Non-Vanishing and Orthogonal Basis of Symmetry Classes of Tensors

M.R. Darafsheh and M.R. Pournaki

Department of Mathematics and Computer Science, University of Tehran, Tehran, Iran
Institute for Studies in Theoretical Physics and Mathematics, Tehran, Iran
E-mail: {darafsheh,pournaki}@vax.ipm.ac.ir

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Abstract. By Cayley’s theorem, any finite group $G$ of order $n$ can be regarded as a subgroup of the symmetric group $S_n$. Let $\chi$ be any irreducible complex character of $G$ and let $V^n_G$ denote the symmetry classes of tensors associated with $G$ and $\chi$. In this paper assuming the Cayley representation of $G$, we obtain a formula for the dimension of $V^n_G$ and discuss its non-vanishing in general. A necessary condition for the existence of the orthogonal basis of decomposable symmetrized tensors for $V^n_G$ is also obtained.

Keywords: symmetry class of tensors, decomposable symmetrized tensor, orthogonal basis, Cayley representation.

1. Introduction

Let $V$ be an $m$-dimensional vector space over the complex field $\mathbb{C}$. Let $\otimes^n V$ be the $n$th tensor power of $V$ and write $v_1 \otimes \cdots \otimes v_n$ for the decomposable tensor product of the indicated vectors. To each permutation $g$ in $S_n$, there corresponds a unique linear operator $P(g) : \otimes^n V \to \otimes^n V$ determined by $P(g)(v_1 \otimes \cdots \otimes v_n) = v_{g^{-1}(1)} \otimes \cdots \otimes v_{g^{-1}(n)}$. Let $G$ be a subgroup of $S_n$ and $I(G)$ the set of all the irreducible complex characters of $G$. It follows from the orthogonality relations for characters that

$$\left\{ T(G, \chi) : \otimes^n V \to \otimes^n V \mid T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) P(g), \chi \in I(G) \right\}$$

is a set of annihilating idempotents which sum to the identity. The image of $\otimes^n V$ under $T(G, \chi)$ is called the symmetry class of tensors associated with $G$ and $\chi$ and is denoted by $V^n_G$. The image of $v_1 \otimes \cdots \otimes v_n$ under $T(G, \chi)$ is denoted by $v_1 \ast \cdots \ast v_n$ and is called a decomposable tensor. It is well known that

$$\dim V^n_G = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g)m^{\chi(g)},$$

(1)
where \( c(g) \) is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of \( g \) (see [7]). Also,

\[
\otimes^n V = \bigoplus_{\chi \in \ell(G)} V^n_\chi(G)
\]

(2)

is a direct sum.

Let \( \Gamma_m^n \) be the set of all sequences \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( 1 \leq \alpha_i \leq m \) so that \( \alpha \) is a mapping from a set of \( n \) elements into a set of \( m \) elements. Then the group \( G \) acts on \( \Gamma_m^n \) by \( g \cdot \alpha := (\alpha_{g^{-1}(1)}, \ldots, \alpha_{g^{-1}(n)}) \), where \( g \in G \) is a permutation on \( n \) letters and \( \alpha \in \Gamma_m^n \) is a mapping from a set of \( n \) elements into a set of \( m \) elements. Therefore, the action may be written as \( g \cdot \alpha = \alpha g^{-1} \) which is a composition of two functions. Let \( O(\alpha) = \{ g \cdot \alpha \mid g \in G \} \) be the orbit with representative \( \alpha \), and also let \( G_\alpha \) be the stabilizer of \( \alpha \), i.e., \( G_\alpha = \{ g \in G \mid g \cdot \alpha = \alpha \} \). Let \( \Delta \) be a system of distinct representatives of the orbits of \( G \) acting on \( \Gamma_m^n \) and define

\[
\Delta = \left\{ \alpha \in \Delta \mid \sum_{g \in G_\alpha} \chi(g) \neq 0 \right\},
\]

and let \( \bar{\Delta} \) be the union of those equivalence classes represented by elements of \( \Delta \).

Let \( \{ e_1, \ldots, e_m \} \) be a basis of \( V \). Denote by \( e^*_\alpha \) the tensor \( e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n} \), where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma_m^n \). For \( \gamma \in \Delta \), \( V^*_\gamma = \langle e^*_\gamma \mid g \in G \rangle \) is called the orbital subspace of \( V^n_\chi(G) \). It follows that

\[
V^n_\chi(G) = \bigoplus_{\gamma \in \bar{\Delta}} V^*_\gamma
\]

(3)

is a direct sum. In [4], Freese proved that

\[
\dim V^*_\gamma = \frac{\chi(1)}{|G|} \sum_{\gamma \in G_\gamma} \chi(g),
\]

(4)
in particular, if \( \chi \) is of degree one, then \( \dim V^*_\gamma = 1 \) for all \( \gamma \in \bar{\Delta} \).

A particular case appears when we assume that \( V \) is an \( m \)-unitary space. In this case, the inner product on \( V \) induces an inner product on \( \otimes^n V \) whose restriction to \( V^n_\chi(G) \) satisfies

\[
\langle u_1 \otimes \cdots \otimes u_n \mid v_1 \otimes \cdots \otimes v_n \rangle = \frac{\chi(1)}{|G|} d^G_\chi(A),
\]

where \( A = [a_{ij}]_{n \times n} = [(\alpha_i)_{j}]_{n \times n} \) and \( d^G_\chi(A) = \sum_{g \in G} \chi(g) a_{\alpha(g(1))} \cdots a_{\alpha(g(n))} \) is the generalized matrix function.

With respect to the above inner product, the sums that appeared in (2) and (3) are orthogonal direct sums. Also, if \( \{ e_1, \ldots, e_m \} \) is an orthonormal basis of \( V \), then we obtain

\[
\langle e^*_\alpha \mid e^*_\beta \rangle = \begin{cases} 
\frac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha} \chi(\sigma^{-1}) & \text{if } \alpha = \tau \cdot \beta \text{ for some } \tau \in G, \\
0 & \text{if } O(\alpha) \neq O(\beta).
\end{cases}
\]

In particular, by taking the norm of \( e^*_\alpha \), with respect to the induced inner product, one can easily obtain the condition \( e^*_\alpha \neq 0 \) if and only if \( \alpha \in \bar{\Delta} \).
If $\alpha = g \cdot \gamma$ and $\beta = g' \cdot \gamma$, then $gg' = g'g^{-1}$, and so if we let $\tau = gg'g^{-1}$ and use the above formula for $(e^\alpha_\gamma | e^\beta_\gamma)$, then we obtain

$$
(e^\gamma_\alpha | e^\alpha_\gamma) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\gamma} \chi(g' \sigma g^{-1}).
$$

An orthogonal basis of the form $\{e^\alpha_\gamma | \alpha \in S\}$, where $S$ is a subset of $\Gamma^n$, is called an **orthogonal basis of decomposable symmetrized tensors** for $V^\alpha_\gamma(G)$. By (3), $V^\alpha_\gamma(G)$ has an orthogonal basis of decomposable symmetrized tensors if and only if, for all $\gamma \in \Delta$, the orbital subspace $V^\alpha_\gamma$ has an orthogonal basis of decomposable symmetrized tensors.

In particular, if $\chi$ is of degree one, since $\dim V^\gamma_\gamma = 1$ for all $\gamma \in \Delta$, then $V^\alpha_\gamma$ has an orthogonal basis of decomposable symmetrized tensors for all $\gamma \in \Delta$ which implies that $V^\alpha_\gamma(G)$ has such a basis. Non-vanishing of $V^\alpha_\gamma(G)$ was studied by several authors and they found a formula for $\dim V^\alpha_\gamma(G)$ in a more closed form than (1). Also, the existence of an orthogonal basis of decomposable symmetrized tensors for these vector spaces was considered (see, for example, [1, 14]). In [8] and [10], a formula for $\dim V^\alpha_\gamma(G)$ is also given when $G$ is equal to the whole group $S_n$, and in [13], a formula for calculating $\dim V^\alpha_\gamma(G)$ is given in the case that $G = \langle \tau_1 \rangle = \langle \tau_r \rangle$ and in [2] for $G = \langle \tau_1 \rangle = \langle \tau_p \rangle$ is given, where $\tau_i$, $1 \leq i \leq p$, are disjoint cycles in $S_n$.

Also, in [5] a necessary and sufficient condition for the existence of orthogonal basis of decomposable symmetrized tensors for $V^\alpha_\gamma(G)$ is given, when $G$ is a cyclic or a dihedral group and in [3] when $G$ is a dicyclic group.

In this paper, we let $G$ be a subgroup of $S_n$ by acting faithfully on a set of $n$ elements. In this case, the vector space $V^\alpha_\gamma(G)$ is meaningful for all $\chi \in I(G)$ and we will discuss the non-vanishing property of these vector spaces. As a special case, that is, when $G$ is a group of order $n$ and acts on $G$ by right multiplication, we can prove that, for all $\chi \in I(G)$, $V^\alpha_\gamma(G) \neq 0$, and we will find a necessary condition for the existence of the orthogonal basis of decomposable symmetrized tensors for these vector spaces.

### 2. Main Results

Let $V$ be an $m$-dimensional vector space over the complex field $\mathbb{C}$ and let $G$ be a finite group and $\Omega$ a set of $n$ elements. Suppose $G$ acts faithfully on $\Omega$, so we can assume that $G$ is a subgroup of $S_n$, i.e., $G = \{g | g \in G\} = \{\sigma_g | g \in G\}$, where $\sigma_g : \Omega \to \Omega$ defined by $\sigma_g(\omega) = g \cdot \omega$ for all $\omega \in \Omega$, is a permutation on $n$ letters. Therefore, the vector space $V^\alpha_\gamma(G)$ is meaningful for all $\chi \in I(G)$, $G$ as a subgroup of $S_n$ acts on $n$ letters and we denote by $\theta$ its permutation character. For $g \in G$, the value $\theta(g)$ is the number of letters fixed by $g$, i.e., the number of cycles of length one in the cycle structure of $g$.

In the following lemma we give a formulation of (1) in terms of $\theta$.

**Lemma 1.** Let $G$ be a finite group and $\Omega$ a set of $n$ elements. Assume that $G$ acts faithfully on $\Omega$ and let $V$ be an $m$-dimensional vector space over the complex field $\mathbb{C}$. Then, for all $\chi \in I(G)$, we have

$$
\dim V^\alpha_\gamma(G) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) m^{(\theta^1(\gamma), \lambda_{\gamma})}.
$$
Proof. Suppose \( g = \sigma_x \in G \). Then by definition, \( c(g) \) is the number of cycles in the cycle structure of \( g \) including cycles of length one. But it is easy to see that this number is equal to the number of orbits of the cyclic group \( \langle g \rangle \) acting on \( \Omega \). By Burnside’s Lemma (see [9, p. 59]) the number of orbits of \( \langle g \rangle \) acting on \( \Omega \) is \( 1/|\langle g \rangle| \Sigma_{\sigma \in \langle g \rangle} |\sigma| \) and so \( c(g) = (0 \downarrow \langle g \rangle, 1_{\langle g \rangle} \langle g \rangle) \), hence the theorem follows by (1).

As an application of the above lemma, we give here a solution to problem 5.18 in [6] and obtain a number theoretical result.

Proposition 2. Let \( G \) be a finite group. Write \( a(k) = |\{ g \in G \mid o(g) = k \}| \). Then the polynomial \( f(x) = 1/|G| \Sigma_k a(k)x^{(G)/k} \) takes on integer values whenever \( x \in \mathbb{Z} \).

Proof. First, we claim that \( \psi : G \rightarrow \mathbb{C} \), defined by \( \psi(g) = (-1)^{|G|/o(g)}, g \in G \), is a generalized character of \( G \). If \( |G| \) is odd, then \( \psi(g) = -1 \), for all \( g \in G \). This leads to \( \psi = -1_G \) and so \( \psi \) is a generalized character of \( G \). Hence, we suppose \( |G| \) is even. Consider \( D : G \rightarrow GL_n(\mathbb{C}) \) the regular representation of \( G \), where \( |G| = n \). Then by problem 2.3 of [6], \( \det : G \rightarrow \mathbb{C} \), defined by \( \det(g) = \det(D(g)), g \in G \), is a linear character of \( G \). But it is easy to see that \( \det(D(g)) = (-1)^{|G|/o(g)} = -1 \), for all \( g \in G \). Since \( |G| \) is even, for all \( g \in G \), we have \( \det(g) = \det(D(g)) = (-1)^{|G|/o(g)} = (-1)^{|G|/o(g)} = \psi(g) \). Therefore, \( \det = \psi \) and so in this case, \( \psi \) is a linear character of \( G \), and our claim is established.

Now by the notation in Lemma 1, we consider the function \( P : g \rightarrow P(g) \). It can be easily shown that \( P \) is a representation of \( G \) affording the character \( \phi_m \), such that \( \phi_m(g) = m^{e(g)} = m^{-\frac{e(g)}{o(g)}}, g \in G \).

Let \( \Omega = G \) and consider that \( G \) acts on \( \Omega \) by right multiplication, and put \( |G| = n \). In this case, \( G \) can be regarded as a subgroup of \( S_n \), whose action on \( \Omega \) is regular and therefore, \( c(1) = n = |G| \) and \( c(g) = |G|/o(g) \) for \( g \neq 1 \). Thereby, for any \( \chi \in I(G) \), we have \( \dim V^\chi_G = \chi(1)/|G| \Sigma_{g \in G} \chi(g)m^{G/o(g)} \). Now, taking \( \chi \) to be the principal character of \( G \), we get \( \dim V^{1_G}_G = 1/|G| \Sigma_{g \in G} \chi(g)m^{G/o(g)} \) which is a natural number.

For all \( x \in \mathbb{Z} \), we have \( f(x) = 1/|G| \Sigma_k a(k)x^{(G)/k} = 1/|G| \Sigma_{g \in G} x^{(G)/o(g)} \). Therefore, if \( x > 0 \), then \( f(x) = (\phi_1, 1_G)_G \in \mathbb{N} \) and if \( x < 0 \), then \( f(x) = (\phi_{-x}, \psi)_G \in \mathbb{Z} \). This is because \( \psi \) is a generalized character of \( G \). Therefore, \( f(x) \) takes on integer values, for all \( x \in \mathbb{Z} \).

If \( G \) is a cyclic group, then for any divisor \( k \) of \( |G| \) we have \( a(k) = \varphi(k) \), where \( \varphi \) is the Euler \( \varphi \)-function, and therefore, Proposition 2 implies that \( 1/|G| \Sigma_{k | G} \varphi(k)x^{(G)/k} \in \mathbb{Z} \) for all \( x \in \mathbb{Z} \). Hence, if \( G \) is a cyclic group of order \( p \), \( p \) prime, then \( 1/p (x^p + (p - 1)x) \in \mathbb{Z} \) which implies Fermat’s little theorem \( x^p \equiv x \mod{p} \).

In [11] and [12], a similar consideration led the authors to obtain Furtwängler’s little theorem as a consequence. Let \( G \) be a subgroup of the symmetric group on a finite set \( \Omega \) and let \( H \leq G \). Let \( \mu \) be a function defined on subgroups of \( G \) recursively by \( \mu(H, H) = 1 \) and \( \mu(H, T) = -\Sigma_{H \leq K < T} \mu(H, K) \). If \( H = 1 \) is the trivial subgroup of \( G \), then it is easy to prove that \( \mu(1, T) = \mu(|T|) \) is the ordinary Möbius function of number theory. For a subgroup \( T \) of \( G \), let \( c(T) \) be the number of orbits of \( T \) on \( \Omega \). Then it is proved in [12] that, for any \( x \in \mathbb{Z} \), we have \( |H|/|N_G(H)| |\Sigma_{T \geq H} \mu(H, T)a^{c(T)} \in \mathbb{Z} | \). If we let \( H \) be the trivial subgroup of \( G \), then the above formula becomes \( 1/|G| \Sigma_{T \geq 1} \mu(1, T)a^{c(T)} \in \mathbb{Z} \). Now, assume that \( G \) is embedded in \( S_3 \) by Cayley representation. If \( G \) is assumed to be a cyclic group, then we have \( 1/|G| \Sigma_{k | G} \mu(k)x^{(G)/k} \in \mathbb{Z} \). Now, if \( G = Z_p, p \) prime,
then by the above formula, we obtain $1/p(a^p - a) \in \mathbb{Z}$ and consequently, we re-obtain Fermat’s little theorem.

In the following we obtain a result about the non-vanishing of the symmetry classes of tensors which is a generalization of the known results.

**Lemma 3.** Let $G$ be a finite group and $\Omega$ a set of $n$ elements $n \geq 2$. Assume that $G$ acts faithfully on $\Omega$ and let $V$ be an $m$-dimensional vector space over the complex field $\mathbb{C}$. If, for all $g \in G - \{1\}$, $|\text{fix}(g)| \leq l'$, where $l'$ is the sharp upper bound, i.e., there is a $g \in G - \{1\}$ such that $|\text{fix}(g)| = l'$. Then, we have $l' \leq l$ and $l' \leq n - 2$. Consider the action of the group $G$ on $\Gamma_{\mathbb{C}}^{n}$ and put $\gamma = (1, 2, \ldots, l', l' + 1, l' + 2, \ldots, l' + 2)$. If $m \geq l + 2$, then $m \geq l' + 2$, but $l' + 2 \leq n$, therefore $\gamma \in \Gamma_{\mathbb{C}}^{n}$, and we can choose $\Delta$ such that $\gamma \in \Delta$. By our hypothesis, one easily obtains that $G_{\gamma} = \{1\}$, so $\sum_{g \in G_{\gamma}} \chi(g) = \chi(1) \neq 0$ and therefore, $\gamma \in \Delta$. But by (4), we have $\dim V_{\gamma}^{\ast} = \chi(1)/|G_{\gamma}| \sum_{g \in G_{\gamma}} \chi(g) = \chi(1) \neq 0$, therefore, $V_{\gamma}^{\ast} \neq 0$, and by (3), we have $V_{\gamma}^{\ast}(G) \neq 0$.

Since, for all non-identity $g \in G$, we have $|\text{fix}(g)| \leq n - 2$, we obtain the following result (see [10]) which is a consequence of the above lemma.

**Corollary 4.** Let $G$ be a finite group and $\Omega$ a set of $n$ elements, $n \geq 2$. Assume that $G$ acts faithfully on $\Omega$ and let $V$ be an $m$-dimensional vector space over the complex field $\mathbb{C}$. Then, for all $m \geq n$ and all $\chi \in I(G)$, we have $V_{\chi}^{\ast}(G) \neq 0$. In particular, for the subgroup $G$ of $S_{n}$, we have $V_{\chi}^{\ast}(G) \neq 0$, for all $m \geq n$.

Now, we consider a special case. Suppose $G$ is a group of order $n$ and $G$ acts on $\Omega = G$ by right multiplication, i.e., for all $g \in G$, $w \in \Omega$, $g \cdot w = gw$. This action is faithful, therefore, $G$ is a subgroup of $S_{n}$ and we say $G$ is a subgroup of $S_{n}$ by Cayley representation. By Lemmas 1 and 3, we re-obtain the following result.

**Theorem 5.** Let $G$ be a group of order $n$, that is, a subgroup of $S_{n}$ by Cayley representation. If $V$ is an $m$-dimensional vector space over the complex field $\mathbb{C}$, then for all $\chi \in I(G)$, we have

$$\dim V_{\chi}^{\ast}(G) = \frac{\chi(1)}{n} \sum_{g \in G} \chi(g)m^{\sigma(g)},$$

in particular, for all $m \geq 2$, $V_{\chi}^{\ast}(G) \neq 0$.

In closing, we give a necessary condition for the existence of the orthogonal basis of decomposable symmetrized tensors for $V_{\chi}^{\ast}(G)$ in the case where the finite group $G$ of order $n$ is embedded in $S_{n}$ by Cayley representation.

**Theorem 6.** Let $G$ be a non-trivial group of order $n$, that is, a subgroup of $S_{n}$ by Cayley representation. If $V$ is an $m$-unitary space, $m \geq 2$, and $\chi \in I(G)$ such that $\chi(1) = |G|/2$, then $V_{\chi}^{\ast}(G)$ does not have an orthogonal basis of decomposable symmetrized tensors.
Proof. Let \( \{e_1, \ldots, e_m\} \) be an orthonormal basis of \( V \). Suppose \( V^*_\chi(G) \) has an orthogonal basis of decomposable symmetrized tensors. Then, by (3), for all \( \gamma \in \overline{\Delta} \), the orbital subspace \( V^*_\chi \) has an orthogonal basis of decomposable symmetrized tensors. Put \( \gamma = (1, 2, \ldots, 2) \in \Gamma^*_n \), then we have \( G_\gamma = \{1\} \). Therefore, \( \Sigma_{\gamma \in G} \chi(g) = \chi(1) \neq 0 \), so we can assume that \( \gamma \in \overline{\Delta} \).

Therefore, by the above discussion, \( V^*_\chi, \gamma = (1, 2, \ldots, 2) \) has an orthogonal basis of decomposable symmetrized tensors. Since \( \dim V^*_\chi = \chi(1)/|G| \Sigma_{\gamma \in G} \chi(g) = \chi(1)^2 = s \), we can assume that \( \{e^*_{\gamma \times \gamma}, e^*_{\gamma \times \gamma}, \ldots, e^*_{\gamma \times \gamma}\} \) is an orthogonal basis of decomposable symmetrized tensors for \( V^*_\chi \).

Define the \( n \times n \) complex matrix \( A = [a_{ij}] \) as below:

\[
a_{ij} = \frac{\chi(1)}{n} \chi(g_i g_j^{-1}),
\]

where \( G = \{g_1, \ldots, g_n\} \). For all \( i, j, 1 \leq i, j \leq s \), by (5), we obtain

\[
a_{ij} = \frac{\chi(1)}{n} \chi(g_i g_j^{-1}) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(g_i \sigma g_j^{-1}) = (e^*_{\gamma, \gamma} | e^*_{\gamma, \gamma}) = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{\chi(1)}{n} \chi(1) & \text{if } i = j, \end{cases} = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{\chi(1)}{n} & \text{if } i = j. \end{cases}
\]

Therefore, \( A \) has the form

\[
A = \left[ \begin{array}{ccc} \frac{s}{n} I_s & A_1 \\ A_2 & A_3 \end{array} \right],
\]

where \( A_1, A_2, \) and \( A_3 \) are matrices of sizes \( s \times (n-s), (n-s) \times s, \) and \( (n-s) \times (n-s) \), respectively, and \( I_s \) is the \( s \times s \) identity matrix. If \( A^2 = [b_{ij}] \), then by the generalized orthogonality relations for characters, we obtain

\[
b_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj} = \frac{\chi(1)^2}{n^2} \sum_{k=1}^{n} \chi(g_i g_k^{-1}) \chi(g_k g_j^{-1}) = \frac{\chi(1)^2}{n^2} \sum_{g \in G} \chi(g) \chi(g^{-1} g_i g_j^{-1}) = \frac{\chi(1)^2}{n^2} \frac{n}{\chi(1)} \chi(g_i g_j^{-1}) = \frac{\chi(1)}{n} \chi(g_i g_j^{-1}) = a_{ij}.
\]
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This leads to $A^2 = A$ and therefore if we apply this condition for the block form of $A$, we obtain $A_1 A_2 = (s/n - s^2/n^2)I_s$. Since $s \neq n$, $A_1 A_2$ is an invertible matrix, and thereby we can easily obtain $s \leq n - s$ or $s \leq n/2$ or $\chi(1)^2 \leq |G|/2$ which is a contradiction. Thus, $V_x^n(G)$ does not have an orthogonal basis of decomposable symmetrized tensors.

References