On the Dimensions of Cyclic Symmetry Classes of Tensors

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Abstract

The dimensions of the symmetry classes of tensors, associated with a certain cyclic subgroup of $S_m$ which is generated by a product of disjoint cycles is explicitly given in terms of the generalized Ramanujan sum. These dimensions can also be expressed as the Euler $\varphi$-function and the Möbius function.

Keywords: Symmetry class of tensors, Ramanujan sum, Euler $\varphi$-function, Möbius function.

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1 Introduction

Let $V$ be an $n$-dimensional vector space over the complex field $C$. Let $\otimes^n V$ be the $n$th tensor power of $V$ and write $\nu_1 \otimes \cdots \otimes \nu_m$ for the decomposable tensor product of the indicated vectors. To each permutation $\sigma$ in $S_m$ there corresponds a unique linear operator $P(\sigma) : \otimes^n V \rightarrow \otimes^n V$ determined by $P(\sigma)(\nu_1 \otimes \cdots \otimes \nu_m) = \nu_{\sigma^{-1}(1)} \otimes \cdots \otimes \nu_{\sigma^{-1}(m)}$. Let $G$ be a subgroup of $S_m$ and let $I(G)$ be the set of all the irreducible complex characters of $G$. It follows from the orthogonality relations for characters that

$$\left\{ T(G, \chi) : \otimes^n V \rightarrow \otimes^n V \mid T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)P(\sigma); \, \chi \in I(G) \right\}$$

is a set of annihilating idempotents which sum to the identity. The image of $\otimes^n V$ under the $T(G, \chi)$ is called the symmetry class of tensors associated with $G$ and $\chi$ and is denoted by $V^m_\chi(G)$. It is well known that

$$\dim V^m_\chi(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)c(\sigma),$$

where $c(\sigma)$ is the number of cycles, including cycles of length 1, in the disjoint cycle decomposition of $\sigma$ (see [4]).
Several papers are devoted to calculating $\dim V^m \chi(G)$ in a more closed form than (1). Cummings [2] in the case that $G$ is a cyclic subgroup of $S_m$ generated by a cycle of length $m$ gives a formula for $\dim V^m \chi(G)$ in terms of the Euler $\varphi$-function and considers the case that $G$ is isomorphic to a direct product of cyclic groups as well. In [3] when $G$ is the dihedral group of order $2m$ is considered and a formula is given when $G$ is equal to the whole group $S_m$ in [5] and [6]. In all cases $\dim V^m \chi(G)$ involves certain functions of $n$.

In [7] there is a formula for calculating $\dim V^m \chi(G)$ in the case that $G = \langle \pi_1 \cdots \pi_p \rangle$, where $\pi_i$s, $1 \leq i \leq p$, are disjoint cycles in $S_m$. This formula involves the Euler $\varphi$-function and Möbius function and is a modification of the formula given in [2]. In this case if the order of $\pi_i$ is $m_i$, $1 \leq i \leq p$, then $G \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_p}$. It is mentioned in [2] and [7] that if two groups are isomorphic, then the dimensions of the symmetry classes of tensors associated with them need not be equal. For example if $G = \langle (12) \rangle$ and $H = \langle (12)(34) \rangle$ are considered as subgroups of $S_4$, then it is easy to calculate that $\dim V^4 \chi_0(G) = n^3(n^4 + 1)/2$ and $\dim V^4 \chi_0(H) = n^2(n^2 + 1)/2$ whereas $G \cong H \cong \mathbb{Z}_2$ and $\chi_0$ is the identity character of $\mathbb{Z}_2$.

Now it is a natural question to consider the cyclic group $G = \langle \pi_1 \cdots \pi_p \rangle$ where the $\pi_i$s, $1 \leq i \leq p$, are disjoint cycles and ask about the dimension of $V^m \chi(G)$, where $\chi \in \mathfrak{I}(G)$. In this case if the order of $\pi_i$, $1 \leq i \leq p$, is $m_i$, then $G \cong \mathbb{Z}_{[m_1, \ldots, m_p]}$, where $[m_1, \ldots, m_p]$ denotes the least common multiple of the integers $m_1, \ldots, m_p$. In this paper we obtain a formula for $\dim V^m \chi(G)$ in the above case and this formula involves the generalized Ramanujan sum which itself involves the Euler $\varphi$-function and Möbius function. Therefore for the rest of this paper let $G < S_m$ be of the form

$$G = \langle \pi_1 \cdots \pi_p \rangle,$$

where $\pi_i$s, $1 \leq i \leq p$, are disjoint cycles in $S_m$ of certain orders, say, $m_1, \ldots, m_p$, respectively. Since $G$ is cyclic, therefore the irreducible characters of $G$ are all linear and are of the form

$$\chi_h : G \to \mathbb{C}^*, \quad \chi_h (\langle \pi_1 \cdots \pi_p \rangle^t) = \exp \left( \frac{2\pi i h t}{[m_1, \ldots, m_p]} \right),$$

$$0 \leq t \leq [m_1, \ldots, m_p] - 1,$$

so

$$\mathfrak{I}(G) = \left\{ \chi_h : G \to \mathbb{C}^* \mid 0 \leq h \leq [m_1, \ldots, m_p] - 1 \right\}.$$
where \([m_1, \ldots, m_p]\) denotes the least common multiple of the integers \(m_1, \ldots, m_p\). The symbol \((m_1, \ldots, m_p)\) denotes the greatest common divisor of \(m_1, \ldots, m_p\).

2 A Result About the Ramanujan Sum

The well known Ramanujan sum is

\[
C_m(h) = \sum_{t=0}^{m-1} \exp \left( \frac{2\pi i ht}{m} \right),
\]

where \(m\) is a positive integer and \(h\) is a nonnegative integer. Ramanujan proved that (see [1])

\[
C_m(h) = \frac{\varphi(m)\mu(m/(m,h))}{\varphi(m/(m,h))},
\]

where \(\varphi\) is the Euler \(\varphi\)-function, i.e., \(\varphi(1) = 1\); for \(m > 1\), \(\varphi(m) =\) the number of positive integers less than \(m\) and relatively prime to \(m\), and \(\mu\) is the Möbius function, i.e., \(\mu(1) = 1\), \(\mu(m) = 0\) if \(p^2\mid m\) for some prime number \(p\), and \(\mu(m) = (-1)^r\) if \(m = p_1 \ldots p_r\), where \(p_1, \ldots, p_r\) are distinct prime numbers.

For our main result, we need to generalize the Ramanujan sum. It seems natural to us to generalize the Ramanujan sum as follows.

**Definition 1** Let \(m_1, \ldots, m_p\) be positive integers and let \(h\) be a nonnegative integer. Suppose \(d_1 \mid m_1, \ldots, d_p \mid m_p\). The *generalized Ramanujan sum* denoted by \(S(h; m_1, \ldots, m_p; d_1, \ldots, d_p)\) is defined by

\[
S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = \sum_{t=0}^{[m_1, \ldots, m_p]-1} \exp \left( \frac{2\pi i ht}{[m_1, \ldots, m_p]} \right).
\]

If the set \(\{0 \leq t \leq [m_1, \ldots, m_p] - 1 \mid (t, m_i) = d_i; 1 \leq i \leq p\}\) is empty, then we define \(S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = 0\).
Remark 1 It is obvious that \( S(h; m; 1) = C_m(h) \), and so the sum appearing in Definition 1 is a generalization of the Ramanujan sum.

In the following lemma we prove that the generalized Ramanujan sum defined in Definition 1 involves the Ramanujan sum.

Lemma 1 Let \( m_1, \ldots, m_p \) be positive integers and let \( h \) be a nonnegative integer. Suppose \( d_1|m_1, \ldots, d_p|m_p \) and set \( m'_i = m_i/d_i, M_i = m_1 \cdots m_p/m_i, M'_i = m'_1 \cdots m'_p/m'_i, \) \( D_i = d_1 \cdots d_p/d_i (1 \leq i \leq p) \) and

\[
l = \frac{(M_1, \ldots, M_p)}{(M'_1, \ldots, M'_p)(D_1, \ldots, D_p)}.
\]

Then we have

\[
S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = \begin{cases} 
\frac{1}{l}C_{[m'_1, \ldots, m'_p]}(hl), & \text{if } \left( \frac{[d_1, \ldots, d_p]}{d_i}, m'_i \right) = 1, \quad 1 \leq i \leq p \\
0, & \text{otherwise}.
\end{cases}
\]

Proof. By Definition 1 and the fact that \( \exp\left(\frac{2\pi i ht}{[m_1, \ldots, m_p]}\right) \) is a periodic function of \( t \) with period \([m_1, \ldots, m_p]\) we have

\[
S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = \sum_{\substack{t = 0 \\
(t, m_1) = d_1 \\
\vdots \\
(t, m_p) = d_p}}^{[m_1, \ldots, m_p]-1} \exp\left(\frac{2\pi i ht}{[m_1, \ldots, m_p]}\right)
\]

\[
= \frac{1}{l} \sum_{\substack{t = 0 \\
(t, m_1) = d_1 \\
\vdots \\
(t, m_p) = d_p}}^{[m_1, \ldots, m_p]-1} \exp\left(\frac{2\pi i ht}{[m_1, \ldots, m_p]}\right),
\]

Now letting \( t = [d_1, \ldots, d_p]t' \), we obtain
\[ S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = \frac{1}{l} \sum_{t' = 0}^{[m'_1, \ldots, m'_p] - 1} \exp \left( \frac{2\pi iht'}{[m'_1, \ldots, m'_p]} \right) \]

\[
\begin{align*}
&= \frac{1}{l} \sum_{\substack{t' = 0 \\
(i \leq i \leq p)} \exp \left( \frac{2\pi iht'}{[m'_1, \ldots, m'_p]} \right) \\
&= \frac{1}{l} \sum_{\substack{t' = 0 \\
(i \leq i \leq p)} \exp \left( \frac{2\pi iht'}{[m'_1, \ldots, m'_p]} \right).
\end{align*}
\]

If \([d_1, \ldots, d_p]/d_i, m'_i = 1\) for all \(i, 1 \leq i \leq p\), then the set of all the \(t'\)'s indexing the above summation is equal to the set of all the \(t'\)'s such that \(0 \leq t' \leq [m'_1, \ldots, m'_p] - 1\) with conditions \((t', m'_i) = 1, 1 \leq i \leq p\). And if there is an \(i\) for which \([d_1, \ldots, d_p]/d_i, m'_i \neq 1\) then the above sum is zero and therefore we obtain:

\[ S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = \begin{cases} 
\frac{1}{l} \sum_{\substack{t' = 0 \\
(i \leq i \leq p)} \exp \left( \frac{2\pi iht'}{[m'_1, \ldots, m'_p]} \right), & \text{if } \left\{ \frac{[d_1, \ldots, d_p]}{d_i}, m'_i \right\} = 1, 1 \leq i \leq p \\
0, & \text{otherwise,}
\end{cases} \]

\[ = \begin{cases} 
\frac{1}{l} \sum_{\substack{t' = 0 \\
(i \leq i \leq p)} \exp \left( \frac{2\pi iht'}{[m'_1, \ldots, m'_p]} \right), & \text{if } \left\{ \frac{[d_1, \ldots, d_p]}{d_i}, m'_i \right\} = 1, 1 \leq i \leq p \\
0, & \text{otherwise,}
\end{cases} \]

\[ = \begin{cases} 
\frac{1}{l} C_{[m'_1, \ldots, m'_p]}(hl), & \text{if } \left\{ \frac{[d_1, \ldots, d_p]}{d_i}, m'_i \right\} = 1, 1 \leq i \leq p \\
0, & \text{otherwise.}
\end{cases} \]
In some special cases the generalized Ramanujan sum is given in the following examples:

**Example 1**  
\( S(0; m; d) = C_{m/d}(0) = \frac{\varphi(m/d)\mu((m/d)/(m/d, 0))}{\varphi((m/d)/(m/d, 0))} = \varphi(m/d). \)

**Example 2**  
If \((h, m) = 1\); we have  
\( S(h; m; d) = C_{m/d}(h) = \frac{\varphi(m/d)\mu((m/d)/(m/d, h))}{\varphi((m/d)/(m/d, h))} = \mu(m/d). \)

### 3 The Dimensions of Some Symmetry Classes of Tensors

In this section, as we mentioned earlier, the group \( G = \langle \pi_1, \ldots, \pi_p \rangle \) is considered, where the \( \pi_i \)'s, \( 1 \leq i \leq p \), are disjoint cycles in \( S_m \). Our aim is to calculate \( \dim V^m_{\chi}(G) \), where \( \chi \in I(G) \), in terms of known functions. Our formula involves the generalized Ramanujan sum.

**Theorem 1**  
Let \( G = \langle \pi_1, \ldots, \pi_p \rangle \), where the \( \pi_i \)'s, \( 1 \leq i \leq p \), are disjoint cycles in \( S_m \) of orders \( m_1, \ldots, m_p \), respectively, and let \( \chi_h \), \( 0 \leq h \leq [m_1, \ldots, m_p] - 1 \), be an irreducible complex character of \( G \). Then  
\[ \dim V^m_{\chi_h}(G) = \frac{n^{m-(m_1+\cdots+m_p)}}{[m_1, \ldots, m_p]} \sum_{d_1|m_1} \cdots \sum_{d_p|m_p} S(h; m_1, \ldots, m_p; d_1, \ldots, d_p)n^{d_1+\cdots+d_p}, \]

where \( S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) \) denotes the generalized Ramanujan sum.

**Proof.** According to (1) the dimension of \( V^m_{\chi_h}(G) \) is  
\[ \frac{1}{[m_1, \ldots, m_p]} \sum_{\sigma \in G} \chi_h(\sigma)c(\sigma), \]

where \( c(\sigma) \) denotes the number of cycles, including cycles of length 1, in the disjoint cycle decomposition of \( \sigma \). But every \( \sigma \in G \) is equal to \( \sigma = (\pi_1 \cdots \pi_p)^t \) for some \( t \), \( 0 \leq t \leq [m_1, \ldots, m_p] - 1 \). Since \( \pi_1, \ldots, \pi_p \) are disjoint, so are \( (\pi_1 \cdots \pi_p)^t = \pi_1^t \cdots \pi_p^t \). Appealing to [7] we can obtain  
\[ c(\pi_1^t \cdots \pi_p^t) = c(\pi_1^t) + \cdots + c(\pi_p^t) + m - (m_1 + \cdots + m_p). \]
Note that if \((t, m_i) = d_i\), then \(\pi^t_i\) has \(d_i\) cycles of length \(m_i/d_i\) and therefore \(c(\pi^t_i) = d = (t, m_i)\). So we have
\[
c(\pi^t_1 \cdots \pi^t_p) = (t, m_1) + \cdots + (t, m_p) + m - (m_1 + \cdots + m_p).
\]

Hence according to (2) we have
\[
\dim V^m_{\chi^h}(G) = \frac{1}{[m_1, \ldots, m_p]} \sum_{\sigma \in G} \chi^h(\sigma) n^{c(\sigma)}
\]
\[
= \frac{1}{[m_1, \ldots, m_p]} \sum_{t=0}^{[m_1, \ldots, m_p]-1} \chi^h(\pi^t_1 \cdots \pi^t_p) n^{c(\pi^t_1 \cdots \pi^t_p)}
\]
\[
= \frac{1}{[m_1, \ldots, m_p]} \sum_{t=0}^{[m_1, \ldots, m_p]-1} \exp \left( \frac{2\pi i h t}{[m_1, \ldots, m_p]} \right) n^{(t, m_1) + \cdots + (t, m_p) + m - (m_1 + \cdots + m_p)}
\]
\[
= \frac{1}{[m_1, \ldots, m_p]} \sum_{t=0}^{[m_1, \ldots, m_p]-1} \exp \left( \frac{2\pi i h t}{[m_1, \ldots, m_p]} \right) n^{(t, m_1) + \cdots + (t, m_p)}.
\]

Now letting \((t, m_i) = d_i, 1 \leq i \leq p\), we obtain
\[
\dim V^m_{\chi^h}(G) = \frac{n^{m-(m_1+\cdots+m_p)}}{[m_1, \ldots, m_p]} \sum_{d_1|m_1} \cdots \sum_{d_p|m_p} \exp \left( \frac{2\pi i h t}{[m_1, \ldots, m_p]} \right) n^{d_1+\cdots+d_p}
\]
\[
= \frac{n^{m-(m_1+\cdots+m_p)}}{[m_1, \ldots, m_p]} S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) n^{d_1+\cdots+d_p}. \quad \Box
\]
Using Theorem 1 we obtain Theorems 1 and 2 of [2] in the following corollaries.

**Corollary 1** If $G$ is a cyclic subgroup of $\mathbb{Z}_m$ generated by an $m$-cycle and $\chi$ is the identity character 1, then $\dim V^m_\chi(G) = \frac{1}{m} \sum_{d|m} \varphi(m/d)n^d$.

**Proof.** Since $\chi = \chi_0$, by Example 1 and using Theorem 1 we obtain

$$\dim V^m_\chi(G) = \frac{n^m}{m} \sum_{d|m} S(0; m; d)n^d = \frac{1}{m} \sum_{d|m} \varphi(m/d)n^d. \square$$

**Remark 2** In Corollary 1, if $\dim V = n = 1$, then $\dim \mathbb{Z}_m = 1$, and so $\dim V^m_\chi(G) = 0$ or 1. So $(1/m) \sum_{d|m} \varphi(m/d) = 0$ or 1. But $(1/m) \sum_{d|m} \varphi(m/d) = 0$ is impossible, therefore $(1/m) \sum_{d|m} \varphi(m/d) = 1$ or $\sum_{d|m} \varphi(d) = m$, which is well known identity in number theory.

**Corollary 2** If $G$ is a cyclic subgroup of $\mathbb{Z}_m$ generated by an $m$-cycle and $\chi$ is a primitive linear character, then $\dim V^m_\chi(G) = \frac{1}{m} \sum_{d|m} \mu(m/d)n^d$.

**Proof.** We know that a linear character of a cyclic subgroup of $\mathbb{Z}_m$ is primitive if its value on a generator of the subgroup is a primitive $m$th root of unity, so $\chi = \chi_h$ where $(h, m) = 1$ and by Example 2 and using Theorem 1, we have

$$\dim V^m_\chi(G) = \frac{n^m}{m} \sum_{d|m} S(h; m; d)n^d = \frac{1}{m} \sum_{d|m} \mu(m/d)n^d. \square$$

**Example 3** Let $G = \langle (12)(34)(5678) \rangle$ be a subgroup of $\mathbb{Z}_9$. Suppose $\chi$ is the identity character 1, i.e., $\chi = \chi_0$. Then by Theorem 1 we have

$$\dim V^9_\chi(G) = \frac{n^9}{4} \sum_{d_1|9} S(0; 2, 2, 4; d_1, d_2, d_3)n^{d_1+d_2+d_3}$$

$$= \frac{n}{4} [S(0; 2, 2, 4; 2, 2, 2)n^8 + S(0; 2, 2, 4; 2, 2, 2)n^6 + S(0; 2, 2, 4; 1, 1, 1)n^3]$$

$$= \frac{n}{4} [n^8 + n^6 + 2n^3].$$
References


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