Zero-Divisor Graph with Respect to an Ideal

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Abstract

Let $R$ be a commutative ring with nonzero identity and let $I$ be an ideal of $R$. The zero-divisor graph of $R$ with respect to $I$, denoted by $\Gamma_I(R)$, is the graph whose vertices are the set \{ $x \in R \setminus I$ | $xy \in I$ for some $y \in R \setminus I$ \} with distinct vertices $x$ and $y$ adjacent if and only if $xy \in I$. In the case $I = 0$, $\Gamma_0(R)$, denoted by $\Gamma(R)$, is the zero-divisor graph which has well known results in the literature. In this article we explore the relationship between $\Gamma_I(R) \cong \Gamma_J(S)$ and $\Gamma(R/I) \cong \Gamma(S/J)$. We also discuss when $\Gamma_I(R)$ is bipartite. Finally we give some results on the subgraphs and the parameters of $\Gamma_I(R)$.

Keywords: Zero-divisor graph, $r$-Partite graph, Clique number, Girth.

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1 Introduction and Preliminaries

Let $R$ be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero divisors. The zero-divisor graph, $\Gamma(R)$, is the graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero divisors of $R$, and for distinct $x, y \in Z(R)^*$, the vertices $x$ and $y$ are adjacent if and only if $xy = 0$. In [7] Beck introduced the concept of a zero-divisor graph of a commutative ring. However, he lets all elements of $R$ be vertices of the graph and his work was mostly concerned with coloring of rings. We adopt the approach used by D. F. Anderson and P. S. Livingston in [6] and consider only nonzero zero divisors as vertices of the graph. The zero-divisor graph of a commutative ring has been studied extensively by several authors, e.g., [7, 6, 4, 10, 5, 1, 2].

In [11] Redmond introduced the definition of the zero-divisor graph with respect to an ideal. Let $I$ be an ideal of $R$. The zero-divisor graph of $R$ with respect to $I$, denoted by $\Gamma_I(R)$, is the graph whose vertices are the set \{ $x \in R \setminus I$ | $xy \in I$ for some $y \in R \setminus I$ \} with distinct vertices $x$ and $y$ adjacent if and only if $xy \in I$. Thus if $I = 0$ then $\Gamma_I(R) = \Gamma(R)$, and $I$ is a nonzero prime ideal of $R$ if and

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only if $\Gamma_I(R) = \emptyset$. In [11] Redmond explored the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$. He gave an example of rings $R$ and $S$ and ideals $I \leq R$ and $J \leq S$, where $\Gamma(R/I) \cong \Gamma(S/J)$ but $\Gamma_I(R) \not\cong \Gamma_J(S)$. Among other things, he showed that for an ideal $I$ of $R$, $\Gamma_I(R)$ contains $|I|$ disjoint subgraphs isomorphic to $\Gamma(R/I)$. In section 2, we show that for finite ideals $I$ and $J$ of $R$ and $S$, respectively, for which $I = \sqrt{I}$ and $J = \sqrt{J}$, if $\Gamma_I(R) \cong \Gamma_J(S)$, then $\Gamma(R/I) \cong \Gamma(S/J)$. Also we will show that the converse of this result holds if $|I| = |J|$ (see Theorem 2.2).

For a graph $G$, the vertices set of $G$ is denoted by $V(G)$. The degree of a vertex $v$ in $G$ is the number of edges of $G$ incident with $v$. We denote by $\delta(G)$ the minimum degree of vertices of $G$. For any nonnegative integer $r$, the graph $G$ is called $r$-regular if the degree of each vertex is equal to $r$. The girth of $G$ is the length of a shortest cycle in $G$ and is denoted by $gr(G)$. If $G$ has no cycles, we define the girth of $G$ to be infinite. An $r$-partite graph is one whose vertex set can be partitioned into $r$ subsets so that no edge has both ends in any one subset. A complete $r$-partite graph is one in which each pair of distinct vertices is jointed by an edge. The complete bipartite graph with part sizes $m$ and $n$ is denoted by $K_{m,n}$. A graph in which each pair of distinct vertices is jointed by an edge is called a complete graph. We use $K_n$ for the complete graph with $n$ vertices. In section 3, we show that $\Gamma_I(R)$ is a complete bipartite graph provided $I = p_1 \cap p_2 \neq 0$ for prime ideals $p_1$ and $p_2$ of $R$ (see Theorem 3.1).

A clique of a graph is a maximal complete subgraph and the number of graph vertices in the largest clique of graph $G$, denoted by $\omega(G)$, is called the clique number of $G$. In section 4, we show that if $I$ is an ideal of $R$ such that $I = \bigcap_{1 \leq i \leq n} p_i$ and for each $1 \leq j \leq n$, $I \neq \bigcap_{1 \leq i \leq n, \ i \neq j} p_i$ where $p_i$’s are prime ideals of $R$, then $\omega(\Gamma_I(R)) = n$ (see Theorem 4.2).

In this article the notations of graph theory are from [8], and the notations of commutative rings are from [9].

2 Some Basic Properties of Zero-Divisor Graphs

One of the main questions in the study of zero-divisor graphs is as follows: Let $R$ and $S$ be two commutative rings. If $\Gamma(R) \cong \Gamma(S)$, then do we have $R \cong S$? Some well known results on this question are as follows:

(i) If $R$ and $S$ are two finite reduced rings which are not fields, and $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$ (see [4, Theorem 4.1]).
Let $\text{Theorem 2.1}$  

(ii) If $R$ is a finite reduced ring which is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_6$, and $S$ is a ring which is not a local integral domain, and $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$ (see [2, Theorem 5]).

(iii) If $R = \prod_{i \in I} F_i$ and $S = \prod_{j \in J} G_j$, where $F_i$’s are finite fields and $G_j$’s are integral domains, and $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$ (see [5, Theorem 2.1]).

Now let $I$ be an ideal of $R$ and $J$ be an ideal of $S$. It is natural to ask the following question. If $\Gamma_I(R) \cong \Gamma_J(S)$, then do we have $R/I \cong S/J$? The main purpose of this section is to focus on this question.

A subgraph $H$ of $G$ is called a spanning subgraph when $V(G) = V(H)$. A $1$-regular spanning subgraph $H$ of $G$ is called a $1$-factor or a perfect matching of $G$. A graph $G$ is $1$-factorable if the edges of $G$ are partitioned into $1$-factors of $G$. Every $r$-regular bipartite graph is $1$-factorable (cf. [8, p. 192]). If the edges of $G$ are partitioned into subgraphs $H_1, \ldots, H_n$, then we write $G \cong H_1 \oplus \ldots \oplus H_n$, and if $H_i \cong H_j$ for all $1 \leq i, j \leq n$, then we write $G \cong nH$, where $H \cong H_i$.

**Theorem 2.1** Let $I$ be a finite ideal of $R$ such that $I = \sqrt{I}$. Then $\Gamma_I(R) \cong |I|^2 \Gamma(R/I)$.

*Proof.* Let $e$ be the edge of $\Gamma(R/I)$ between the vertices $a$ and $b$. Since every element of coset $a + I$ is adjacent to every element of coset $b + I$, it is easy to see that there exists a subgraph of $\Gamma_I(R)$, denoted by $H^{(e)}$, which is isomorphic to complete bipartite graph $K_{|I|,|I|}$. On the other hand, by [8, p. 192], we have $K_{|I|,|I|} \cong M_{|I|}^{(e)} \oplus \ldots \oplus M_{|I|}^{(e)}$, where each of $M_{|I|}^{(e)}$ is a perfect matching of $K_{|I|,|I|}$. Now consider $K_i := \oplus_{e \in E(\Gamma(R/I))} M_{|I|}^{(e)}$ which is a subgraph of $\Gamma_I(R)$. Since $I = \sqrt{I}$, $\Gamma_I(R) \cong K_1 \oplus \ldots \oplus K_{|I|}$. Now the assertion follows from the fact that each $K_i$ is partitioned into $|I|$ edge-disjoint subgraphs, where each of them is isomorphic to $\Gamma(R/I)$. $\square$

Let $S$ be a nonempty set of vertices of a graph $G$. The subgraph induced by $S$ is the maximal subgraph of $G$ with vertex set $S$, and is denoted by $\langle S \rangle$, that is, $\langle S \rangle$ contains precisely those edges of $G$ joining two vertices in $S$.

**Theorem 2.2** Let $I$ be a finite ideal of $R$ and let $J$ be a finite ideal of $S$ such that $I = \sqrt{I}$ and $J = \sqrt{J}$. Then the following hold:

(a) If $|I| = |J|$ and $\Gamma(R/I) \cong \Gamma(S/J)$, then $\Gamma_I(R) \cong \Gamma_J(S)$.

(b) If $\Gamma_I(R) \cong \Gamma_J(S)$, then $\Gamma(R/I) \cong \Gamma(S/J)$.

*Proof.* Part (a) is an easy consequence of Theorem 2.1. For proving part (b),

\[ \text{...} \]
let \( \varphi : \Gamma_I(R) \to \Gamma_J(S) \) be an isomorphism. Now consider \( K \subseteq R \) to be a set of distinct representatives of the vertices of \( \Gamma(R/I) \). Clearly, the subgraph induced by \( K \) is isomorphic to \( \Gamma(R/I) \). Now consider the restriction of \( \varphi \) to \( K \). Suppose that \( \varphi(K) = K' \) and \( \langle K' \rangle = H \). Now, if \( a, b \in V(K') \), then \( a + J \neq b + J \); otherwise, \( a^2 \in J = \sqrt{J} \), and hence \( a \in J \), which is a contradiction. Hence, \( K' \) is a distinct representation of the vertices of \( \Gamma(S/J) \), and hence \( \langle K' \rangle = H \cong \Gamma(J) \). Therefore, \( \varphi \) induced an isomorphism from \( \Gamma(R/I) \) to \( \Gamma(S/J) \). \( \square \)

Note that in Theorem 2.2 (a), the condition “\(|I| = |J|\)” is not superficial, as the following example shows.

**Example 2.3** Let \( R = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \) and \( S = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \), and consider \( I = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \) and \( J = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \). Hence, \( \Gamma(R/I) \cong \Gamma(S/J) \). But by computing the number of edges in each graph we have \( \Gamma(R) \neq \Gamma(S/J) \).

The conditions “\( I = \sqrt{T} \)” and “\( J = \sqrt{J} \)” on ideals \( I \) and \( J \) are also necessary in Theorem 2.2 (see [11, Remark 2.3]).

**Theorem 2.4** Let \( I \) be a nonzero ideal of \( R \) and \( a \in \Gamma_I(R) \), adjacent to every vertex of \( \Gamma_I(R) \). Then \( (I : a) \) is a maximal element of the set \( \{(I : x) \mid x \in R\} \).

Moreover, \( (I : a) \) is a prime ideal.

**Proof.** Let \( V = V(\Gamma_I(R)) \). Choose \( 0 \neq x \in I \). It is easy to see that \( a \neq a + x \in \Gamma_I(R) \). Thus \( a(a + x) \in I \) and hence \( a^2 \in I \). Therefore, \( V \cup I = (I : a) \), and so for any \( x \in R \), \( (I : x) \) is contained in \( V \cup I = (I : a) \). Thus the first assertion holds.

Now, we prove that \( (I : a) \) is a prime ideal. Let \( xy \in (I : a) \) and \( x, y \notin (I : a) \). Therefore, \( xy^a \in I \). If \( ya \notin I \), then \( x \in (I : ya) \). We know that \( (I : a) \subseteq (I : ya) \), and therefore, \( (I : a) = (I : ya) \). Hence, \( x \in (I : a) \), which is a contradiction. \( \square \)

**Theorem 2.5** Let \( I \) be an ideal of \( R \) and let \( S \) be a clique in \( \Gamma_I(R) \) such that \( x^2 = 0 \) for all \( x \in S \). Then \( S \cup I \) is an ideal of \( R \).

**Proof.** Suppose that \( x, y \in S \cup I \). Consider the following three cases.

Case 1: If \( x, y \in I \), then \( x - y \in S \cup I \).

Case 2: If \( x, y \in S \) with \( x - y \notin I \), then for all \( c \in S \), \( c(x - y) \in I \) and hence \( S \cup \{x - y\} \) is a clique. Now, since \( S \) is a clique, \( x - y \in S \).

Case 3: If \( x \in I \) and \( y \in S \), then \( x - y \notin I \), and hence for any \( c \in S \), \( c(x - y) \in I \).

Therefore, \( x - y \in S \).

Now, let \( x \in S \cup I \) and \( r \in R \). Suppose that \( r, x \notin I \). If \( rx \in I \), then \( rx \in S \cup I \).
If \( rx \notin I \), since for any \( c \in S \), \( rxc \in I \), we have \( rx \in S \). \( \square \)

**Theorem 2.6** Let \( I \) be an ideal of \( R \) and consider \( S = \sqrt{I} \setminus I \). If \( S \) is a nonempty set, then \( \langle S \rangle \) is connected.

*Proof.* Let \( x, y \in S \). If \( xy \not\in I \), then the result is obtained. Suppose that \( xy \not\in I \), where \( x^n, y^m \in I \) and \( x^{n-1}, y^{m-1} \not\in I \). Hence, the path

\[
x - x^{n-1} - xy - y^{m-1} - y
\]

is a path of length four from \( x \) to \( y \). \( \square \)

**Corollary 2.7** Suppose either \( N \) is the nil radical of \( R \), or is a nilpotent ideal of \( R \). If \( N \) is nontrivial, then \( \langle N \setminus \{0\} \rangle \) is a connected subgraph of \( \Gamma(R) \).

## 3 Complete \( r \)-Partite Graph

It is easy to see that if \( I \) is a prime ideal of \( R \), then we have \( \Gamma_I(R) = \emptyset \). In the following, we show that if \( I = p_1 \cap p_2 \), where \( p_1 \) and \( p_2 \) are prime ideals of \( R \), then \( \Gamma_I(R) \) is a complete bipartite graph. In section 4, we study the girth and the clique number of \( \Gamma_I(R) \) for \( I = p_1 \cap \ldots \cap p_n \), where \( p_i \)'s are prime ideals of \( R \).

**Theorem 3.1** Let \( I \) be a nonzero ideal of \( R \). Then the following hold:

(a) If \( p_1 \) and \( p_2 \) are prime ideals of \( R \) and \( I = p_1 \cap p_2 \not= 0 \), then \( \Gamma_I(R) \) is a complete bipartite graph.

(b) If \( I \not= 0 \) is an ideal of \( R \) for which \( I = \sqrt{I} \), then \( \Gamma_I(R) \) is a complete bipartite graph if and only if there exist prime ideals \( p_1 \) and \( p_2 \) of \( R \) such that \( I = p_1 \cap p_2 \).

*Proof.* (a): Let \( a, b \in R \setminus I \) with \( ab \in I \). Then \( ab \in p_1 \) and \( ab \in p_2 \). Since \( p_1 \) and \( p_2 \) are prime, we have \( a \in p_1 \) or \( b \in p_1 \) and \( a \in p_2 \) or \( b \in p_2 \). Therefore, suppose \( a \in p_1 \setminus p_2 \) and \( b \in p_2 \setminus p_1 \). Thus, \( \Gamma_I(R) \) is a complete bipartite graph with parts \( p_1 \setminus p_2 \) and \( p_2 \setminus p_1 \).

(b): Suppose that the parts of \( \Gamma_I(R) \) are \( V_1 \) and \( V_2 \). Set \( p_1 = V_1 \cup I \) and \( p_2 = V_2 \cup I \). It is clear that \( I = p_1 \cap p_2 \). We now prove that \( p_1 \) is an ideal of \( R \). To show this let \( a, b \in p_1 \).

Case 1: If \( a, b \in I \), then \( a - b \in I \) and so \( a - b \in p_1 \).

Case 2: If \( a, b \in V_1 \), then there is \( c \in V_2 \) such that \( ca \in I \) and \( cb \in I \). So, \( c(a - b) \in I \). If \( a - b \in I \), then \( a - b \in p_1 \). Otherwise, \( a - b \in V_1 \), which implies \( a - b \in p_1 \).
I c that is a contradiction. Hence, $p$.

Theorem 4.1 Let $I$ be a nonzero proper ideal of $R$. If $\Gamma_I(R)$ is a complete $r$-partite graph, $r \geq 3$, then at most one of the parts has more than one vertex.

Proof. Assume that $V_1, \ldots, V_r$ are parts of $\Gamma_I(R)$. Let $V_i$ and $V_s$ have more than one element. Choose $x \in V_i$ and $y \in V_s$. Let $V_i$ be a part of $\Gamma_I(R)$ such that $V_i \neq V_i$ and $V_i \neq V_s$. Let $z \in V_i$. Since $\Gamma_I(R)$ is a complete $r$-partite graph, $(I : x) = (\bigcup_{1 \leq i \leq r, i \neq t} V_i) \cup I$, $(I : y) = (\bigcup_{1 \leq i \leq r, i \neq s} V_i) \cup I$, and $(I : z) = (\bigcup_{1 \leq i \leq r, i \neq i} V_i) \cup I$. Therefore, $(I : z) \subseteq (I : x) \cup (I : y)$, and so we have $(I : z) \subseteq (I : x)$ or $(I : z) \subseteq (I : y)$. Let $(I : z) \subseteq (I : x)$ and choose $x' \in V_i$ such that $x' \neq x$. Then we have $x' \in (I : z) \setminus (I : x)$. This is a contradiction. □

4 Girth and Clique Number

In this section we study the girth and the clique number of $\Gamma_I(R)$, when $I$ is an intersection of prime ideals.

Theorem 4.1 Let $p_1$ and $p_2$ be prime ideals of $R$ and $I = p_1 \cap p_2$. Then either $\text{gr}(\Gamma_I(R)) = 4$ or $R/I \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. 

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Proof. If $|p_1 \setminus p_2| = 1$ and $|p_2 \setminus p_1| \geq 2$, then $\Gamma_I(R)$ is a star graph and so has a cut point. This is a contradiction by [11, Theorem 3.2]. Therefore, this case cannot happen. The case $|p_2 \setminus p_1| = 1$ and $|p_1 \setminus p_2| \geq 2$ is similar. So there are two other possibilities.

Case 1: $|p_1 \setminus p_j| \geq 2$ for $i \neq j$ and $1 \leq i, j \leq 2$. In this case, Theorem 3.1 implies that $\text{gr}(\Gamma_I(R)) = 4$.

Case 2: $|p_1 \setminus p_j| < 2$ for $i \neq j$ and $1 \leq i, j \leq 2$. In this case, there is $x \in R$ for which $p_1 \setminus p_2 = \{x\}$ and so $p_1 = \{x\} \cup I$. For any $r \in R \setminus p_2$ we have $rx \in p_1 \setminus I$ and so $rx = x$. Therefore, $(1-r)x = 0 \in p_2$ and hence $(1-r) \in p_2$. Thus $|R/p_2| = 2$. That implies $p_2$ is a maximal ideal of $R$ and $R/p_2 \cong \mathbb{Z}_2$. But $p_1 + p_2 = R$, so that implies $R/I \cong R/p_1 \times R/p_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. □

Theorem 4.2 Let $I$ be an ideal of $R$ such that $I = \bigcap_{1 \leq i \leq n} p_i$ and for each $1 \leq j \leq n$, $I \neq \bigcap_{1 \leq i \leq n, i \neq j} p_i$ where $p_i$’s are prime ideals of $R$. Then $\omega(\Gamma_I(R)) = n$.

Proof. Consider $x_j \in \bigcap_{1 \leq i \leq n, i \neq j} p_i \setminus p_j$. It is easy to see that $X = \{x_1, \ldots, x_n\}$ is a clique in $\Gamma_I(R)$. Hence, $\omega(\Gamma_I(R)) \geq n$ and so it is sufficient to show that $\omega(\Gamma_I(R)) \leq n$. In order to do this, we use induction on $n$. For $n = 2$, by Theorem 3.1, $\Gamma_I(R)$ is a bipartite graph and hence $\omega(\Gamma_I(R)) = 2$. Suppose $n > 2$ and the result is true for any integer less than $n$. Let $I = \bigcap_{1 \leq i \leq n} p_i$ and for each $1 \leq j \leq n$, $I \neq \bigcap_{1 \leq i \leq n, i \neq j} p_i$. Let $\{x_1, \ldots, x_m\}$ be a clique in $\Gamma_I(R)$. Hence, $x_1x_j \in \bigcap_{1 \leq i \leq n} p_i$ for any $2 \leq j \leq m$. Without loss of generality, suppose that $x_1 \notin p_1$. Therefore, $x_2, \ldots, x_m \in p_1$, so $x_2, \ldots, x_m \notin \bigcap_{2 \leq i \leq n} p_i$. Let $J = \bigcap_{2 \leq i \leq n} p_i$. Hence, $\{x_2 \ldots, x_m\}$ is a clique in $\Gamma_J(R)$. Therefore, $m - 1 \leq n - 1$, and the result is obtained. □

Corollary 4.3 The following hold:

(a) If $I = \bigcap_{1 \leq i \leq n} p_i \neq 0$ and $J = \bigcap_{1 \leq j \leq m} q_j$ where $p_i$’s and $q_j$’s are prime ideals such that $\Gamma_I(R) = \Gamma_J(R)$, then $m = n$.

(b) If for any $p \in \text{Min}(R)$, $p$ is a finitely generated ideal, then $\omega(\Gamma_{\text{nil}(R)}(R)) = |\text{Min}(R)|$ (which is finite by the main theorem of [3]).

(c) If $R$ is a semi-local ring and not local, then $\omega(\Gamma_{I(R)}(R)) = |\text{Max}(R)|$.

(d) If $n$ is a square-free integer, then $\omega(\Gamma_{n\mathbb{Z}}(\mathbb{Z})) = k$, where $k$ is the number of primes in the decomposition of $n$ into primes.

Theorem 4.4 Let $I$ be an ideal of $R$. Suppose either $I$ is a primary ideal of $R$ that is not prime and $|I| \geq 3$, or $|\text{Ass}(R/I)| \geq 3$. Then $\text{gr}(\Gamma_I(R)) = 3$.

Proof. For the first case, let $a, b \in R \setminus I$ such that $ab \in I$. Then there exists
\( n \in \mathbb{N} \) such that \( b^n \in I \), so we can choose \( t \in \mathbb{N} \) for which \( b^t \in I \) and \( b^t - 1 \notin I \). Since \( a, b^t - 1 \notin I \), we have the chain
\[
a - b - b^t - 1 - a
\]
in the graph \( \Gamma_I(R) \). Therefore, \( \text{gr}(\Gamma_I(R)) = 3 \).

For the second case, \( |\text{Ass}(R/I)| \geq 3 \) implies that \( \text{gr}(\Gamma(R/I)) = 3 \) (see [1, Corollary 2.2]), and hence \( \text{gr}(\Gamma_I(R)) = 3 \). □

In the above theorem, one of the conditions “\( |I| \geq 3 \)” or “\( |\text{Ass}(R/I)| \geq 3 \)” are necessary. To see this, for example let \( R = \mathbb{Z}_8 \) and consider \( I = \langle 4 \rangle \); and note that we have \( |\text{Ass}(R/I)| = 1 \) and \( \text{gr}(\Gamma(R/I)) = \infty \).

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