On the Dimensions of Cyclic Symmetry Classes of Tensors

M. R. Darafsheh* and M. R. Pournaki†

Department of Mathematics and Computer Science, University of Tehran, and Institute for Studies in Theoretical Physics and Mathematics, Tehran, Iran

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The dimensions of the symmetry classes of tensors, associated with a certain cyclic subgroup of $S_m$ which is generated by a product of disjoint cycles is explicitly given in terms of the generalized Ramanujan sum. These dimensions can also be expressed as the Euler $\phi$-function and the Möbius function.

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1. INTRODUCTION

Let $V$ be an $n$-dimensional vector space over the complex field $\mathbb{C}$. Let $\otimes V$ be the $m$th tensor power of $V$ and write $\nu_1 \otimes \cdots \otimes \nu_m$ for the decomposable tensor product of the indicated vectors. To each permutation $\sigma$ in $S_m$ there corresponds a unique linear operator $P(\sigma): \otimes V \rightarrow \otimes V$ determined by $P(\sigma)(\nu_1 \otimes \cdots \otimes \nu_m) = \nu_{\sigma^{-1}(1)} \otimes \cdots \otimes \nu_{\sigma^{-1}(m)}$. Let $G$ be a subgroup of $S_m$ and let $I(G)$ be the set of all the irreducible complex characters of $G$. It follows from the orthogonality relations for characters that

$$\left\{ T(G, \chi): \otimes V \rightarrow \otimes V | T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma); \chi \in I(G) \right\}$$

is a set of annihilating idempotents which sum to the identity. The image

*E-mail: darafshe@vax.ipm.ac.ir.
†E-mail: pournaki@vax.ipm.ac.ir.
of $\otimes V$ under the $T(G, \chi)$ is called the symmetry class of tensors associated with $G$ and $\chi$ and is denoted by $V^m_\chi(G)$. It is well known that

$$\dim V^m_\chi(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \ell(\sigma),$$

(1)

where $\ell(\sigma)$ is the number of cycles, including cycles of length 1, in the disjoint cycle decomposition of $\sigma$ (see [4]).

Several papers are devoted to calculating $\dim V^m_\chi(G)$ in a more closed form than (1). Cummings [2] in the case that $G$ is a cyclic subgroup of $\mathbb{Z}_m$ generated by a cycle of length $m$ gives a formula for $\dim V^m_\chi(G)$ in terms of the Euler $\phi$-function and considers the case that $G$ is isomorphic to a direct product of cyclic groups as well. In [3] when $G$ is the dihedral group of order $2m$ is considered and a formula is given when $G$ is equal to the whole group $\mathbb{Z}_m$ in [5] and [6]. In all cases $\dim V^m_\chi(G)$ involves certain functions of $n$.

In [7] there is a formula for calculating $\dim V^m_\chi(G)$ in the case that $G = \langle \pi_1 \cdots \pi_p \rangle$, where $\pi_i, 1 \leq i \leq p$, are disjoint cycles in $\mathbb{Z}_m$. This formula involves the Euler $\phi$-function and Möbius function and is a modification of the formula given in [2]. In this case if the order of $\pi_i$ is $m_i, 1 \leq i \leq p$, then $G \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_p}$, $m_1, \ldots, m_p$ are linear and are of the form

$$\chi_h: G \to \mathbb{C}^*, \quad \chi_h((\pi_1 \cdots \pi_p)^t) = \exp\left(\frac{2\pi iht}{[m_1, \ldots, m_p]}\right),$$

$$0 \leq t \leq [m_1, \ldots, m_p] - 1,$$
so

\[ I(G) = \{ \chi_h: G \to \mathbb{C}^* \mid 0 \leq h \leq [m_1, \ldots, m_p] - 1 \}, \quad \text{where } [m_1, \ldots, m_p] \]

denotes the least common multiple of the integers \( m_1, \ldots, m_p \). The symbol \((m_1, \ldots, m_p)\) denotes the greatest common divisor of \( m_1, \ldots, m_p \).

### 2. A Result About the Ramanujan Sum

The well known Ramanujan sum is

\[
C_m(h) = \sum_{t=1}^{m-1} \exp \left( \frac{2\pi iht}{m} \right),
\]

where \( m \) is a positive integer and \( h \) is a nonnegative integer. Ramanujan proved that (see [1])

\[
C_m(h) = \frac{\varphi(m)\mu(m/(m, h))}{\varphi(m/(m, h))},
\]

where \( \varphi \) is the Euler \( \varphi \)-function, i.e., \( \varphi(1) = 1 \); for \( m > 1 \), \( \varphi(m) \) is the number of positive integers less than \( m \) and relatively prime to \( m \), and \( \mu \) is the Möbius function, i.e., \( \mu(1) = 1 \), \( \mu(m) = 0 \) if \( p^2 \mid m \) for some prime number \( p \), and \( \mu(m) = (-1)^y \) if \( m = p_1 \cdots p_r \), where \( p_1, \ldots, p_r \) are distinct prime numbers.

For our main result, we need to generalize the Ramanujan sum. It seems natural to us to generalize the Ramanujan sum as follows.

**Definition 1.** Let \( m_1, \ldots, m_p \) be positive integers and let \( h \) be a nonnegative integer. Suppose \( d_1 \mid m_1, \ldots, d_p \mid m_p \). The **generalized Ramanujan sum** denoted by \( S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) \) is defined by

\[
S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = \sum_{t=0}^{[m_1, \ldots, m_p] - 1} \exp \left( \frac{2\pi iht}{[m_1, \ldots, m_p]} \right),
\]

where

\[
\begin{align*}
&\quad (t, m_1) = d_1 \\
&\quad \vdots \\
&\quad (t, m_p) = d_p
\end{align*}
\]
If the set \(0 \leq t \leq [m_1, \ldots, m_p] - 1 \mid (t, m_i) = d_i, \ 1 \leq i \leq p\) is empty, then we define \(S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = 0\).

**Remark 1.** It is obvious that \(S(h; m; 1) = C_m(h)\), and so the sum appearing in Definition 1 is a generalization of the Ramanujan sum.

In the following lemma we prove that the generalized Ramanujan sum defined in Definition 1 involves the Ramanujan sum.

**Lemma 1.** Let \(m_1, \ldots, m_p\) be positive integers and let \(h\) be a nonnegative integer. Suppose \(d_i|m_1, \ldots, d_p|m_p\) and set \(m'_i = m_i/d_i, M_i = m_1 \cdots m_p/m_i, M'_i = m'_1 \cdots m'_p/m'_i, D_i = d_1 \cdots d_p/d_i (1 \leq i \leq p)\) and

\[
l = \frac{(M_1, \ldots, M_p)}{(M'_1, \ldots, M'_p)(D_1, \ldots, D_p)}.
\]

Then we have

\[
S(h; m_1, \ldots, m_p; d_1, \ldots, d_p)
= \begin{cases} 
\frac{1}{l} C_{[m_1, \ldots, m_p]}(hl), & \text{if } \left\lfloor \frac{d_1, \ldots, d_p}{d_i} \right\rfloor, m'_i = 1, \ 1 \leq i \leq p \\
0, & \text{otherwise}.
\end{cases}
\]

**Proof.** By Definition 1 and the fact that \(\exp(2\pi iht/[m_1, \ldots, m_p])\) is a periodic function of \(t\) with period \([m_1, \ldots, m_p]\) we have

\[
S(h; m_1, \ldots, m_p; d_1, \ldots, d_p)
= \sum_{t=0}^{[m_1, \ldots, m_p]-1} \exp\left(\frac{2\pi iht}{[m_1, \ldots, m_p]}\right)
= \frac{1}{l} \sum_{t=0}^{[[m_1, \ldots, m_p]-1} \exp\left(\frac{2\pi iht}{[m_1, \ldots, m_p]}\right).
\]
Now letting \( t = [d_1, \ldots, d_p]' \), we obtain

\[
S(h; m_1, \ldots, m_p; d_1, \ldots, d_p)
= \frac{1}{l} \sum_{t' = 0}^{[m'_1, \ldots, m'_p] - 1} \exp \left( \frac{2\pi i hl'}{[m'_1, \ldots, m'_p]} \right).
\]

If \( ([d_1, \ldots, d_p]/d_i, m'_i) = 1 \) for all \( i, 1 \leq i \leq p \), then the set of all the \( t' \)'s indexing the above summation is equal to the set of all the \( t' \)'s such that \( 0 \leq t' \leq [m'_1, \ldots, m'_p] - 1 \) with conditions \((t', m'_i) = 1, 1 \leq i \leq p \). And if there is an \( i \) for which \( ([d_1, \ldots, d_p]/d_i, m'_i) \neq 1 \) then the above sum is zero and therefore we obtain:

\[
S(h; m_1, \ldots, m_p; d_1, \ldots, d_p)
= \begin{cases} 
\frac{1}{l} \sum_{t' = 0}^{[m'_1, \ldots, m'_p] - 1} \exp \left( \frac{2\pi i hl'}{[m'_1, \ldots, m'_p]} \right), & \text{if } \left( \frac{d_1, \ldots, d_p}{d_i}, m'_i \right) = 1, \ 1 \leq i \leq p \\
0, & \text{otherwise},
\end{cases}
\]

\[
= \begin{cases} 
\frac{1}{l} \sum_{t' = 0}^{[m'_1, \ldots, m'_p] - 1} \exp \left( \frac{2\pi i hl'}{[m'_1, \ldots, m'_p]} \right), & \text{if } \left( \frac{d_1, \ldots, d_p}{d_i}, m'_i \right) = 1, \ 1 \leq i \leq p \\
0, & \text{otherwise},
\end{cases}
\]
In some special cases the generalized Ramanujan sum is given in the following examples:

**Example 1.**

\[
S(0; m; d) = C_{m/d}(0) = \varphi(m/d) \mu((m/d)/(m/d, 0)) \varphi((m/d)/(m/d, 0)) = \varphi(m/d).
\]

**Example 2.** If \((h, m) = 1\); we have

\[
S(h; m; d) = C_{m/d}(h) = \frac{\varphi(m/d) \mu((m/d)/(m/d, h))}{\varphi((m/d)/(m/d, h))} = \mu(m/d).
\]

3. **The Dimensions of Some Symmetry Classes of Tensors**

In this section, as we mentioned earlier, the group \(G = \langle \pi_1, \ldots, \pi_p \rangle\) is considered, where the \(\pi_i\)'s, \(1 \leq i \leq p\), are disjoint cycles in \(S_m\). Our aim is to calculate \(\dim V^m(G)\), where \(\chi \in \mathcal{I}(G)\), in terms of known functions. Our formula involves the generalized Ramanujan sum.

**Theorem 1.** Let \(G = \langle \pi_1, \ldots, \pi_p \rangle\), where the \(\pi_i\)'s, \(1 \leq i \leq p\), are disjoint cycles in \(S_m\) of orders \(m_1, \ldots, m_p\), respectively, and let \(\chi_h, 0 \leq h \leq \left\lfloor \frac{m}{m_1, \ldots, m_p} \right\rfloor - 1\), be an irreducible complex character of \(G\). Then

\[
\dim V^m_{\chi_h}(G) = \frac{n^{m-(m_1+\cdots+m_p)}}{[m_1, \ldots, m_p]} \times \sum_{d_1|m_1} \ldots \sum_{d_p|m_p} S(h; m_1, \ldots, m_p; d_1, \ldots, d_p)n^{d_1+\cdots+d_p},
\]
where $S(h; m_1, \ldots, m_p; d_1, \ldots, d_p)$ denotes the generalized Ramanujan sum.

Proof. According to (1) the dimension of $V_{X_h}^m(G)$ is

$$\frac{1}{[m_1, \ldots, m_p]} \sum_{\sigma \in G} \chi_h(\sigma)n^{c(\sigma)},$$

where $c(\sigma)$ denotes the number of cycles, including cycles of length 1, in the disjoint cycle decomposition of $\sigma$. But every $\sigma \in G$ is equal to $\sigma = (\pi_1 \cdots \pi_p)^t$ for some $0 \leq t \leq [m_1, \ldots, m_p] - 1$ since $\pi_1, \ldots, \pi_p$ are disjoint, so are $(\pi_1 \cdots \pi_p)^t = \pi_1^t \cdots \pi_p^t$. Appealing to [7] we can obtain

$$c(\pi_1^t \cdots \pi_p^t) = c(\pi_1^t) + \cdots + c(\pi_p^t) + m - (m_1 + \cdots + m_p).$$

Note that if $(t, m) = d$, then $\pi_i^t$ has $d$ cycles of length $m_i/d$ and therefore $c(\pi_i^t) = d = (t, m)$. So we have

$$c(\pi_1^t \cdots \pi_p^t) = (t, m_1) + \cdots + (t, m_p) + m - (m_1 + \cdots + m_p).$$

Hence according to (2) we have

$$\dim V_{X_h}^m(G)$$

$$= \frac{1}{[m_1, \ldots, m_p]} \sum_{\sigma \in G} \chi_h(\sigma)n^{c(\sigma)}$$

$$= \frac{1}{[m_1, \ldots, m_p]} \sum_{t=0}^{[m_1, \ldots, m_p] - 1} \chi_h(\pi_1^t \cdots \pi_p^t)n^{c(\pi_1^t \cdots \pi_p^t)}$$

$$= \frac{1}{[m_1, \ldots, m_p]}$$

$$\times \sum_{t=0}^{[m_1, \ldots, m_p] - 1} \exp\left(\frac{2\pi iht}{[m_1, \ldots, m_p]}\right) n^{(t, m_1) + \cdots + (t, m_p) + m - (m_1 + \cdots + m_p)}$$

$$= \frac{1}{[m_1, \ldots, m_p]}$$

$$\times \sum_{t=0}^{[m_1, \ldots, m_p] - 1} \exp\left(\frac{2\pi iht}{[m_1, \ldots, m_p]}\right) n^{(t, m_1) + \cdots + (t, m_p)}.$$
Now letting \((t, m_i) = d_i, 1 \leq i \leq p\), we obtain

\[
\dim V^m_\chi(G) = \frac{n^{m-(m_1+\cdots+m_p)}}{[m_1, \ldots, m_p]} \sum_{d_i|m_1} \left( \frac{[m_1, \ldots, m_p]-1}{\prod_{t=0}^{i-1} (t, m_t)} \right) \sum_{d|d_i} \exp \left( \frac{2\pi i t d}{[m_1, \ldots, m_p]} \right) n^{d_1+\cdots+d_p}.
\]

Using Theorem 1 we obtain Theorems 1 and 2 of [2] in the following corollaries.

**Corollary 1.** If \(G\) is a cyclic subgroup of \(\mathbb{Z}_m\) generated by an \(m\)-cycle and \(\chi\) is the identity character 1, then \(\dim V^m_\chi(G) = (1/m) \sum_{d|m} \varphi(m/d)n^d\).

**Proof.** Since \(\chi = \chi_0\), by Example 1 and using Theorem 1 we obtain

\[
\dim V^m_\chi(G) = \frac{n^{m-m}}{m} \sum_{d|m} S(0; m; d)n^d = \frac{1}{m} \sum_{d|m} \varphi(m/d)n^d.
\]

**Remark 2.** In Corollary 1, if \(\dim V = n = 1\), then \(\dim \bigotimes V = 1\), and so \(\dim V^m_\chi(G) = 0\) or 1. so \((1/m)\sum_{d|m} \varphi(m/d) = 0\) or 1. But \((1/m)\sum_{d|m} \varphi(m/d) = 0\) is impossible, therefore \((1/m)\sum_{d|m} \varphi(m/d) = 1\) or \(\sum_{d|m} \varphi(d) = m\), which is well known identity in number theory.

**Corollary 2.** If \(G\) is a cyclic subgroup of \(\mathbb{Z}_m\) generated by an \(m\)-cycle and \(\chi\) is a primitive linear character, then \(\dim V^m_\chi(G) = (1/m) \sum_{d|m} \mu(m/d)n^d\).
Proof. We know that a linear character of a cyclic subgroup of $S_m$ is primitive if its value on a generator of the subgroup is a primitive $m$th root of unity, so $\chi = \chi_h$ where $(h, m) = 1$ and by Example 2 and using Theorem 1, we have

$$\dim V^{m\chi}_\chi(G) = \frac{n^{m-m}}{m} \sum_{d|m} S(h; m; d) n^d = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) n^d.$$ 

Example 3. Let $G = \langle (12)(34)(5678) \rangle$ be a subgroup of $S_9$. Suppose $\chi$ is the identity character 1, i.e., $\chi = \chi_0$. Then by Theorem 1 we have

$$\dim V^{3\chi}_\chi(G) = \frac{n^{9-(2+2+4)}}{4} \sum_{d_1|2} S(0; 2, 2, 4; d_1, d_2, d_3) n^{d_1+d_2+d_3}$$

$$= \frac{n}{4} \left[ S(0; 2, 2, 4; 2, 2, 4) n^8 + S(0; 2, 2, 4; 2, 2, 2) n^6 + S(0; 2, 2, 4; 1, 1, 1) n^3 \right]$$

$$= \frac{n}{4} \left[ n^8 + n^6 + 2n^3 \right].$$

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