Tensor Products of Approximately Cohen-Macaulay Rings

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Abstract

Our aim in this article is to study a problem originally raised by Grothendieck. We show that the approximately Cohen-Macaulay property is preserved for the tensor product of algebras over a field $k$. We also discuss the converse problem.

Keywords: Approximately Cohen-Macaulay ring, Flat homomorphism of rings, Tensor product.

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1 Introduction

All rings and algebras considered in this article are commutative Noetherian with identity element, and all ring homomorphisms are unital. Throughout, $k$ stands for a field. Let $(R, \mathfrak{m})$ be a local ring with $\dim(R) = d$. Recall that $R$ is a Gorenstein ring if and only if there is an element $a$ of $\mathfrak{m}$ such that $R/a^nR$ is a Gorenstein ring of dimension $d-1$ for every integer $n > 0$ (cf. [7]). Clearly, this is not true for Cohen-Macaulay rings. The local ring $R$ is called an approximately Cohen-Macaulay ring if either $\dim(R) = 0$ or there exists an element $a$ of $\mathfrak{m}$ such that $R/a^nR$ is a Cohen-Macaulay ring of dimension $d-1$ for every integer $n > 0$ (cf. [5]). It is shown that if $R$ is an approximately Cohen-Macaulay ring, then so is the ring $R_p$ for any prime ideal $p$ (see Theorem 2). Therefore, the concept of approximately Cohen-Macaulay is extended to nonlocal rings as follows. A ring $R$ is an approximately Cohen-Macaulay ring if for all prime ideals $p$ of $R$, the ring $R_p$ is an approximately Cohen-Macaulay ring. It is well known that the tensor product $R \otimes_A S$ of regular rings is not regular in general, even if we assume $R$ and $S$ are $A$-algebras, where $A$ is a field (cf. [10, Remark 7]). In [11, Remark 1.7], Watanabe, Ishikawa, Tachibana and Otsuka showed that under a suitable condition, tensor products of regular rings are complete intersections. It is proven in [6] that the tensor product $R \otimes_A S$ of Cohen-Macaulay rings are again Cohen-Macaulay, if we assume that $R$ is a flat $A$-module and $S$ is a finitely generated $A$-module, and in [11], it is shown that the

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same is true for Gorenstein rings. In [1], Bouchiba and Kabbaj showed that if $R$ and $S$ are $k$-algebras such that $R \otimes_k S$ is Noetherian, then $R \otimes_k S$ is a Cohen-Macaulay ring if and only if $R$ and $S$ are Cohen-Macaulay rings. Recently, in [10], Tousi and Yassemi showed that if $R$ and $S$ are nonzero $k$-algebras such that $R \otimes_k S$ is Noetherian, then $R \otimes_k S$ is a locally complete intersection (resp. Gorenstein, Cohen-Macaulay) if and only if $R$ and $S$ are locally complete intersections (resp. Gorenstein, Cohen-Macaulay).

In this article we shall investigate if the approximately Cohen-Macaulay property is conserved under tensor product operations. It is shown that if $\varphi : (R, m) \rightarrow (S, n)$ is a flat local homomorphism and $R$ is not a Cohen-Macaulay ring, then the following are equivalent (see Theorem 6):

(a) $R$ is an approximately Cohen-Macaulay ring and $S/mS$ is a Cohen-Macaulay ring.

(b) $S$ is an approximately Cohen-Macaulay ring and $Ass_S(S/pS) = Assh_S(S/pS)$ for every $p \in Ass(R)$.

Further, if $R$ is a homomorphic image of a Cohen-Macaulay local ring, then the next condition is also equivalent:

(c) $S$ is an approximately Cohen-Macaulay ring.

We will also prove the following result. Let $R$ and $S$ be nonzero $k$-algebras such that $T := R \otimes_k S$ is Noetherian. Assume that $R$ is not a Cohen-Macaulay ring. Then the following hold (see Theorem 10):

(i) If $R$ is an approximately Cohen-Macaulay ring and $S$ is a Cohen-Macaulay ring, then $T$ is an approximately Cohen-Macaulay ring.

(ii) If $T$ is an approximately Cohen-Macaulay ring, then $S$ is a Cohen-Macaulay ring.

(iii) If $R$ is a homomorphic image of a Cohen-Macaulay ring or $k$ is algebraically closed, then the following conditions are equivalent:

(a) $T$ is an approximately Cohen-Macaulay ring.

(b) $R$ is an approximately Cohen-Macaulay ring and $S$ is a Cohen-Macaulay ring.

\section{Main Results}

For a finitely generated $R$-module $M$ of finite Krull dimension, recall that

$$Assh_R(M) = \{p \in \text{Supp}_R(M) | \dim(R/p) = \dim(M)\},$$

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and denote \( \text{Assh}_R(R) = \text{Assh}(R) \). Let \( U_R(0) = \bigcap_{p \in \text{Assh}(R)} I(p) \), where \( (0) = \bigcap_{p \in \text{Ass}(R)} I(p) \) denotes a minimal primary decomposition of the zero ideal of \( R \) (cf. [5]).

Let \( M \) be a finitely generated \( R \)-module and \( I \) an ideal of \( R \) such that \( IM \neq M \). Then the common length of the maximal \( M \)-sequences in \( I \) is called the \textit{grade} of \( I \) on \( M \), denoted by \( \text{grade}_M(I) \). If \((R, m)\) is a local ring, and \( M \) is a finitely generated nonzero \( R \)-module, then the grade of \( m \) on \( M \) is called the \textit{depth} of \( M \), denoted by \( \text{depth}(M) \).

**Theorem 1** (see [5])  
Let \( R \) be a local ring with maximal ideal \( m \) and \( \dim(R) = d \). Suppose that \( R \) is not a Cohen-Macaulay ring. Then the following conditions are equivalent:

(i) \( R \) is an approximately Cohen-Macaulay ring.

(ii) \( R \) contains an ideal \( I \) such that \( I \) is a Cohen-Macaulay \( R \)-module of dimension \( d - 1 \) and \( R/I \) is a Cohen-Macaulay ring of dimension \( d \).

(iii) \( R/U_R(0) \) is a Cohen-Macaulay ring and \( \text{depth}(R) = d - 1 \).

(iv) (a) \( H^i_m(R) = (0) \) for \( i \neq d - 1, d \).

(b) \( \text{Hom}_R(H^{d-1}_m(R), E_R(R/m)) \) is a Cohen-Macaulay \( \hat{R} \)-module of dimension \( d - 1 \).

(c) The local ring \( R/p \) is unmixed for every \( p \in \text{Assh}(R) \), i.e., the equality \( \dim(\hat{R}/P) = d \) holds for every \( P \in \text{Ass}(\hat{R}/p\hat{R}) \) and for every \( p \in \text{Assh}(R) \).

In this case, the ideal \( I \) appearing in assertion (ii) is uniquely determined and equals \( U_R(0) \). Here \( \hat{R} \) (resp. \( E_R(R/m) \)) denotes the \( m \)-adic completion of \( R \) (resp. the injective hull of \( R/m \)).

In the following theorem we consider the behavior of approximately Cohen-Macaulay property by passing to localizations.

**Theorem 2**  
Let \((R, m)\) be an approximately Cohen-Macaulay ring. Then

(i) For any \( p \in \text{Spec}(R) \), \( \dim(R_p) - \text{depth}(R_p) \leq 1 \).

(ii) Suppose \( R \) is not a Cohen-Macaulay ring. Then for any \( p \in \text{Spec}(R) \) such that \( R_p \) is not a Cohen-Macaulay ring, \( \text{ht}(p) + \dim(R/p) = \dim(R) \).

(iii) Suppose \( R \) is not a Cohen-Macaulay ring. Then for any \( p \in \text{Spec}(R) \) such that \( R_p \) is not a Cohen-Macaulay ring, \( U_{R_p}(0) = U_R(0)R_p \).

(iv) For any \( p \in \text{Spec}(R) \), \( R_p \) is an approximately Cohen-Macaulay ring.

**Proof.** (i): This follows from the fact that \( \dim(R_p) - \text{depth}(R_p) \leq \dim(R) - \text{depth}(R) \) for any \( p \in \text{Spec}(R) \) (see [8, Exercise 17.5(ii)]).

(ii): Let \( p \in \text{Spec}(R) \) such that \( R_p \) is not a Cohen-Macaulay ring. By (i), we
have depth$(R_p) = \text{ht}(p) - 1$. Also, in view of [8, Exercise 17.5(i)], we have

\[
\text{ht}(p) + \dim(R/p) - 1 \leq \dim(R) - 1 \\
= \text{depth}(R) \\
\leq \text{grade}_R(p) + \dim(R/p) \\
\leq \text{depth}(R_p) + \dim(R/p).
\]

Therefore, \(\text{ht}(p) + \dim(R/p) = \dim(R)\).

(iii): Let \(p \in \text{Spec}(R)\) such that \(R_p\) is not a Cohen-Macaulay ring. We claim that \(U_R(0)R_p = U_{R_p}(0)\). If \((0) = \bigcap_{q \in \text{Ass}(R)} I(q)\) is an irredundant primary decomposition for the zero ideal of \(R\), then

\[
(0) = \bigcap_{q \in \text{Ass}(R)} I(q)R_p
\]

is a minimal primary decomposition for the zero ideal of \(R_p\). Thus it is enough to show that

\[
\text{Assh}(R_p) = \{ qR_p \mid q \in \text{Assh}(R), \ q \subseteq p \}.
\]

Let \(q \in \text{Spec}(R)\). We have

\[
qR_p \in \text{Assh}(R_p) \iff q \subseteq p \text{ and } \dim(R_p/qR_p) = \dim(R_p) \\
\iff q \subseteq p \text{ and } \text{ht}(p/q) = \text{ht}(p) \\
\iff q \subseteq p \text{ and } \text{ht}(p/q) + \dim(R/p) = \dim(R).
\]

Let \(qR_p \in \text{Assh}(R_p)\). Since

\[
\dim(R/q) \geq \text{ht}(p/q) + \dim(R/p) \\
= \dim(R),
\]

\(\dim(R/q) = \dim(R)\) and hence \(q \in \text{Assh}(R)\).

Now, let \(q \subseteq p\) and \(q \in \text{Assh}(R)\). Since \(R/U_R(0)\) is a catenary ring, \(R/q\) is also catenary and hence by [8, Theorem 31.4], \(\text{ht}(p/q) + \dim(R/p) = \dim(R/q)\). Thus \(\text{ht}(p/q) + \dim(R/p) = \dim(R)\). Therefore, \(qR_p \in \text{Assh}(R_p)\).

(iv): Let \(p \in \text{Spec}(R)\). If \(R_p\) is a Cohen-Macaulay ring, then \(R_p\) is an approximately Cohen-Macaulay ring. If \(R_p\) is not a Cohen-Macaulay ring, then \(R\) is not a Cohen-Macaulay ring, and so \(R/U_R(0)\) is a Cohen-Macaulay ring. Now, by (iii), \(R_p/U_{R_p}(0)\) is Cohen-Macaulay. Thus the assertion follows from (i) and Theorem 1(iii). \(\Box\)

By using Theorem 2, the concept of approximately Cohen-Macaulay ring can
be extended to nonlocal rings by defining that a ring $R$ is approximately Cohen-Macaulay if for all prime ideals $p$ of $R$, the ring $R_p$ is an approximately Cohen-Macaulay ring.

In the following result we consider the behavior of $\text{Assh}(-)$ and the primary decomposition of the zero submodule under base change.

**Proposition 3** Let $\varphi : (R, m) \rightarrow (S, n)$ be a flat local homomorphism. Then the following hold:

(i) $\text{Assh}(R) = \{ q \cap R | q \in \text{Assh}(S) \}$.

(ii) $U_R(0)S \subseteq U_S(0)$.

(iii) The following conditions are equivalent:

(a) $U_R(0)S = U_S(0)$.

(b) $\text{Assh}(S) = \{ q \in \text{Ass}(S) | q \cap R \in \text{Assh}(R) \}$.

(c) $\text{Ass}_S(S/pS) = \text{Assh}_S(S/pS)$ for every $p \in \text{Assh}(R)$.

**Proof.** (i): Let $p \in \text{Assh}(R)$. Then $\dim(R) = \dim(R/p)$. Consider the flat local homomorphism $\tilde{\varphi} : R/p \rightarrow S/pS$. Hence,

$$\dim(R) = \dim(S/pS) - \dim(S/mS),$$

and therefore $\dim(S) = \dim(S/pS)$. So there exists a minimal prime ideal of $pS$, say $q$, such that $\dim(S) = \dim(S/q)$. Since $q$ is a minimal prime ideal of $pS$, $q \cap R = p$.

Let $q \in \text{Assh}(S)$ and $q \cap R = p$. We have $\dim(S/q) = \dim(S)$ and $pS \subseteq q$; that means $\dim(S/pS) = \dim(S)$. Thus,

$$\dim(R/p) + \dim(S/mS) = \dim(S),$$

and so $\dim(R/p) = \dim(R)$.

(ii): It is enough to show that $U_R(0)S_q = (0)S_q$ for every $q \in \text{Assh}(S)$. Indeed, since $S$ is flat,

$$U_R(0)S_q = \bigcap_{p \in \text{Assh}(R)} (I(p)S_q) = \bigcap_{q \subseteq p \cap R} (I(p)S_q),$$

If $q \in \text{Assh}(S)$, then $q \cap R = p$. We know that $(I(p)R_p)S_q$ and $(0)S_q$ are $pR_p$-primary submodules of the $R_p$-module $S_q$. Thus $I(p)S_q = (I(p)R_p)S_q = (0)S_q$, as required.

(iii): It is known that (cf. [8, Theorem 23.2])

$$\{ p \} = \{ \varphi^{-1}(q) | q \in \text{Ass}_S(S/pS) \}$$

for each $p \in \text{Spec}(R)$,
\[
\text{Ass}(S) = \bigcup_{p \in \text{Ass}(R)} \text{Ass}_S(S/pS),
\]
\[
\text{Ass}_S(S/U_R(0)S) = \bigcup_{p \in \text{Assh}(R)} \text{Ass}_S(S/pS),
\]
\[
\text{Assh}(S) = \text{Ass}_S(S/U_S(0)) = \bigcup_{p \in \text{Assh}(R)} \text{Assh}_S(S/pS).
\]

Also note that \( \cap_{p \in \text{Ass}(R)} I(p)S = (0) \) is a minimal primary decomposition of the zero submodule of the \( R \)-module \( S \) and if \( \cap_{1 \leq i \leq n} Q_i = (0) \) is a minimal primary decomposition of the zero ideal in \( S \), then \( \cap_{1 \leq i \leq n} Q_i = (0) \) is a primary decomposition of the zero submodule of the \( R \)-module \( S \). We can now easily obtain (iii). □

The next result shows that the approximately Cohen-Macaulay property is stable under specialization.

**Lemma 4** Let \((R, m)\) be a Noetherian local ring with \( \dim(R) = d \). Let \( x \in m \setminus Z(R) \). If \( R \) is an approximately Cohen-Macaulay ring, then \( R/xR \) is an approximately Cohen-Macaulay ring.

**Proof.** We may assume that \( R \) is not a Cohen-Macaulay ring. Then there exists an ideal \( I \) of \( R \) such that \( I \) is a Cohen-Macaulay \( R \)-module, \( \dim(I) = d - 1 \), and \( R/I \) is a Cohen-Macaulay ring of dimension \( d \). Therefore, \( I/xI \) is a Cohen-Macaulay \( R/xR \)-module of dimension \( d - 2 \). Since \( \text{Ass}_R(R/I) = \{ q \in \text{Assh}(R) | I \subseteq q \} \), \( x \notin Z_R(R/I) \) and hence \( R/(I + xR) \) is a Cohen-Macaulay ring of dimension \( d - 1 \) and \( I \cap xR = xI \). By using the isomorphism \( (I + xR)/xR \cong I/(I \cap xR) \), we obtain that \( (I + xR)/xR \) is a Cohen-Macaulay \( R/xR \)-module of dimension \( d - 2 \). The assertion now follows from Theorem 1. □

**Lemma 5** Let \( \varphi : (R, m) \rightarrow (S, n) \) be a flat local homomorphism. Let \( S \) be an approximately Cohen-Macaulay ring. Then either \( R \) or \( S/mS \) is Cohen-Macaulay.

**Proof.** Assume that \( R \) is not Cohen-Macaulay. Then we have
\[
\text{depth}(R) = \text{depth}(S) - \text{depth}(S/mS)
= \dim(S) - 1 - \text{depth}(S/mS)
\geq \dim(S) - 1 - \dim(S/mS)
= \dim(R) - 1.
\]
Since \( R \) is not Cohen-Macaulay, \( \text{depth}(R) = \dim(R) - 1 \) and hence \( S/mS \) is Cohen-Macaulay. □

We are now ready to prove that the approximately Cohen-Macaulay property is
stable (in some sense) under change of ring. This result is somehow parallel to the results on properties like regular, complete intersection and Cohen-Macaulay (cf. [10, Theorem 1]).

**Theorem 6** Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism. Assume that $R$ is not a Cohen-Macaulay ring. Then the following are equivalent:

(a) $R$ is an approximately Cohen-Macaulay ring and $S/\mathfrak{m}S$ is a Cohen-Macaulay ring.

(b) $S$ is an approximately Cohen-Macaulay ring and $\text{Ass}_S(S/\mathfrak{p}S) = \text{Assh}_S(S/\mathfrak{p}S)$ for every $\mathfrak{p} \in \text{Ass}(R)$.

Further, if $R$ is a homomorphic image of a Cohen-Macaulay local ring, then the next condition is also equivalent:

(c) $S$ is an approximately Cohen-Macaulay ring.

**Proof.** Consider the induced flat local homomorphism $\tilde{\varphi} : R/U_R(0) \to S/U_R(0)S$.

(a) $\implies$ (b): By [3, Theorem 2.1.7], $S/U_R(0)S$ is a Cohen-Macaulay ring. Also the following (in)equalities hold by Proposition 3(ii):

$$\dim(S) \geq \dim(S/U_R(0)S)$$
$$\geq \dim(S/U_S(0))$$
$$= \dim(S).$$

On the other hand, $U_R(0)$ is a Cohen-Macaulay $R$-module of dimension $\dim(R) - 1$. Thus $U_R(0)S \cong U_R(0) \otimes_R S$ is a Cohen-Macaulay $S$-module of dimension $\dim(S) - 1$, because

$$\dim(S) - 1 = \dim(R) - 1 + \dim(S/\mathfrak{m}S)$$
$$= \dim(U_R(0)) + \dim(S/\mathfrak{m}S)$$
$$= \dim(U_R(0) \otimes_R S).$$

The last paragraph of Theorem 1 implies that $U_R(0)S = U_S(0)$. Now, the assertions follow from Proposition 3 and Theorem 1(ii).

(b) $\implies$ (a): By Lemma 5, $S/\mathfrak{m}S$ is a Cohen-Macaulay ring. We have $U_S(0) = U_R(0)S$ by Proposition 3(iii). On the other hand, $S$ is not a Cohen-Macaulay ring and so $S/U_R(0)S$ is a Cohen-Macaulay ring. Therefore, $R/U_R(0)$ is a Cohen-Macaulay ring. Now, the assertion follows from the following equalities:

$$\dim(R) = \dim(S) - \dim(S/\mathfrak{m}S)$$
$$= \text{depth}(S) + 1 - \text{depth}(S/\mathfrak{m}S)$$
$$= \text{depth}(R) + 1.$$
(b) $\implies$ (c): It is clear.

(c) $\implies$ (a): By Lemma 5, we know that $S/mS$ is Cohen-Macaulay. Assume that $q$ is a minimal ideal of $V(mS)$. Then $q \cap R = m$ and $\dim(S_q/mS_q) = 0$. By considering the induced homomorphism $\bar{\phi}: R \rightarrow S_q$ one can reduce to the case where $\dim(S/mS) = 0$. By using “(a) $\implies$ (b)” and [5, Corollary 2.6], we may assume that $R$ and $S$ are complete. Note that $S$ is not a Cohen-Macaulay ring. We use induction on $\dim(S) = n$. If $n = 1$, then $\dim(R) + \dim(S/mS) = 1$ and hence $\dim(R) = 1$. Thus, $R$ is an approximately Cohen-Macaulay ring. Now suppose, inductively, that $n \geq 2$ and we have established the result for $n - 1$. Set $N = \text{Hom}_R(H^{n-1}_m(R), E_R(R/m))$. Since

$$\text{depth}(R) = \text{depth}(S) = \dim(S) - 1 = \dim(R) - 1,$$

$H^{n}_m(R) = (0)$ for $i \notin \{n-1, n\}$. Therefore, it is enough to show that $N$ is a Cohen-Macaulay $R$-module of dimension $n - 1$.

We claim that $m \notin \text{Ass}_R(N)$. Otherwise $m \in \text{Att}_R(H^{n-1}_m(R))$ and so by [2, Exercise 11.3.7], $n \in \text{Att}_S(H^{n-1}_m(S))$. Therefore, $n \in \text{Ass}_S\left(\text{Hom}_S(H^{n-1}_m(S), E_S(S/n))\right)$. But $\text{Hom}_S(H^{n-1}_m(S), E_S(S/n))$ is a Cohen-Macaulay $S$-module of dimension $n - 1 \geq 1$. That is a contradiction.

Since $N$ is a finitely generated $R$-module, the set $\text{Ass}_R(N)$ is finite and hence there exists $x \in m \setminus \left(Z(R) \cup Z_R(N)\right)$. Consider the induced flat local homomorphism $\bar{\phi}: R/xR \rightarrow S/\varphi(x)S$. Since $\varphi(x) \notin Z(S)$, by Lemma 4, $S/\varphi(x)S$ is an approximately Cohen-Macaulay ring of dimension $n - 1$. Therefore, by the inductive hypothesis $R/xR$ is an approximately Cohen-Macaulay module of dimension $n - 1$. Set $\bar{R} = R/xR$ and $\bar{m} = m/xR$. The $R/xR$-module $\text{Hom}_R(H^{n-2}_m(\bar{R}), E_R(\bar{R}/\bar{m}))$ is a Cohen-Macaulay module of dimension $n - 2$. The exact sequence

$$0 \rightarrow R \xrightarrow{x} R \rightarrow R/xR \rightarrow 0$$

induces the exact sequence

$$0 \rightarrow H^{n-2}_m(R/xR) \rightarrow H^{n-1}_m(R) \xrightarrow{x} H^{n-1}_m(R).$$

Therefore, $H^{n-2}_m(\bar{R}) \cong \text{Hom}_R(R/xR, H^{n-1}_m(R))$. By [2, Lemma 10.1.15],

$$\text{Hom}_R(H^{n-2}_m(\bar{R}), E_R(\bar{R}/\bar{m})) \cong \text{Hom}_R\left(\text{Hom}_R(\bar{R}, H^{n-1}_m(R)), \text{Hom}_R(\bar{R}, E_R(R/m))\right)$$

$$\cong \text{Hom}_R\left(\text{Hom}_R(\bar{R}, H^{n-1}_m(R)), E_R(R/m)\right)$$

$$\cong \bar{R} \otimes_R \text{Hom}_R(H^{n-1}_m(R), E_R(R/m)).$$
Thus $N/xN$ is a Cohen-Macaulay $R/xR$-module of dimension $n - 2$. Since $x \notin Z_R(N)$, $N$ is a Cohen-Macaulay $R$-module of dimension $n - 1$. □

Note that in Theorem 6, the condition “$\text{Ass}_S(S/pS) = \text{Assh}_S(S/pS)$ for every $p \in \text{Assh}(R)$" is not superficial, as the following example shows.

**Example 7** (See [9]) Let $(R, m)$ be a 2-dimensional local domain for which the $m$-adic completion $\hat{R} \cong k[[x, y, z]]/(xy, xz)$. Put $S = \hat{R}$. Let $\varphi : R \rightarrow S$ be a natural ring homomorphism. Then

(i) $R$ is not approximately Cohen-Macaulay local domain. In particular, $\text{Assh}(R) = \{(0)\}$.

(ii) $S/mS = k$ is regular, and thus is Cohen-Macaulay.

(iii) $S$ is approximately Cohen-Macaulay, but not unmixed.

(iv) $R$ is not a homomorphic image of a Cohen-Macaulay local ring.

**Corollary 8** Let $\varphi : R \rightarrow S$ be a flat homomorphism. If $R$ is an approximately Cohen-Macaulay ring and $(R_p/pR_p) \otimes_R S$ is a Cohen-Macaulay ring for every $p \in \text{Spec}(R)$, then $S$ is an approximately Cohen-Macaulay ring.

*Proof.* Let $q \in \text{Spec}(S)$. Set $p = q \cap R \in \text{Spec}(R)$. The induced homomorphism $\tilde{\varphi} : R_p \rightarrow S_q$ is a flat local homomorphism. It is clear that $S_q/(pR_p)S_q$ is a localization of $(R_p/pR_p) \otimes_R S$. Now, the assertion follows from Theorem 6. □

**Corollary 9** Let $\varphi : R \rightarrow S$ be a faithfully flat homomorphism. Suppose that $R$ is not a Cohen-Macaulay ring, but a homomorphic image of a Cohen-Macaulay ring. If $S$ is an approximately Cohen-Macaulay ring, then $R$ is an approximately Cohen-Macaulay ring.

*Proof.* Assume that $p \in \text{Spec}(R)$ and $q$ is a minimal ideal of $V(pS)$. Then $q \cap R = p$ and $\dim(S_q/pS_q) = 0$. Consider the induced homomorphism $\tilde{\varphi} : R_p \rightarrow S_q$. The assertion now follows from Theorem 6. □

**Theorem 10** Let $R$ and $S$ be nonzero $k$-algebras such that $T := R \otimes_k S$ is Noetherian. Assume that $R$ is not a Cohen-Macaulay ring. Then the following hold:

(i) If $R$ is an approximately Cohen-Macaulay ring and $S$ is a Cohen-Macaulay ring, then $T$ is an approximately Cohen-Macaulay ring.

(ii) If $T$ is an approximately Cohen-Macaulay ring, then $S$ is a Cohen-Macaulay ring.

(iii) If $R$ is a homomorphic image of a Cohen-Macaulay ring or $k$ is algebraically closed, then the following conditions are equivalent:

(a) $T$ is an approximately Cohen-Macaulay ring.
(b) $R$ is an approximately Cohen-Macaulay ring and $S$ is a Cohen-Macaulay ring.

Proof. (i): Consider the faithfully flat homomorphism $\varphi : R \longrightarrow (R \otimes_k S)$. It is enough to show that the fibers $(R_p/pR_p) \otimes_R (R \otimes_k S) \cong (R_p/pR_p) \otimes_k S$ over every prime ideal $p$ of $R$ are Cohen-Macaulay rings. Since $R_p/pR_p$ is a Cohen-Macaulay ring (it is actually a field), $(R_p/pR_p) \otimes_k S$ is also a Cohen-Macaulay ring by [10, Theorem 6].

(ii): Assume that $S$ is not a Cohen-Macaulay ring, Then there exist $p \in \text{Spec}(R)$ and $q \in \text{Spec}(S)$ such that $R_p$ and $S_q$ are not Cohen-Macaulay rings, and hence $\text{grade}_{R_p}(pR_p) \leq \text{ht}(p) - 1$ and $\text{grade}_{S_q}(qS_q) \leq \text{ht}(q) - 1$. Therefore,

\[
\text{grade}_{R_p}(pR_p) + \text{grade}_{S_q}(qS_q) \leq \text{ht}(p) + \text{ht}(q) - 2. \tag{*}
\]

There exists $Q \in \text{Spec}(R \otimes_k S)$ such that $Q \cap R = p$ and $Q \cap S = q$. On the other hand, by [1, Proposition 2.3],

$$\text{ht}(Q) = \text{ht}(p) + \text{ht}(q) + \text{ht}\left(Q/(p \otimes_k S + (R \otimes_k q))\right)$$

and

$$\text{grade}_{(R \otimes_k S)_Q}(Q(R \otimes_k S)_Q) = \text{grade}_{R_p}(pR_p) + \text{grade}_{S_q}(qS_q) + \text{ht}\left(Q/(p \otimes_k S + (R \otimes_k q))\right).$$

But

$$\text{grade}_{(R \otimes_k S)_Q}(Q(R \otimes_k S)_Q) \geq \text{ht}(Q) - 1,$$

so we have

$$\text{grade}_{R_p}(pR_p) + \text{grade}_{S_q}(qS_q) \geq \text{ht}(p) + \text{ht}(q) - 1$$

and by using (*), $\text{ht}(p) + \text{ht}(q) - 1 \leq \text{ht}(p) + \text{ht}(q) - 2$. That is a contradiction.

(iii): In fact, (b) $\implies$ (a) is just (i), and for proving (a) $\implies$ (b); by (ii), it is enough to show that $R$ is an approximately Cohen-Macaulay ring.

First, let $R$ be a homomorphic image of a Cohen-Macaulay ring. Consider the faithfully flat homomorphism $\varphi : R \longrightarrow (R \otimes_k S)$. The assertion now follows form Corollary 9.

Next, let $k$ be algebraically closed. Let $p \in \text{Spec}(R)$. Then there exists $q \in \text{Spec}(R \otimes_k S)$ with $q \cap R = p$. Consider the induced flat local homomorphism $\tilde{\varphi} : R_p \longrightarrow (R \otimes_k S)_q$. Let $q' \in \text{Ass}(R \otimes_k S)$ with $q' \subseteq q$ and $\text{ht}(p/(q' \cap R)) = \text{ht}(p)$. By using Proposition 3 and Theorem 6, it is enough to show that $\text{ht}(q/q') = \text{ht}(q)$. Set $q' \cap R = p'$, $q \cap S = p_2$, and $q' \cap S = p_1$. Then by [8, Theorem 23.2], $q' \in \text{Ass}_T(T/p'T)$. On the other hand, $S$ is Cohen-Macaulay and hence by [10, Theorem
Therefore, \( q' \in \text{Min}(p'T) \) and so \( q' \in \text{Min}(p'T + p_1 T) \). On the other hand, \( T/(p'T + p_1 T) \equiv (R/p') \otimes_k (S/p_1) \) is an integral domain (see [4, Exercise A1.2(a), p. 562]). Thus \( q' = p'T + p_1 T \). Now, the following equalities hold:

\[
\begin{align*}
\text{ht}(q/q') &= \text{ht}(q/(p'T + p_1 T)) \\
&= \text{ht}(p/p') + \text{ht}(p_2/p_1) + \text{ht}(q/(pT + p_2 T)) \\
&= \text{ht}(p) + \text{ht}(p_2) + \text{ht}(q/(pT + p_2 T)) \\
&= \text{ht}(q),
\end{align*}
\]

where the second and the last equalities hold by [1, Proposition 2.3] and the third one uses the fact that \( p_1 \in \text{Ass}(S) \). \( \square \)

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