alpha, \\ 1 - p_\beta(z) & Otherwise. \end{cases}$$
Since $D$ is dense, we can find $\bar{q} \in D$ such that $\bar{q} \leq q$. Set $p_\alpha = p_\beta \cup (\bar{q} \setminus q)$.
- $\alpha$ **is a limit ordinal:** Let $p_\alpha = \bigcup_{\beta<\alpha} p_\beta$.

Then $q = p_\mu$ is as required. □

**Lemma 1.5.** *$E$ is dense in $j(\mathbb{P})$.*

*Proof.* Let $p \in j(\mathbb{P})$. Using the above claim $\kappa^+-$times, we can produce a decreasing sequence $\langle p_\alpha : \alpha < \kappa^+\rangle$ of conditions extending $p$ such that for any $\alpha < \kappa^+$ if $r \sim p_\alpha$, then $r \cup (p_{\alpha+1} \setminus p_\alpha) \in D$. Let $q = \bigcup_{\alpha<\kappa^+} p_\alpha$. Then $q \leq p$ and $q \in E$. To see this just note that if $r \sim q$, then for some $\alpha < \kappa^+, X(r,q) \subseteq dom(p_\alpha)$, so $X(r,q) = X(r,p_\alpha)$. □

Let $p \in H \cap E$. Then $p^* \sim p$, so $p^* \in H^* \cap D$. The theorem follows. □

**Second proof.** Let $H^*$ be as defined above. We show that it is $j(\mathbb{P})-$generic over $N$. Thus let $A \in N$ be a maximal antichain of $j(\mathbb{P})$. Then $|A| \leq j(\kappa)$. Set

$$S = \bigcup\{dom(p) : p \in A\}.$$

then $N \models$" $S \subseteq j(\nu)$ and $|S| \leq j(\kappa)$". Let $S = j(F)(a)$, where $a \in [\lambda]^{<\omega}, F \in M, F : [\kappa]^{|a|} \to M$. We may further suppose that $M \models$" For each $x \in dom(F), f(x) \subseteq \nu$ and $|f(x)| \leq \kappa$". Set $T = \bigcup\{f(x) : x \in [\kappa^{|a|}\}$. Then $T \in M$ and $M \models$" $T \subseteq \nu$ and $|T| \leq \kappa$". It is easily seen that

$$M[G] \models \text{"}S \cap j[\nu] \subseteq j[T]\text{"}.$$

Hence by clause (2), $S \cap j[\nu] \in N$. Let $X_0 = S \cap j[\nu]$ and $X_1 = j(\nu) \setminus X_0$. Also set $\mathbb{P}_i = \{p \in j(\mathbb{P}) : dom(p) \subseteq X_i\}, i = 0, 1$. Then we have a natural forcing isomorphism

$$\pi : j(\mathbb{P}) \to \mathbb{P}_0 \times \mathbb{P}_1,$$

given by

$$\pi(p) = \langle p \restriction X_0, p \restriction X_1\rangle.$$

Note that $H^* \restriction X_0 \in \mathbb{P}_0$. Set

$$A_1 = \{p \restriction X_1 : p \in A \text{ and } p \text{ is compatible with } h^* \restriction X_0\}.$$

The following lemma can be proved quite easily.

**Lemma 1.6.** *$A_1$ is a maximal antichain in $\mathbb{P}_1$.*

On the other hand $H_1 = \{p \restriction X_1 : p \in H\}$ is $\mathbb{P}_1-$generic, so $H_1 \cap A_1 \neq \emptyset$. Let $p \in A$ be such that $p$ is compatible with $h^* \restriction X_0$ and $p \restriction X_1 \in H_1 \cap A_1$. But then $p \in H^* \cap A$, and hence $H^* \cap A \neq \emptyset$. The theorem follows. □

## 2. Surgery method for supercompact cardinals

In this section we prove the following theorem, which is an analogue of Theorem 1.1 for supercompact cardinals.

**Theorem 2.1.** *Let $j : M \to N$ be an elementary embedding with $crit(j) = \kappa$, where $\kappa$ is inaccessible in $M$, and $N = \{j(F)(j[\lambda]) : F \in M, F : P_\kappa(\lambda) \to M\}$. Let $\mathbb{P} = Add(\kappa, \nu)_M$, where $\nu$ is a cardinal in $M$ and suppose that $j \restriction \nu \in M$. Let $G$ be $\mathbb{P}-$generic over $M$, and suppose that there exists $H$ such that:*

1. *$N \subseteq M[G]$,*
2. *$M[G] \models N^\lambda \subseteq N$,*
3. *$H$ is $j(\mathbb{P})-$generic over $M[G]$.*

*Then there exists $H^* \in M[G][H]$ such that $H^*$ is $j(\mathbb{P})-$generic over $N$ and $j[G] \subseteq H^*$.*

*Proof.* Let $g, H$ and $H^*$ be defined as before. We show that $H^*$ is as required.

Let $A \in N$ be a maximal antichain of $j(\mathbb{P})$. Then $|A| \leq j(\kappa)$. Set

$$S = \bigcup\{dom(p) : p \in A\}.$$

Then $N \models$" $S \subseteq j(\nu)$ and $|S| \leq j(\kappa)$". Let $S = j(F)(j[\lambda])$, where $F \in M, F : P_\kappa(\lambda) \to M$. We may further suppose that $M \models$" For each $x \in dom(F), f(x) \subseteq \nu$ and $|f(x)| \leq \kappa$". Set $T = \bigcup\{f(x) : x \in P_\kappa(\lambda)\}$. Then $T \in M$ and $M \models$" $T \subseteq \nu$ and $|T| \leq \lambda$". It is easily seen that

$$M[G] \models \text{"}S \cap j[\nu] \subseteq j[T]\text{"}.$$

Thus by clause (2), $S \cap j[\nu] \in N$. The rest of the argument is as before. □

As an application of the above theorem, we give a proof of the following theorem (compare with [2], Section 13).

**Theorem 2.2.** *Assume GCH holds and $\kappa$ is $\kappa^+-$supercompact. Then there is a generic extension in which $\kappa$ remains $\kappa^+-$supercompact and $2^\kappa = \kappa^{++}$.*

*Proof.* Let $\mathbb{P} = \mathbb{P}_\kappa * \underset{\sim}{\mathrm{Add}}(\kappa, \kappa^{++})$ be the reverse Easton iteration for adding $\alpha^{++}-$many new Cohen subsets of $\alpha$, using $\mathrm{Add}(\alpha, \alpha^{++})$, for all inaccessible cardinals $\alpha \leq \kappa$, and let $G * g$ be $\mathbb{P}-$generic over $V$. Let $j : V \to M \simeq Ult(V, U)$, where $U$ is a normal measure on $P_\kappa(\kappa^+)$, so that $M = \{j(F)(j[\kappa^+]) : F \in V, F : P_\kappa(\kappa^+) \to V\}$. Also let $j(\mathbb{P}) = \mathbb{P} * \underset{\sim}{\mathbb{R}} * \underset{\sim}{\mathrm{Add}}(j(\kappa), j(\kappa^{++}))$.

By standard forcing arguments we can find $H \in V[G * g]$ which is $j(\mathbb{P}_\kappa) = \mathbb{P} * \underset{\sim}{\mathbb{R}}-$generic over $M$, and since $j[G] \subseteq G * g * H$, We can lift $j$ to some $j : V[G] \to M[G * g * H]$.

Let $h$ be $\mathrm{Add}(j(\kappa), j(\kappa^{++}))_{M[G*g*H]}-$generic over $V[G * g * H]$. Applying Theorem 2.1, there exists $h^*$ such that we have the lifting $j^* : V[G * g] \to M[G * g * H * h^*]$. Working in $V[G * g * H * h]$, define $U^*$ on $P_\kappa(\kappa^+)$ by

$$X \in U^* \Leftrightarrow j[\kappa^+] \in j^*(X).$$

Note that

$$(*) \qquad V^{\mathbb{P}} \models \text{“}\mathbb{R} * \underset{\sim}{\mathrm{Add}}(j(\kappa), j(\kappa^{++})) \text{ is } \leq \kappa^+-\text{closed”}.$$

Using $(*)$, $U^* \in V[G * g]$, and $V[G * g] \models$“$U^*$ is a normal measure on $P_\kappa(\kappa^+)$”. The theorem follows. □

## References

[1] J. Cummings, A model in which *GCH* holds at successors but fails at limits. Trans. Am. Math. Soc. 329, No.1, 1–39 (1992).

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[3] H. Woodin, Personal communication.

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