On the Orthogonal Basis of the Symmetry Classes of Tensors Associated with the Dicyclic Group

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Abstract

A necessary and sufficient condition for the existence of orthogonal basis of decomposable symmetrized tensors for the symmetry classes of tensors associated with the dicyclic group is given. In particular we apply these conditions to the generalized quaternion group, for which the dimensions of the symmetry classes of tensors are computed.

Keywords: Symmetry class of tensors, Orthogonal basis, Dicyclic group, 2-Adic valuation.

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1 Introduction

Let \( V \) be an \( m \)-unitary space. Let \( \otimes^n V \) be the \( n \)-th tensor power of \( V \) and write \( v_1 \otimes \cdots \otimes v_n \) for the decomposable tensor product of the indicated vectors. To each permutation \( \sigma \) in \( S_n \) there corresponds a unique linear operator \( P(\sigma): \otimes^n V \to \otimes^n V \) determined by

\[
P(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.
\]

Let \( G \) be a subgroup of \( S_n \) and let \( I(G) \) be the set of all the irreducible complex characters of \( G \). It follows from the orthogonality relations for characters that

\[
\left\{ T(G, \chi): \otimes^n V \to \otimes^n V \mid T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)P(\sigma), \ \chi \in I(G) \right\}
\]

is a set of annihilating idempotents which sum to the identity. The image of \( \otimes^n V \) under the \( T(G, \chi) \) is called the symmetry class of tensors associated with \( G \) and \( \chi \) and it is denoted by \( V_n^\chi(G) \). The image of \( v_1 \otimes \cdots \otimes v_n \) under \( T(G, \chi) \) is denoted by \( v_1 \ast \cdots \ast v_n \) and it is called a decomposable tensor. It is well-known that

\[
\dim V_n^\chi(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)m^c(\sigma)
\]

(1)

where \( c(\sigma) \) is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of \( \sigma \) (see [5]).
The inner product on $V$ induces an inner product on $\otimes^n V$ whose restriction to $V^n_\chi(G)$ satisfies

$$\langle u_1 \ast \ldots \ast u_n \mid v_1 \ast \ldots \ast v_n \rangle = \frac{\chi(1)}{|G|} d^G_\chi(A)$$

where $A = [a_{ij}]_{n \times n} = [(u_i|v_j)]_{n \times n}$ and $d^G_\chi(A) = \sum_{\sigma \in G} \chi(\sigma) a_{1\sigma(1)} \ldots a_{n\sigma(n)}$ is the generalized matrix function.

With respect to this inner product we have

$$\otimes^n V = \bigoplus_{\chi \in \Pi(G)} V^n_\chi(G)$$

which is an orthogonal direct sum.

Let $\Gamma^n_m$ be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $1 \leq \alpha_i \leq m$, so $\alpha$ is a mapping from a set of $n$ elements into a set of $m$ elements. Then the group $G$ acts on $\Gamma^n_m$ by $\sigma.\alpha := (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(n)})$ where $\sigma \in G$ is a permutation on $n$ letters and $\alpha \in \Gamma^n_m$ is a mapping from a set of $n$ elements into a set of $m$ elements. Therefore the action may be written as $\sigma.\alpha = \alpha \sigma^{-1}$ which is a composition of two functions. Let $O(\alpha) = \{\sigma.\alpha \mid \sigma \in G\}$ be the orbit with representative $\alpha$, i.e., $G_\alpha = \{\sigma \in G \mid \sigma.\alpha = \alpha\}$. In this setting if $\alpha \in \Gamma^n_m$ and $\sigma \in G$, then we have $G_{\sigma.\alpha} = \sigma G_\alpha \sigma^{-1}$.

Let $\Delta$ be a system of distinct representatives of the orbits of $G$ acting on $\Gamma^n_m$ and define

$$\overline{\Delta} = \left\{\alpha \in \Delta \mid \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0\right\},$$

and let $\Omega$ be the union of those equivalence classes represented by elements of $\overline{\Delta}$.

Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $V$. Denote by $e^*_\alpha$ the tensor $e_{\alpha_1} \ast \ldots \ast e_{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma^n_m$. We have

$$\langle e^*_\alpha \mid e^*_\beta \rangle = \begin{cases} \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\beta} \chi(\sigma \tau^{-1}) & \text{if } \alpha = \tau.\beta \text{ for some } \tau \in G, \\ 0 & \text{if } O(\alpha) \neq O(\beta), \end{cases}$$

in particular, taking the norm of $e^*_\alpha$, with respect to the induced inner product, one easily obtains the condition $e^*_\alpha \neq 0$ if and only if $\alpha \in \Omega$.

For $\gamma \in \overline{\Delta}$, $V^*_\gamma = \langle e^*_\sigma \mid \sigma \in G\rangle$ is called the orbital subspace of $V^n_\chi(G)$. It follows that

$$V^n_\chi(G) = \bigoplus_{\gamma \in \overline{\Delta}} V^*_\gamma \quad (2)$$
is an orthogonal direct sum. In [2] Freese proved that

$$\dim V^*_\gamma = \frac{\chi(1)}{|G_{\gamma}|} \sum_{\sigma \in G_{\gamma}} \chi(\sigma)$$

(3)

in particular, if $\chi$ is of degree one, then $\dim V^*_\gamma = 1$ for all $\gamma \in \Delta$.

If $\alpha = g.\gamma$ and $\beta = g'.\gamma$, then $gg'^{-1}\beta = \alpha$, so if we set $\tau = gg'^{-1}$ and use the above formula for $\langle e^*_\alpha | e^*_\beta \rangle$, then we obtain

$$\langle e^*_g.\gamma | e^*_g'.\gamma \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in gG_{g'-1}} \chi(\sigma).$$

(4)

An orthogonal basis of the form $\{e^*_\alpha | \alpha \in S\}$, where $S$ is a subset of $\Gamma_m\times m$, is called an orthogonal basis of decomposable symmetrized tensors for $V^\chi_{\times n}(G)$. By (2) $V^\chi_{\times n}(G)$ has an orthogonal basis of decomposable symmetrized tensors if and only if for all $\gamma \in \Delta$, the orbital subspace $V^*_\gamma$ has an orthogonal basis of decomposable symmetrized tensors. In particular, if $\chi$ is of degree one, since $\dim V^*_\gamma = 1$ for all $\gamma \in \Delta$, then $V^*_\gamma$ has an orthogonal basis of decomposable symmetrized tensors for all $\gamma \in \Delta$ which implies that $V^\chi_{\times n}(G)$ has such a basis.

Several papers are devoted in investigation of the existence of an orthogonal basis of decomposable symmetrized tensors for $V^\chi_{\times n}(G)$, see for example [10]. In [3] a necessary and sufficient condition for the existence of orthogonal basis of decomposable symmetrized tensors for $V^\chi_{\times n}(G)$ is given, where $G$ is a cyclic or a dihedral group. In [6] it was claimed that if $\{e_1, \ldots, e_m\}$ is an orthogonal basis of $V$, then there exists a subset $S$ of $\Gamma_m$ such that the set $\{e^*_\alpha | \alpha \in S\}$ forms an orthogonal basis of $V^\chi_{\times n}(G)$ if and only if $\chi(1) = 1$. But later in [9] a counter-example to this claim was presented.

Now it is natural to consider a group $G$ and an irreducible character $\chi \in I(G)$ and obtain necessary and sufficient conditions for the existence of an orthogonal basis for $V^\chi_{\times n}(G)$. With respect to this we consider the dicyclic group which will be explained below.

In this paper we find a necessary and sufficient condition for the existence of an orthogonal basis of decomposable symmetrized tensors of symmetry classes of tensors associated with the dicyclic group. Throughout this paper all characters are considered over the complex field $C$. We adopt notations from [7] in this paper.
2 Dicyclic Group

The group $T_{4n}$, $n \geq 1$, generated by the elements $r, s$ such that $r^{2n} = 1$, $r^n = s^2$, $s^{-1}rs = r^{-1}$ is called the dicyclic group of degree $n$, i.e., $T_{4n} = \langle r, s \mid r^{2n} = 1, r^n = s^2, s^{-1}rs = r^{-1} \rangle$ (see [4]). This group is of order $4n$ and in [1] page 7 it is denoted by $\langle 2, 2, n \rangle$ and it is proved that $T_{4n} = \langle r, s \mid r^{2n} = 1, r^n = s^2 = (rs)^2 \rangle$. In any case we have $T_{4n} = \{r^l, r^ls \mid 0 \leq l < 2n\}$.

By [4] $T_{4n}$ has $n + 3$ conjugacy classes which are

\[
\{1\}, \{r^n\}, \{r^k, r^{2n-k}\}, 1 \leq k \leq n - 1, \{r^{2k}s \mid 0 \leq k \leq n - 1\},
\]

and the character table of $T_{4n}$ is indicated in Tables I and II.

Table I: The character table of $T_{4n}$ where $n$ is odd

| $|C_{T_{4n}}(g)|$ | 4n | 4n | 2n | 4 | 4 |
|-----------------|----|----|----|---|---|
| $g$             |    |    |    |   |   |
| $\psi_1$        | 1  | 1  | 1  | 1 | 1 |
| $\psi_2$        | 1  | 1  | 1  | 1 | 1 |
| $\psi_3$        | 1  | -1 | 1  | 1 | -1|
| $\psi_4$        | 1  | -1 | (-1)$^k$ | -1 | -1|
| $\chi_h$        | 2  | 2(-1)$^h$ | 2 cos$(kh\pi/n)$ | 0 | 0 |

$(1 \leq h \leq n - 1)$

Table II: The character table of $T_{4n}$ where $n$ is even

| $|C_{T_{4n}}(g)|$ | 4n | 4n | 2n | 4 | 4 |
|-----------------|----|----|----|---|---|
| $g$             |    |    |    |   |   |
| $\phi_1$        | 1  | 1  | 1  | 1 | 1 |
| $\phi_2$        | 1  | 1  | 1  | 1 | 1 |
| $\phi_3$        | 1  | 1  | 1  | 1 | 1 |
| $\phi_4$        | 1  | 1  | (-1)$^k$ | -1 | -1|
| $\chi_h$        | 2  | 2(-1)$^h$ | 2 cos$(kh\pi/n)$ | 0 | 0 |

$(1 \leq h \leq n - 1)$

From the above tables we see that $T_{4n}$ has four irreducible characters of degree 1 namely $\psi_1, \psi_2, \psi_3, \psi_4$ if $n$ is odd and $\phi_1, \phi_2, \phi_3, \phi_4$ if $n$ is even and $n - 1$ irreducible
characters of degree 2 which are denoted by \( \chi_h, 1 \leq h \leq n - 1 \).

By classical Cayley Theorem \( T_{4n} \) can be embed in \( S_{4n} \) and so we assume that \( T_{4n} \) is a subgroup of \( S_{4n} \). In this case generators \( r, s \) of \( T_{4n} \) as permutations on \( 4n \) letters are given by

\[
\begin{align*}
    r &= (1 \ 2 \ 3 \ \ldots \ 2n)(2n + 1 \ 2n + 2 \ 2n + 3 \ \ldots \ 4n), \\
    s &= (1 \ 2n + 1 \ n + 1 \ 3n + 1)(2 \ 4n \ n + 2 \ 3n)(3 \ 4n - 1 \ n + 3 \ 3n - 1) \ldots \\
        &\quad (n - 1 \ 3n + 3 \ 2n - 1 \ 2n + 3)(n \ 3n + 2 \ 2n + 2) .
\end{align*}
\]

In particular, the dicyclic group of degree \( 2^{n-1} \) is called the generalized quaternion group and denoted by \( Q_{2^{n+1}} \), i.e., \( Q_{2^{n+1}} = T_{4(2^{n-1})} = T_{2^{n+1}} \), and \( \chi = \chi_h, 1 \leq h \leq 2^{n-1} - 1 \), are characters of degree 2 for \( Q_{2^{n+1}} \). In the following theorems we find the dimensions of the symmetry classes of tensors associated with the dicyclic group \( T_{4n} \).

**Theorem 1** Let \( G = T_{4n}, n \) odd, and assume that \( V \) is an \( m \)-unitary space. Then considering \( G \) as a subgroup of the symmetric group on \( 4n \) letters we have the following, where \( (2n, k) \) denotes the greatest common divisor of \( 2n \) and \( k \).

\[
\begin{align*}
    \dim V_{\psi_1}^{4n}(G) &= \frac{1}{4n} \left[ m^{4n} + m^{2n} + 2 \sum_{k=1}^{n-1} m^{2(2n,k)} + 2nm^n \right], \\
    \dim V_{\psi_2}^{4n}(G) &= \frac{1}{4n} \left[ m^{4n} - m^{2n} + 2 \sum_{k=1}^{n-1} (-1)^k m^{2(2n,k)} \right], \\
    \dim V_{\psi_3}^{4n}(G) &= \frac{1}{4n} \left[ m^{4n} + m^{2n} + 2 \sum_{k=1}^{n-1} m^{2(2n,k)} - 2nm^n \right], \\
    \dim V_{\psi_4}^{4n}(G) &= \frac{1}{4n} \left[ m^{4n} - m^{2n} + 2 \sum_{k=1}^{n-1} (-1)^k m^{2(2n,k)} \right], \\
    \dim V_{\chi_h}^{4n}(G) &= \frac{1}{2n} \left[ 2m^{4n} + 2(-1)^h m^{2n} + 4 \sum_{k=1}^{n-1} \cos(kh\pi/n)m^{2(2n,k)} \right], \quad 1 \leq h \leq n - 1.
\end{align*}
\]

**Proof.** Note that if \( \pi \) is a cycle of length \( a \) and \( (k, a) = d \), then \( \pi^k \) has \( d \) cycles of length \( a/d \) and therefore \( c(\pi^k) = d = (k, a) \). So \( c(1) = 4n, c(r^n) = 2n, c(r^k) = 2(2n, k) \) and \( c(s) = n \), where \( c(\pi) \) denotes the number of cycles in the cycle structure of \( \pi \) including
cycles of length one. Considering the cycle structures of \( r \) and \( s \) given above we obtain:

\[
rs = (1 \ 2n+2 \ n+1 \ 3n+2)(2 \ 2n+1 \ n+2 \ 3n+1)(3 \ 4n \ n+3 \ 3n)\ldots
\]

\[
\quad(n \ 3n+3 \ 2n \ 2n+3),
\]

therefore \( c(rs) = n \). Now using the character table of \( T_{4n} \) given in Table I the theorem holds by (1). □

**Theorem 2** Let \( G = T_{4n} \), \( n \) even, and assume that \( V \) is an \( m \)-unitary space. Then considering \( G \) as a subgroup of the symmetric group on \( 4n \) letters we have the following, where \( (2n,k) \) denotes the greatest common divisor of \( 2n \) and \( k \).

\[
\dim V_{\phi_1}^{4n}(G) = \frac{1}{4n} \left[ m^{4n} + m^{2n} + 2 \sum_{k=1}^{n-1} m^{2(2n,k)} + 2nm^n \right],
\]

\[
\dim V_{\phi_2}^{4n}(G) = \frac{1}{4n} \left[ m^{4n} + m^{2n} + 2 \sum_{k=1}^{n-1} (-1)^k m^{2(2n,k)} \right],
\]

\[
\dim V_{\phi_3}^{4n}(G) = \frac{1}{4n} \left[ m^{4n} + m^{2n} + 2 \sum_{k=1}^{n-1} m^{2(2n,k)} - 2nm^n \right],
\]

\[
\dim V_{\phi_4}^{4n}(G) = \frac{1}{4n} \left[ m^{4n} + m^{2n} + 2 \sum_{k=1}^{n-1} (-1)^k m^{2(2n,k)} \right],
\]

\[
\dim V_{\chi_h}^{4n}(G) = \frac{1}{2n} \left[ 2m^{4n} + 2(-1)^h m^{2n} + 4 \sum_{k=1}^{n-1} \cos(kh\pi/n) m^{2(2n,k)} \right], \quad 1 \leq h \leq n-1.
\]

**Proof.** Similar to the proof of Theorem 1. □

The following lemma is useful in later considerations.

**Lemma 1** Let \( H \) be a subgroup of \( T_{4n} \). Then there is a natural number \( k \), \( 0 \leq k < 2n \), such that \( H = \langle r^k \rangle \) or \( \langle r^k \rangle \leq H \) and \( H \cap \langle r \rangle = \langle r^k \rangle \). In the second case we have \(|H| \geq 2|\langle r^k \rangle|\).

**Proof.** By definition of \( T_{4n} \) we see that elements of \( T_{4n} \) are of the forms \( r^l \) or \( r^l s \) where \( 0 \leq l < 2n \). If \( H \) is an arbitrary subgroup of \( T_{4n} \), then \( H \cap \langle r \rangle \) is a cyclic subgroup of \( \langle r \rangle \) and therefore there is a natural number \( k \), \( 0 \leq k < 2n \), such that \( H \cap \langle r \rangle = \langle r^k \rangle \). The rest by our lemma follows immediately. □
3 On the Existence of Orthogonal Basis for the Symmetry Classes of Tensors Associated with the Dicyclic Group

Let $G = T_{4n}$, $n \geq 1$, and $V$ be an $m$-unitary space, with orthonormal basis $\{e_1, \ldots, e_m\}$. For $n = 1$, the dicyclic group $T_4$ is cyclic, $T_4 \cong \mathbb{Z}_4$, therefore all irreducible characters are of degree 1 and so $V^4(T_4)$ has an orthogonal basis of decomposable symmetrized tensors for all $\chi \in I(T_4)$. Therefore we assume that $n \geq 2$. If $m = 1$, then $\dim \otimes^4 V = 1$, so $\dim V^4(T_{4n}) = 0$ or 1, therefore we don’t have any problem about the existence of orthogonal basis of decomposable symmetrized tensors for $V^4(T_{4n})$ for all $\chi \in I(G)$, therefore we assume that $m \geq 2$.

For irreducible characters of $T_{4n}$ of degree 1, $\psi_i$, $\phi_i$, $1 \leq i \leq 4$, $V^4(T_{4n})$ and $V^4(T_{4n})$ have an orthogonal basis of decomposable symmetrized tensors and so we don’t deal with the $\psi_i$’s and $\phi_i$’s.

Therefore we investigate the problem for irreducible characters of degree 2 of $T_{4n}$, i.e., $\chi_h$, $1 \leq h \leq n - 1$, which are given by

$$\chi_h(r^k) = 2\cos \frac{kh\pi}{n}, \quad \chi_h(r^ks) = 0, \quad 0 \leq k < 2n.$$

Lemma 2 Suppose $n \geq 2$, $1 \leq h \leq n - 1$, $0 \leq k < 2n$. Let $l = (2n/(2n,k))$, where $(2n,k)$ denotes the greatest common divisor of $2n$ and $k$. Then we have

$$\sum_{l=1}^{l} \cos \frac{tkh\pi}{n} = \begin{cases} l & \text{if } kh \equiv 0, \\ 0 & \text{if } kh \not\equiv 0. \end{cases}$$

Proof. It is straightforward. □

Lemma 3 Suppose $G = T_{4n}$, $n \geq 2$, and $\chi = \chi_h$, $1 \leq h \leq n - 1$. Let $H$ be any subgroup of $T_{4n}$, i.e., $H = \langle r^k \rangle$ or $\langle r^k \rangle \leq H$ and $H \cap \langle r \rangle = \langle r^k \rangle$, for some $k$, $0 \leq k < 2n$. If $l = (2n/(2n,k))$, where $(2n,k)$ denotes the greatest common divisor of $2n$ and $k$, then we have

$$\sum_{g \in H} \chi(g) = \begin{cases} 2l & \text{if } kh \equiv 0, \\ 0 & \text{if } kh \not\equiv 0. \end{cases}$$
Proof. We have $o(r^k) = (2n/(2n,k)) = l$, so $H = \{r^k, r^{2k}, \ldots, r^{lk}\}$ or $\{r^k, r^{2k}, \ldots, r^{lk}\} \subseteq H$ and $H \cap \langle r \rangle = \{r^k, r^{2k}, \ldots, r^{lk}\}$. But $\chi$ vanishes outside $\langle r \rangle$, therefore by Lemma 2 we have
\[
\sum_{g \in H} \chi(g) = \sum_{l=1}^l \chi(r^{lk}) = 2 \sum_{l=1}^l \cos \frac{tkh \pi}{n} = \begin{cases} 
2l & \text{if } kh \frac{2n}{2n} = 0, \\
0 & \text{if } kh \neq 0.
\end{cases}
\]
\[
\square
\]

Lemma 4 Let $G = T_{4n}$, $n \geq 2$, and $\chi = \chi_h$, $1 \leq h \leq n - 1$. Then for $\gamma \in \overline{\Delta}$, we have $G_\gamma = \langle r^k \rangle$ or $\langle r^k \rangle \leq G_\gamma$ and $G_\gamma \cap \langle r \rangle = \langle r^k \rangle$, for some $k$, $0 \leq k < 2n$, where $kh \frac{2n}{2n} = 0$. In particular, if $\langle r^k \rangle \leq G_\gamma$, then we have $|G_\gamma| \geq 2|\langle r^k \rangle|$.

Proof. $G_\gamma$ is a subgroup of $G$ so by Lemma 1, $G_\gamma = \langle r^k \rangle$ or $\langle r^k \rangle \leq G_\gamma$ and $G_\gamma \cap \langle r \rangle = \langle r^k \rangle$, for some $k$, $0 \leq k < 2n$. In particular if $\langle r^k \rangle \leq G_\gamma$, then $|G_\gamma| \geq 2|\langle r^k \rangle|$. But by Lemma 3 if $kh \frac{2n}{2n} \neq 0$, then $\sum_{g \in G_\gamma} \chi(g) = 0$ and so $\gamma \not\in \overline{\Delta}$. This contradiction show that $kh \frac{2n}{2n} = 0$. \square

Lemma 5 Let $n \geq 2$ and $1 \leq h \leq n - 1$. Then there exist $t$, $t'$, $0 \leq t$, $t' < 2n$, such that $\cos ((t - t')h \pi/n) = 0$ if and only if $\nu_2(h/n) < 0$, where $\nu_2$ is the 2-adic valuation.

Proof. It is straightforward. For the definition of $p$-adic valuation we refer the reader to [8]. \square

Lemma 6 Suppose $G = T_{4n}$, $n \geq 2$, and let $\chi = \chi_h$, $1 \leq h \leq n - 1$. Let $\gamma \in \overline{\Delta}$ and suppose that $G_\gamma$ is of the form $G_\gamma = \langle r^k \rangle$, where $0 \leq k < 2n$, $kh \frac{2n}{2n} = 0$. If $\nu_2(h/n) < 0$, where $\nu_2$ is the 2-adic valuation, then the orbital subspace $V_\gamma^*$ has an orthogonal basis of decomposable symmetrized tensors.

Proof. We have $o(r^k) = (2n/(2n,k)) = l$, so $G_\gamma = \{r^k, r^{2k}, \ldots, r^{lk}\}$ and therefore by (3) and Lemma 3
\[
\dim V_\gamma^* = \frac{\chi(1)}{|G_\gamma|} \sum_{g \in G_\gamma} \chi(g) = \frac{2}{l} \cdot (2l) = 4.
\]
Now for all $g$, $g' \in G$ we have
\[
g' G_\gamma g^{-1} = \begin{cases} 
\{r^{k+b-a}, r^{2k+b-a}, \ldots, r^{lk+b-a}\} & \text{if } g = r^a, g' = r^b, \\
\{r^{k+n+a+b} r^{2k+n+a+b} s, r^{lk+n+a+b} s, \ldots, r^{2k+n+a+b} s\} & \text{if } g = r^a s, g' = r^b s, \\
\{r^{-k+b-a}, r^{-2k+b-a}, \ldots, r^{-lk+b-a}\} & \text{if } g = r^a s, g' = r^b s.
\end{cases}
\]
For $g = r^a$, $g' = r^b$ by (4) we have

$$\langle e^*_g, e^*_{g'} \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in gG, g^{-1}} \chi(\sigma)$$

$$= \frac{1}{n} \sum_{t=1}^{l} \cos \left( \frac{(tk + b - a)h^\pi}{n} \right)$$

If $g = r^a s$ and $g' = r^b s$ then by the same computation we obtain $\langle e^*_g, e^*_{g'} \rangle = 0$ or $\langle e^*_g, e^*_{g'} \rangle = (l/n) \cos \left( (b-a)h^\pi/n \right)$, respectively. Therefore

$$\langle e^*_g, e^*_{g'} \rangle = \begin{cases} \frac{1}{n} \cos \left( \frac{(b-a)h^\pi}{n} \right) & \text{if } g = r^a, g' = r^b, \\ 0 & \text{if } g = r^a s, g' = r^b s, \\ \frac{1}{n} \cos \left( \frac{(b-a)h^\pi}{n} \right) & \text{if } g = r^a s, g' = r^b s. \end{cases}$$

Since by the assumption $\nu_2(\frac{h}{n}) < 0$, hence by Lemma 5 there exist $t, t'$, $0 \leq t, t' < 2n$, such that $\cos \left( (t-t')h^\pi/n \right) = 0$. Let $S = \{r^t \cdot \gamma, r^{t'} \cdot \gamma, r^s \cdot \gamma, r^{t'} \cdot \gamma \} \subseteq \Gamma_m$, then by the above computation, we have $\langle e^*_g, e^*_{g'} \rangle = 0$ for all $\alpha, \beta \in S$, $\alpha \neq \beta$. But $\dim V_\gamma^* = 4$, so $\{e^*_\alpha \mid \alpha \in S\}$ is an orthogonal basis of decomposable symmetrized tensors for $V_\gamma^*$. □

Lemma 7  Suppose $G = T_{4n}$, $n \geq 2$, and $\chi = \chi_h$, $1 \leq h \leq n-1$. Let $\gamma \in \Delta$ be such that $\langle r^k \rangle \subseteq G_\gamma$ and $G_\gamma \cap \langle r \rangle = \langle r^k \rangle$, where $0 \leq k < 2n$, $kh \equiv 0$. If $\nu_2(\frac{h}{n}) < 0$, where $\nu_2$ is 2-adic valuation, then the orbital subspace $V_\gamma^*$ has an orthogonal basis of decomposable symmetrized tensors.

Proof. We have $o(r^k) = (2n/(2n,k)) = l$, so $\{r^k, r^{2k}, \ldots, r^{lk}\} \subseteq G_\gamma$ and $G_\gamma \cap \langle r \rangle = \{r^k, r^{2k}, \ldots, r^{lk}\}$. Note that, by Lemma 4, in this case we have $|G_\gamma| \geq 2l$, and therefore by (3) and Lemma 3

$$\dim V_\gamma^* = \frac{\chi(1)}{|G_\gamma|} \sum_{g \in G_\gamma} \chi(g) \leq \frac{2}{2l} \cdot (2l) = 2$$
so \( \dim V^*_\gamma = 1 \) or \( \dim V^*_\gamma = 2 \). If \( \dim V^*_\gamma = 1 \), then we don’t have any problem about the existence of orthogonal basis of decomposable symmetrized tensors for \( V^*_\gamma \), therefore we assume that \( \dim V^*_\gamma = 2 \). For \( g = r^a \), \( g' = r^b \) we have \( \{r^{k+b-a}, r^{2k+b-a}, \ldots, r^{lk+b-a}\} \leq g'G_\gamma g^{-1} \) and \( g'G_\gamma g^{-1} \cap (r) = \{r^{k+b-a}, r^{2k+b-a}, \ldots, r^{lk+b-a}\} \) therefore by (4) we have

\[
\langle e^*_{g,\gamma} e^*_{g',\gamma} \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in g'G_\gamma g^{-1}} \chi(\sigma)
\]

\[
= \frac{2}{4n} \sum_{l=1}^{n} \cos \left( \frac{(tk+b-a)h\pi}{n} \right)
\]

\[
= \frac{1}{n} \sum_{l=1}^{n} \cos \left( \frac{t(h\pi)}{n} + \frac{(b-a)h\pi}{n} \right)
\]

\[
= \frac{1}{n} \sum_{l=1}^{n} \cos \left( \frac{(b-a)h\pi}{n} \right)
\]

\[
= \frac{1}{n} \cos \left( \frac{(b-a)h\pi}{n} \right).
\]

Since by assumption \( \nu_2 \left( \frac{b}{n} \right) < 0 \), hence by Lemma 5 there exist \( t, t', 0 \leq t, t' < 2n \), such that \( \cos \left( \left( t - t' \right)h\pi/n \right) = 0 \), therefore by the above computation \( \langle e^*_{r^t,\gamma} e^*_{r^{t'},\gamma} \rangle = 0 \).

For \( S = \{r^t, \gamma, r^{t'}, \gamma\} \subset \Gamma^{4n}_m \), since \( \dim V^*_\gamma = 2 \), so \( \{e^*_{\alpha}| \alpha \in S\} \) is an orthogonal basis of decomposable symmetrized tensors for \( V^*_\gamma \). \( \square \)

**Theorem 3** Let \( G = T_{4n}, \ n \geq 2 \), and \( \chi = \chi_h, \ 1 \leq h \leq n - 1 \), \( \dim V = m \geq 2 \). Then \( V^*_\chi(G) \) has an orthogonal basis of decomposable symmetrized tensors if and only if \( \nu_2(h/n) < 0 \), where \( \nu_2 \) is 2-adic valuation.

**Proof.** Assume \( V^*_\chi(G) \) has an orthogonal basis of decomposable symmetrized tensors, therefore by (2) for all \( \gamma \in \Delta \), the orbital subspace \( V^*_\gamma \) has an orthogonal basis of decomposable symmetrized tensors, in particular for \( \gamma = (1,2,2,\ldots,2) \). Note that in this case \( G_\gamma = \{1\} \), and \( \sum_{g \in G_\gamma} \chi(g) = 2 \neq 0 \), so \( \gamma \in \Delta \). For all \( g, g' \in G \), we have

\[
g'G_\gamma g^{-1} = \begin{cases} 
\{r^{b-a}\} & \text{if } g = r^a, \ g' = r^b, \\
\{r^{n+a+b}\} & \text{if } g = r^{a}s, \ g' = r^b, \\
\{r^{b-a}\} & \text{if } g = r^a, \ g' = r^b.s
\end{cases}
\]
Therefore by (4) we have
\[
\langle e_{g,\gamma}^* | e_{g',\gamma}^* \rangle = \begin{cases} 
\frac{1}{n} \cos \left( \frac{(b-a)h\pi}{n} \right) & \text{if } g = r^a, \ g' = r^b, \\
0 & \text{if } g = r^a s, \ g' = r^b, \\
\frac{1}{n} \cos \left( \frac{(b-a)h\pi}{n} \right) & \text{if } g = r^a s, \ g' = r^b s.
\end{cases}
\]
But by (3)
\[
\dim V^*_{\gamma} = \frac{\chi(1)}{|G_{\gamma}|} \sum_{g \in G_{\gamma}} \chi(g) = \frac{2}{1}(2) = 4.
\]
By the above computation if there are 4 decomposable symmetrized tensors for which any distinct pair are mutually orthogonal, then there should exist \( t, t' \), \( 0 \leq t, t' < 2n \), such that \( \cos \left( (t - t')h\pi/n \right) = 0 \). Therefore by Lemma 5 we obtain \( \nu_2(h/n) < 0 \).

Conversely assume \( \nu_2(h/n) < 0 \), then by Lemmas 4, 6 and 7, for all \( \gamma \in \Delta \), \( V^*_{\gamma} \) has the orthogonal basis of decomposable symmetrized tensors, and therefore by (2) so does \( V^{4n}_\chi(G) \).

**Corollary 1** Let \( G = T_{4n}, \ n \geq 2 \) is odd, and \( \chi = \chi_h, \ 1 \leq h \leq n - 1 \), \( \dim V = m \geq 2 \). Then \( V^{4n}_\chi(G) \) does not have an orthogonal basis of decomposable symmetrized tensors.

**Proof.** Since \( n \) is odd, therefore \( \nu_2(h/n) \geq 0 \), and by Theorem 3 the corollary holds. \( \square \)

**Theorem 4** Let \( G = Q_{2^n+1}, \ n \geq 2 \), the generalized quaternion group, and \( \chi = \chi_h, 1 \leq h \leq 2^n - 1 \), \( \dim V = m \geq 2 \). Then \( V^{2n+1}_\chi(G) \) has an orthogonal basis of decomposable symmetrized tensors.

**Proof.** Note that \( G = Q_{2^n+1} = T_{4(2^n-1)} \), and since \( 1 \leq h \leq 2^n - 1 \) therefore \( \nu_2(h/2^{n-1}) < 0 \) and by Theorem 3, this result follows. \( \square \)

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**References**


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