A SURVEY ON THE ESTIMATION OF COMMUTATIVITY IN FINITE GROUPS

A. K. DAS, R. K. NATH, AND M. R. POURNAKI

Abstract. Let $G$ be a finite group and let $C = \{(x, y) \in G \times G \mid xy = yx\}$. Then $\Pr(G) = |C|/|G|^2$ is the probability that two elements of $G$, chosen randomly with replacement, commute. This probability is a well known quantity, called commutativity degree of $G$, and indeed gives us an estimation of commutativity in $G$. In the last four decades this subject has enjoyed a flourishing development. In this article, we give a brief survey on the development of this subject and then we collect several of our results concerning $\Pr(G)$ as well as its various generalizations.

1. Introduction

The study of group theoretical problems related to discrete probability has long been a topic of interest. Indeed, the story goes back to a sequence of papers all of them entitled “On some problems of a statistical group-theory” written by P. Erdős and P. Turán during the years 1965 to 1972 (see [7, 8, 9, 10, 11, 12, 13]). In Theorem IV of [10] Erdős and Turán have shown that in an arbitrary finite group $G$, the number of elements of

$$C = \{(x, y) \in G \times G \mid xy = yx\}$$

is equal to $|G|k(G)$, where $k(G)$ is the number of conjugacy classes of $G$. Later, in 1973, W. H. Gustafson [18] reproved this theorem adopting the same technique which was used by Erdős and Turán [10]. Here, for convenience of the readers, we reproduce the proof as follows:

Note that

$$C = \bigsqcup_{x \in G} \{x\} \times C_G(x),$$

where the symbol $\bigsqcup$ denotes the disjoint union of sets and $C_G(x)$ the centralizer of $x$ in $G$. Therefore, we have

$$|C| = \sum_{x \in G} |\{x\} \times C_G(x)| = \sum_{x \in G} |C_G(x)|.$$

Recall that if $x$ and $y$ are conjugate elements of $G$, then $C_G(x)$ and $C_G(y)$ are conjugate subgroups of $G$. Also the number of elements in the conjugacy class of $x$ is equal to the

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index of $C_G(x)$ in $G$, i.e., $|G : C_G(x)|$. Therefore, if $x_1, \ldots, x_{k(G)}$ are representatives of the distinct conjugacy classes in $G$, then we have

$$|C| = \sum_{i=1}^{k(G)} |G : C_G(x_i)| \leq \sum_{i=1}^{k(G)} |G| = |G|k(G),$$

as shown by Erdős, Turán, and Gustafson.

Gustafson in [18] has also asked the following question which not only formed the title of his paper but also became the leading problem for the subsequent development of the subject:

*What is the probability that two group elements commute?*

Using the above-mentioned argument, Gustafson himself has provided an elementary but significant answer to this question. More precisely, if $Pr(G)$ denotes the probability that two elements of $G$, chosen randomly with replacement, commute, then

$$Pr(G) = \frac{|C|}{|G|^2} = \frac{|G|k(G)}{|G|^2} = \frac{k(G)}{|G|}.$$

The quantity $Pr(G)$ has been termed by some authors as the *commutativity degree* of $G$. Indeed, it gives us an estimation of commutativity in finite groups. Clearly, a finite group $G$ is abelian if and only if $Pr(G) = 1$. Now, assuming $G$ to be nonabelian, one can obtain (see [18]) an upper bound for $Pr(G)$ as follows:

Consider the class equation of $G$ given by

$$|G| = |Z(G)| + \sum_{i=1}^{t} |G : C_G(x_i)|,$$

where $Z(G)$ denotes the center of $G$ and $x_1, \ldots, x_t$ are representatives of the distinct nontrivial conjugacy classes in $G$. For every $i$ with $1 \leq i \leq t$, we have $|G : C_G(x_i)| \geq 2$ and so $t \leq (|G| - |Z(G)|)/2$. This implies that $k(G) = |Z(G)| + t \leq (|G| + |Z(G)|)/2$. Since $G$ is nonabelian, it is easy to see that $G/Z(G)$ is not cyclic and so $|G/Z(G)| \geq 4$. Therefore, it follows that

$$Pr(G) = \frac{k(G)}{|G|} \leq \frac{|G| + |Z(G)|}{2|G|} = \frac{1}{2} + \frac{|Z(G)|}{2|G|} \leq \frac{1}{2} + \frac{1}{8} = \frac{5}{8}.$$
In 1975, G. Sherman [33] generalized the above probability. Let $G$ be a finite group acting on a finite set $\Omega$. Put $\Pr_G(\Omega) = |\text{Fix}(G, \Omega)|/|G||\Omega|$, where $\text{Fix}(G, \Omega)$ is the set of pairs $(g, \omega)$ of $G \times \Omega$ such that $g \omega = \omega$. Note that $\Pr_G(\Omega)$ is the probability that an element of $G$ leaves an element of $\Omega$ fixed. Sherman [33] also considered the case for which $G$ is abelian and $A$ is the group of its automorphisms. $\Pr_A(G)$ is the probability that an automorphism leaves an element fixed. He proved that for groups of order $p^n$, $p$ a prime, $\Pr_A(G) \leq 2(3/p^2)^{n/2}$. He further showed that if $G_n$ is a sequence of abelian groups with $|G_n| \to \infty$, then $\Pr_A(G_n) \to 0$. Considering the action of $G$ on itself by conjugation one obtains that $\Pr_G(G)$ is equal to $\Pr(G)$, i.e., the probability that two elements of $G$, chosen randomly with replacement, commute. In this sense $\Pr_G(\Omega)$ is a generalization of $\Pr(G)$.

In 1979, D. J. Rusin [32] continued the study of $\Pr(G)$. He considered, for a finite group $G$, two situations, namely, $G' \subseteq Z(G)$ and $G' \cap Z(G) = \{1\}$, where $G'$ is the commutator subgroup of $G$. First, he obtained an explicit computation of $\Pr(G)$ for groups $G$ with $G' \subseteq Z(G)$. In this case, $G$ being the direct product of its Sylow subgroups, the product formula $\Pr(H \times K) = \Pr(H) \Pr(K)$ sufficed him to obtain Theorem 1 of [32]: If $G$ is a $p$-group with $G' \subseteq Z(G)$, then

$$\Pr(G) = \frac{1}{|G'|} \left(1 + \sum_{G'/K \text{ cyclic}} \frac{(p-1)|G'| : K|}{p^{n(K)+1}}\right),$$

with $n(K)$ defined by $|\{x \in G \mid gxg^{-1}x^{-1} \in K \forall g \in G\}| = |G|/p^{n(K)}$. In the second case, i.e., when $G' \cap Z(G) = \{1\}$, he showed that $\Pr(G) = \Pr(K)$ holds true for some groups $K$ such that $K' \cong G'$ and $Z(K) = \{1\}$. On the other hand, for every group $G$, there exist at most a finite number of groups $K$ with $K' \cong G'$ and $Z(K) = \{1\}$ (see [32] Proposition 4]). Under an additional condition that $|G'|$ is a prime, he obtained an explicit computation of $\Pr(G)$ in this second case as well.

In Section IV, Rusin has classified the groups $G$ for which $\Pr(G)$ is greater than $11/32$.

In 1995, P. Lescot [22] generalized $\Pr(G)$ in the other direction. Let $G$ be a finite group and let $n \geq 0$ be an integer. Define

$$d_n(G) = \frac{|\{(x_1, \ldots, x_{n+1}) \in G^{n+1} \mid x_i x_j = x_j x_i, \ 1 \leq i, j \leq n + 1\}|}{|G|^{n+1}}.$$ 

Note that $d_1(G)$ is equal to $\Pr(G)$, i.e., the probability that two elements of $G$, chosen randomly with replacement, commute. Therefore, this later quantity is a generalization of $\Pr(G)$. He showed that $d_n(G) = d_n(H)$ provided $G$ and $H$ are isoclinic in the sense of P. Hall (see [19], Page 133]). He also showed that if $G$ is a finite nonabelian group, then $d_n(G) \leq (3 \times 2^n - 1)/2^{2n+1}$. Moreover, the equality holds if and only if $G$ is isoclinic to the group of quaternions of order eight. Also in 2001, Lescot [23] characterized all finite groups $G$ with commutativity degree at least $1/2$. This is done by first determining all finite groups $G$ whose central factor is a nonabelian group of order $pq$ and then using a result of S. R. Blackburn [2] determining all finite 2-groups with derived subgroup of order two.
In 2008, M. R. Pournaki and R. Sobhani [31] gave a new generalization of \( \text{Pr}(G) \).
For a finite group \( G \) and \( g \in G \) define
\[
\text{Pr}_g(G) = \frac{|\{(x, y) \in G \times G \mid [x, y] = g\}|}{|G|^2},
\]
where \([x, y]\) denotes the commutator of \( x \) and \( y \). Note that \( \text{Pr}_1(G) \) is equal to \( \text{Pr}(G) \), i.e., the probability that two elements of \( G \), chosen randomly with replacement, commute. Therefore, this later quantity is a generalization of \( \text{Pr}(G) \). They found some results concerning \( \text{Pr}_g(G) \) and after that A. K. Das and R. K. Nath in a sequence of different papers gave new results and other new generalizations regarding \( \text{Pr}(G) \) and \( \text{Pr}_g(G) \) (see [3, 4, 5, 6, 28, 29]).

In the rest of this article we collect several of our results concerning \( \text{Pr}(G) \) as well as its various generalizations.

2. Commutativity of two elements

In this section we deal with probability that two elements of a finite group \( G \), chosen randomly with replacement, commute. As we mentioned in Section 1, this probability is \( \text{Pr}(G) = |C|/|G|^2 \), where \( C = \{(x, y) \in G \times G \mid xy = yx\} \). Here we collect several results dealing with this probability.

In 1979, Rusin [32] computed, for a finite group \( G \), the values of \( \text{Pr}(G) \) when \( G' \subseteq \text{Z}(G) \), and also when \( G' \cap \text{Z}(G) \) is trivial. In this direction we have the following result.

**Theorem 2.1** ([5], Theorem 3.5). Let \( G \) be a finite group and let \( p \) be a prime number such that \( \text{gcd}(p - 1, |G|) = 1 \). If \( |G'| = p^2 \) and \( |G' \cap \text{Z}(G)| = p \), then the following statements hold:

\[
(1) \quad \text{Pr}(G) = \begin{cases} 
\frac{2p^2 - 1}{p^4} & \text{if } C_G(G') \text{ is abelian,} \\
\frac{1}{p^4} \left( \frac{p - 1}{p^{2s-1}} + p^2 + p - 1 \right) & \text{otherwise,}
\end{cases}
\]

\[
(2) \quad \frac{|G|}{|\text{Z}(G)|} = \begin{cases} 
p^3 & \text{if } C_G(G') \text{ is abelian,} \\
p^{2s+2} \text{ or } p^{2s+3} & \text{otherwise,}
\end{cases}
\]

where \( p^{2s} = |C_G(G') : \text{Z}(C_G(G'))| \). Moreover, we have

\[
\left| \frac{G}{G' \cap \text{Z}(G)} : \text{Z} \left( \frac{G}{G' \cap \text{Z}(G)} \right) \right| = \left| \frac{G}{\text{Z}(G)} : \text{Z} \left( \frac{G}{\text{Z}(G)} \right) \right| = p^2.
\]
It may be noted here that, on a number of occasions, the structure of $G/Z(G)$ is determined by its size. For example, using GAP [35] and the notion of semidirect product `$\rtimes$’, one can see that if $G$ is a finite group with $G' \nsubseteq Z(G)$, then $G/Z(G)$ is isomorphic to $C_7 \times C_3$, $(C_3 \times C_3) \rtimes C_3$, $C_{13} \rtimes C_3$, $C_{19} \rtimes C_3$, $C_3 \rtimes (C_7 \rtimes C_3)$, or $(C_5 \times C_5) \rtimes C_3$ according as $|G/Z(G)|$ is equal to 21, 27, 39, 57, 63, or 75. Here, $C_n$ denotes the cyclic group of order $n$.

Rusin also determined all numbers lying in the interval $(11/32, 1]$ that can be realized as the commutativity degree of some finite groups, and also classified all finite groups whose commutativity degrees lie in the interval $(11/32, 1]$. In 2006, F. Barry, D. MacHale, and A. Ní Shé [1] have shown that if $G$ is a finite group with $|G|$ odd and Pr($G$) $>$ 11/75, then $G$ is supersolvable. They also proved that if Pr($G$) $>$ 1/3, then $G$ is supersolvable. It may be mentioned here that a group $G$ is said to be supersolvable if there is a series of the form

$$\{1\} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r = G,$$

where $A_i \leq G$ and $A_{i+1}/A_i$ is cyclic for every $i$ with $0 \leq i \leq r - 1$ (see [16, 24, 34]). The previous theorem enables us to obtain the following characterization.

**Theorem 2.2** [5. Theorem 4.3]. Let $G$ be a finite group. If $|G|$ is odd and Pr($G$) $\geq 11/75$, then the possible values of Pr($G$) and the corresponding structures of $G'$, $G' \cap Z(G)$, and $G/Z(G)$ are given by the following Table [1].

<table>
<thead>
<tr>
<th>Pr($G$)</th>
<th>$G'$</th>
<th>$G' \cap Z(G)$</th>
<th>$G/Z(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1)</td>
<td>(1)</td>
<td>(1)</td>
</tr>
<tr>
<td>$\frac{1}{3}(1 + \frac{2}{37})$</td>
<td>$C_3$</td>
<td>$C_3$</td>
<td>$(C_3 \times C_3)^s$, $s \geq 1$</td>
</tr>
<tr>
<td>$\frac{1}{7}(1 + \frac{2}{37})$</td>
<td>$C_5$</td>
<td>$C_5$</td>
<td>$(C_5 \times C_5)^s$, $s \geq 1$</td>
</tr>
<tr>
<td>$\frac{55}{87}$</td>
<td>$C_7$</td>
<td>$C_7$</td>
<td>$C_7 \times C_7$</td>
</tr>
<tr>
<td>$\frac{17}{27}$</td>
<td>$C_9$ or $C_3 \times C_3$</td>
<td>$C_3$ or $C_3 \times C_3$</td>
<td>$(C_4 \times C_3) \rtimes C_3$ or $C_3^3$</td>
</tr>
<tr>
<td>$\frac{121}{129}$</td>
<td>$C_4 \times C_3$</td>
<td>$C_3 \times C_3$</td>
<td>$C_4$</td>
</tr>
<tr>
<td>$\frac{27}{37}$</td>
<td>$C_{13}$</td>
<td>(1)</td>
<td>$C_{13} \times C_3$</td>
</tr>
<tr>
<td>$\frac{3}{17}$</td>
<td>$C_{19}$</td>
<td>(1)</td>
<td>$C_{19} \times C_3$</td>
</tr>
<tr>
<td>$\frac{29}{189}$</td>
<td>$C_{21}$</td>
<td>$C_3$</td>
<td>$C_3 \times (C_7 \times C_3)$</td>
</tr>
<tr>
<td>$\frac{11}{45}$</td>
<td>$C_5 \times C_5$</td>
<td>(1)</td>
<td>$(C_5 \times C_5) \rtimes C_3$</td>
</tr>
</tbody>
</table>

Table 1. The possible values of Pr($G$) and the corresponding structures of $G'$, $G' \cap Z(G)$, and $G/Z(G)$ provided $|G|$ is odd and Pr($G$) $\geq 11/75$. 
The following result gives a universal lower bound for $\Pr(G)$.

**Theorem 2.3** ([28], Theorem 1). Let $G$ be a finite group. Then we have

$$\Pr(G) \geq \frac{1}{|G'|} \left(1 + \frac{|G'| - 1}{|G : Z(G)|}\right).$$

In particular, $\Pr(G) > 1/|G'|$ provided $G$ is nonabelian.

There are several equivalent necessary as well as sufficient conditions for equality to hold in the above theorem. These are listed as follows.

**Theorem 2.4** ([28], Theorem 2). Let $G$ be a finite nonabelian group. Then the following statements are equivalent:

1. The equality $\Pr(G) = 1/|G'| \left(1 + \frac{|G'| - 1}{|G : Z(G)|}\right)$ holds.
2. We have $\text{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$, which means that $G$ is of central type with $|\text{cd}(G)| = 2$. Here, $\text{cd}(G)$ denotes the set of irreducible complex character degrees of $G$.
3. For every $x \in G \setminus Z(G)$, we have $|\text{Cl}_G(x)| = |G'|$. Here, $\text{Cl}_G(x)$ denotes the conjugacy class of $x$.
4. For every $x \in G \setminus Z(G)$, we have $\text{Cl}_G(x) = G'x$. In particular, $G$ is a nilpotent group of class 2.
5. For every $x \in G \setminus Z(G)$, we have $C_G(x) \leq G$ and $G' \cong G/C_G(x)$. In particular, $G$ is a $CN$-group, i.e., a group in which the centralizer of every element is normal.
6. For every $x \in G \setminus Z(G)$, we have $G' = \{[y,x] \mid y \in G\}$. In particular, every element of $G'$ is a commutator.

Theorem 2.3 and Theorem 2.4 not only allow us to obtain some characterizations for finite nilpotent groups of class 2 whose commutator subgroups have prime order, but also enable us to re-establish certain facts (essentially due to K. S. Joseph [21]) concerning the smallest prime divisors of the orders of finite groups.

**Theorem 2.5** ([28], Proposition 1). Let $G$ be a finite group and let $p$ be the smallest prime divisor of $|G|$. Then the following statements hold:

1. If $p \neq 2$, then $\Pr(G) \neq 1/p$.
2. In the case of $G$ is nonabelian, $\Pr(G) > 1/p$ if and only if $|G'| = p$ and $G' \subseteq Z(G)$.

Some of the consequences of the above results are as given below.

**Theorem 2.6** ([28], Corollary 1). Let $G$ be a finite group with $\Pr(G) = 1/3$. Then $|G|$ is even.
Theorem 2.7 ([28], Corollary 2). Let $G$ be a finite group and let $p \neq 2$ be the smallest prime divisor of $|G|$. If $G$ is nonabelian with $G' \cap Z(G) = \{1\}$, then $\Pr(G) < 1/p$.

For a group $K$ and an element $k$ of that group consider $K = K/Z(K)$ and $k = kZ(K)$, and define the map $a_K : K \times K \rightarrow K'$ by $a_K(x, y) = [x, y]$ which is obviously well defined. We now introduce the notion of isoclinism which is originally due to P. Hall (see [19, Page 133]). Let $G$ and $H$ be two groups. A pair $(\varphi, \psi)$ is called an isoclinism from $G$ to $H$ if $\varphi$ is an isomorphism from $G$ to $H$ and $\psi$ is an isomorphism from $G'$ to $H'$ for which the following diagram commutes.

\[
\begin{array}{ccc}
G \times G & \xrightarrow{\varphi \times \varphi} & H \times H \\
\downarrow a_G & & \downarrow a_H \\
G' & \xrightarrow{\psi} & H'
\end{array}
\]

If there is an isoclinism from $G$ to $H$, we shall say that $G$ and $H$ are isoclinic. Clearly, isoclinism is an equivalence relation between groups. It is well known that if $G$ and $H$ are isomorphic groups, then they are isoclinic.

Theorem 2.8 ([28], Proposition 2). Let $G$ be a finite group and let $p$ be a prime. Then the following statements are equivalent:

1. We have $|G'| = p$ and $G' \subseteq Z(G)$.
2. The group $G$ is of central type with $|\text{cd}(G)| = 2$ and $|G'| = p$.
3. The group $G$ is a direct product of a $p$-group $P$ and an abelian group $A$ such that $|P'| = p$ and $\gcd(p, |A|) = 1$.
4. The group $G$ is isoclinic to an extra-special $p$-group, and consequently, the equality $|G : Z(G)| = p^{2k}$ holds true for some positive integer $k$.

In particular, if $G$ is nonabelian and $p$ is the smallest prime divisor of $|G|$, then the above statements are also equivalent to the condition $\Pr(G) > 1/p$.

If $|G'| = p$, $p$ prime, and $G' \subseteq Z(G)$, then we call the integer $2k + 1$, where $k$ is as in Part 4 of Theorem 2.8, as the isoclinic exponent of $G$ and denote it by iso.exp$(G)$.

Thus, we have, in particular, the following theorem:

Theorem 2.9 ([31], Proposition 3.4). Let $G$ be a finite group such that $G' \subseteq Z(G)$ and let $|G'| = p$ be a prime. If iso.exp$(G) = n$, then

\[
\Pr(G) = \frac{1}{p} \left( 1 + \frac{p - 1}{p^n - 1} \right).
\]

Now consider a finite group $G$ such that $|G'| = p$ is a prime and $G' \cap Z(G) = \{1\}$. We have $(G/Z(G))' \cong G'$. On the other hand, if we consider $Z(G/Z(G)) = H/Z(G)$ for some $H$, $Z(G) \leq H \leq G$, then $[G, H] \leq G' \cap Z(G) = \{1\}$ implies that $H = Z(G)$.
and so we have $Z(G/Z(G)) = 1$. Therefore by Proposition 5, there is a positive integer $n$ depending only on $G$ for which $G/Z(G) = \langle a, b \mid a^n = b^n = 1, bab^{-1} = a^r \rangle$. We call $n$ as the invariant number of $G$ and denote it by $\text{inv}(G)$.

**Theorem 2.10** ([31], Proposition 4.4). Let $G$ be a finite group such that $G' \cap Z(G) = \{1\}$ and let $|G'| = p$ be a prime. If $\text{inv}(G) = n$, then

$$\text{Pr}(G) = \frac{n^2 + p - 1}{pn^2}.$$ 

In [15, Corollary 1.2], I. V. Erovenko and B. Sury have shown, in particular, that for every integer $k > 1$ there exists a family $\{G_n\}$ of finite groups such that $\text{Pr}(G_n) \to 1/k^2$ as $n \to \infty$. In this connection, we have the following observations.

**Theorem 2.11** ([27], Proposition 2.5.1). For any $k \in \mathbb{N}$ with $k > 1$, there exists a family $\{G_n\}$ of finite groups such that $\text{Pr}(G_n) \to 1/k$ as $n \to \infty$.

**Theorem 2.12** ([27], Proposition 2.5.2). For any $n \in \mathbb{N}$, there exists a finite group $G$ such that $\text{Pr}(G) = 1/n$.

The following result gives us a formula as well as an upper bound for $\text{Pr}(G)$ when $G$ has exactly two irreducible complex character degrees.

**Theorem 2.13** ([31], Theorem 2.2). Let $G$ be a finite group such that $\text{cd}(G) = \{1, m\}$, $m > 1$. Then we have

$$\text{Pr}(G) = \frac{1}{|G'|} \left(1 + \frac{|G'| - 1}{m^2}\right) \leq \frac{|G'| + 3}{4|G'|}.$$ 

3. **Commutators of two elements which are equal to a given element**

In this section we deal with the probability that the commutator of two elements, chosen randomly with replacement, in a finite group is equal to a given element of that group. For a finite group $G$ and a given element $g$ of $G$, this probability is $\text{Pr}_g(G) = |C_g|/|G|^2$, where $C_g = \{(x, y) \in G \times G \mid [x, y] = g\}$. Clearly, this notion is a generalization of the previous one, mentioned in Section 2, as we have $\text{Pr}_1(G) = \text{Pr}(G)$.

In the following we collect several results dealing with this probability. Obviously for those $g$’s lie in $G \setminus G'$, we have $\text{Pr}_g(G) = 0$. Therefore in the sequel we deal only with $g$’s which lie in $G'$. Note that there are examples of groups $G$ of order 96 due to R. M. Guralnick [17] where $\text{Pr}_g(G) = 0$ even when $g$ belongs to $G'$.

The first result shows that $\text{Pr}_g(G)$ is an invariant under isoclinism in the following sense.

**Theorem 3.1** ([31], Lemma 3.5). Let $G$ and $H$ be two isoclinic finite groups and let $(\varphi, \psi)$ be an isoclinism from $G$ to $H$. If $g \in G'$, then we have

$$\text{Pr}_g(G) = \text{Pr}_{\psi(g)}(H).$$ 

The following result gives us a character theoretical formula for $\text{Pr}_g(G)$.
Theorem 3.2 ([31], Theorem 2.1). Let $G$ be a finite group and $g \in G'$. Then
\[
Pr_g(G) = \frac{1}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(g)}{\chi(1)},
\]
where $\text{Irr}(G)$ denotes the set of irreducible complex characters of $G$.

We also derive a formula as well as a lower bound for $Pr_g(G)$ when $G$ has exactly two irreducible complex character degrees.

Theorem 3.3 ([31], Theorem 2.2). Let $G$ be a finite group with $\text{cd}(G) = \{1, m\}$, $m > 1$. If $g \in G'$, $g \neq 1$, then we have
\[
Pr_g(G) = \frac{1}{|G'|} \left(1 - \frac{1}{m^2}\right) \geq \frac{3}{4|G'|}.
\]

In view of [20, Problem 2.13], we obtain the following theorem.

Theorem 3.4 ([31], Propositions 3.1 and 3.4). Let $G$ be a finite group such that $G' \subseteq Z(G)$ and let $|G'| = p$ be a prime. If $g \in G'$, $g \neq 1$, then we have
\[
Pr_g(G) = \frac{1}{p} \left(1 - \frac{1}{|G : Z(G)|}\right) \geq \frac{3}{4p}.
\]

In particular, if $\text{iso.exp}(G) = n$, then we obtain that
\[
Pr_g(G) = \frac{1}{p} \left(1 - \frac{1}{p^{n-1}}\right).
\]

A universal upper bound for $Pr_g(G)$, $g \in G'$, $g \neq 1$, is given as follows.

Theorem 3.5 ([31], Corollary 2.3). Let $G$ be a finite group with $|\text{cd}(G)| = 2$. Let $g \in G'$, $g \neq 1$. Then we have
\[
Pr_g(G) \leq \frac{1}{|G'|} \left(1 - \frac{1}{|G : Z(G)|}\right).
\]
Moreover, the equality holds if and only if $G$ is of central type.

The following result gives us a formula for $Pr_g(G)$, $g \in G'$, $g \neq 1$, when $|G'|$ is prime and $G' \cap Z(G)$ is trivial.

Theorem 3.6 ([31], Proposition 4.4). Let $G$ be a finite group such that $G' \cap Z(G) = \{1\}$ and let $|G'| = p$ be a prime. If $\text{inv}(G) = n$ and $g \in G'$, $g \neq 1$, then we have
\[
Pr_g(G) = \frac{n^2 - 1}{pn^2}.
\]

We close this section by listing some more properties of $Pr_g(G)$.

Theorem 3.7 ([31], Proposition 5.1). Let $G$ be a finite group and let $g \in G'$. Then we have $Pr_g(G) \leq Pr(G)$. Moreover, the equality holds if and only if $g = 1$. 
Theorem 3.8 ([31], Proposition 5.2). Let $G$ be a finite group and let $g \in G'$, $g \neq 1$. Then we have $\Pr_g(G) < 1/2$.

Theorem 3.9 ([31], Proposition 5.3). For any $\epsilon \in \mathbb{R}$ with $\epsilon > 0$, there exists a finite group $G$ and $g \in G$ such that $1/2 - \epsilon < \Pr_g(G) < 1/2$.

4. Admissible words which are equal to a given element

In this section we deal with a probability which is a generalization of the previous one mentioned in Section 3. For a given positive integer $n$, consider the free group of words on $n$ generators $x_1, x_2, \ldots, x_n$. A word $\omega(x_1, x_2, \ldots, x_n)$ is called admissible (see [3]) if each letter in it has precisely two nonzero indices, namely, $+1$ and $-1$.

Given a nontrivial admissible word $\omega(x_1, x_2, \ldots, x_n)$, $n \geq 2$, and an element $g \in G$, consider the ratio

$$\Pr_\omega^g(G) = \frac{|\{(g_1, g_2, \ldots, g_n) \in G^n \mid \omega(g_1, g_2, \ldots, g_n) = g\}|}{|G^n|}.$$ 

Note that, for $\omega(x_1, x_2) = x_1x_2^{-1}x_2^{-1}$, we have $\Pr_\omega^1(G) = \Pr_g(G)$, and so this later probability is a generalization of $\Pr_g(G)$.

In the following we collect several results dealing with this probability.

Theorem 4.1 ([6], Proposition 2.2). Let $G$ be a finite group and let $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. Then the following statements hold:

(1) $\Pr_\omega^1(G) = 1$ if and only if $G$ is abelian.

(2) $\Pr_\omega^g(G) = \Pr_\omega^h(G)$ if $g, h \in G'$ are conjugate in $G$.

(3) $\Pr_\omega^g(G) = \Pr_\omega^h(G)$ if $g, h \in G'$ generate the same cyclic subgroups of $G$. Consequently, we obtain that

$$\Pr_\omega^g(G) = \frac{1 - \Pr_\omega^1(G)}{p - 1}$$

provided $|G'| = p$ is a prime and $g \in G'$, $g \neq 1$.

The quantity $\Pr_\omega^g(G)$ respects the cartesian product in the following sense.

Theorem 4.2 ([6], Proposition 2.3). Let $H$ and $K$ be two finite groups, let $(h, k) \in H' \times K'$ and $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. Then we have

$$\Pr_{(h,k)}^\omega(H \times K) = \Pr_h^\omega(H)\Pr_k^\omega(K).$$

The following result shows that $\Pr_\omega^g(G)$ is an invariant under isoclinism of finite groups.

Theorem 4.3 ([6], Proposition 2.4). Let $G$ and $H$ be two finite groups and $(\phi, \psi)$ be an isoclinism from $G$ to $H$. If $g \in G'$ and $\omega(x_1, x_2, \ldots, x_n)$ is a nontrivial admissible word, then we have

$$\Pr_\omega^g(G) = \Pr_\omega^{\psi(g)}(H).$$
Theorem 4.4 ([6], Proposition 3.1). Let $G$ be a finite group and let $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. If $g \in G'$, then we have

$$\frac{\Pr_g(G)}{|G : Z(G)|^{n-2}} \leq \Pr_1^\omega(G).$$

If $G$ is a finite nonabelian simple group and $\omega(x_1, x_2, \ldots, x_n)$ is a nontrivial admissible word, then using Ore conjecture (see [30]), which has been established recently in [25] Theorem 1, it follows from the above result that every element of $G$ is of the form $\omega(g_1, g_2, \ldots, g_n)$ for some $g_1, g_2, \ldots, g_n \in G$.

As a generalization of Theorem 3.7, we have the following result.

Theorem 4.5 ([6], Proposition 3.6). Let $G$ be a finite group and let $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. If $g \in G'$, then the following statements hold:

(1) $\Pr_g^\omega(G) \leq \Pr_1^\omega(G) \leq \Pr(G)$.

(2) $\Pr_g^\omega(G) = \Pr_1^\omega(G)$ if and only if $g = 1$.

Consequently, $\Pr_g^\omega(G) = 1$ if and only if $g = 1$ and $G$ is abelian.

The following result gives us a universal lower bound for $\Pr_1^\omega(G)$.

Theorem 4.6 ([6], Proposition 3.7). Let $G$ be a finite group and let $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. Then we have

$$\Pr_1^\omega(G) \geq \frac{1}{|G'|} \left( 1 + \frac{|G'| - 1}{|G : Z(G)|^{n/2}} \right).$$

In particular, $\Pr_1^\omega(G) > \frac{1}{|G'|}$ provided $G$ is nonabelian.

The following result generalizes Theorem 3.8.

Theorem 4.7 ([6], Proposition 3.8). Let $G$ be a finite nonabelian group, let $g \in G'$ and $p$ be the smallest prime divisor of $|G|$. If $\omega(x_1, x_2, \ldots, x_n)$ is a nontrivial admissible word and $g \neq 1$, then $\Pr_g^n(G) < 1/p$. In particular, we have $\Pr_g^n(G) < 1/2$.

Let us now take the nontrivial admissible word $\omega(x_1, x_2, \ldots, x_n)$ to be $x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}$, and write $\Pr_g^n(G)$ in place of $\Pr_1^\omega(G)$. It has been observed that $\Pr_g^n(G) = \Pr_g^{n+1}(G)$. Hence, without any loss we may assume that $n$ is even.

Theorem 4.8 ([29], Equations 8, 11, and 12). Let $G$ be a finite nonabelian group, let $g \in G'$ and $2 \leq d \leq \min\{\chi(1) \mid \chi \in \text{Irr}(G), \chi(1) \neq 1\}$. Then the following statements hold:

(1) $\left| \Pr_g^n(G) - \frac{1}{|G'|} \right| \leq \frac{1}{d^{n-2}} \left( \Pr(G) - \frac{1}{|G'|} \right)$.
(2) \( \left| \Pr_g^n(G) - \frac{1}{|G'|} \right| \leq \frac{1}{d^n} \left( 1 - \frac{1}{|G'|} \right) \). In particular, we have
\[
\Pr_g^n(G) \leq \frac{2^n + 1}{2^{n+1}}.
\]

**Theorem 4.9** ([29], Equation 14). Let \( G \) be a finite nonabelian simple group and let \( g \in G \). Then we have
\[
\left| \Pr_g^n(G) - \frac{1}{|G|} \right| \leq \frac{1}{3^{n-2}} \left( \frac{1}{12} - \frac{1}{|G|} \right).
\]
In particular, we conclude that
\[
\Pr_g^n(G) \leq \frac{3^{n-2} + 4}{3^{n-1} \times 20}.
\]

The following result generalizes Theorem 3.3.

**Theorem 4.10** ([29], Proposition 5.1). Let \( G \) be a finite nonabelian group and let \( g \in G' \), \( g \neq 1 \). If \( \text{cd}(G) = \{1, d\} \), \( d > 1 \), then we have
\[
\Pr_g^n(G) = \frac{1}{|G'|} \left( 1 - \frac{1}{d^n} \right).
\]

The next two results give us some necessary and sufficient conditions for equality to hold in Theorem 4.8.

**Theorem 4.11** ([29], Propositions 4.2 and 5.2). Let \( G \) be a finite nonabelian group, let \( g \in G' \) and \( 2 \leq d \leq \min \{ \chi(1) \mid \chi \in \text{Irr}(G), \chi(1) \neq 1 \} \). Then the following statements hold:

1. \( \Pr_g^n(G) = \frac{1}{d^{n-2}} \left( \Pr(G) + \frac{d^{n-2} - 1}{|G'|} \right) \) if and only if \( g = 1 \) and \( \text{cd}(G) = \{1, d\} \).

2. \( \Pr_g^n(G) = \frac{1}{d^{n-2}} \left( -\Pr(G) + \frac{d^{n-2} + 1}{|G'|} \right) \) if and only if \( g \neq 1 \), \( \text{cd}(G) = \{1, d\} \), and \( |G'| = 2 \).

**Theorem 4.12** ([29], Proposition 4.4 and Corollary 5.3). Let \( G \) be a finite nonabelian group, let \( g \in G' \) and \( 2 \leq d \leq \min \{ \chi(1) \mid \chi \in \text{Irr}(G), \chi(1) \neq 1 \} \). Then the following statements hold:

1. \( \Pr_g^n(G) = \frac{1}{d^n} \left( 1 + \frac{d^n - 1}{|G'|} \right) \) if and only if \( g = 1 \) and \( \text{cd}(G) = \{1, d\} \).
Theorem 4.13 ([29], Proposition 4.6). Let $G$ be a finite nonabelian group, let $g \in G'$ and $p$ be the smallest prime divisor of $|G|$. Then we have

$$\Pr_{g}^{n}(G) = \frac{p^n + p - 1}{p^{n+1}},$$

if and only if $g = 1$ and $G$ is isoclinic to

$$\langle x, y \mid x^{p^2} = 1 = y^p, y^{-1}xy = x^{p+1} \rangle.$$

In particular, putting $p = 2$, we conclude that

$$\Pr_{g}^{n}(G) = \frac{2^n + 1}{2^{n+1}},$$

if and only if $g = 1$ and $G$ is isoclinic to $D_8$, the dihedral group, and hence, to $Q_8$, the group of quaternions.

We now list a few results which are basically generalizations of some of the results obtained in [31].

Theorem 4.14 ([29], Proposition 6.1). Let $G$ be a finite nonabelian group with $|\text{cd}(G)| = 2$ and let $g \in G'$. Then we have

$$\Pr_{1}^{n}(G) \geq \frac{1}{|G'|} \left( 1 + \frac{|G'| - 1}{|G : Z(G)|^{n/2}} \right)$$

and

$$\Pr_{g}^{n}(G) \leq \frac{1}{|G'|} \left( 1 - \frac{1}{|G : Z(G)|^{n/2}} \right)$$

provided $g \neq 1$.

Moreover, in each case, the equality holds if and only if $G$ is of central type.

Theorem 4.15 ([29], Corollary 6.2). Let $G$ be a finite nonabelian group and $g \in G'$. Let $G$ be of central type with $|\text{cd}(G)| = 2$. Then we have

$$\Pr_{1}^{n}(G) \leq \frac{1}{|G'|} \left( 1 + \frac{|G'| - 1}{2^n} \right)$$

and

$$\Pr_{g}^{n}(G) \geq \frac{1}{|G'|} \left( 1 - \frac{1}{2^n} \right)$$

provided $g \neq 1$.

Theorem 4.16 ([29], Proposition 6.3). Let $G$ be a finite nonabelian group and let $G' \subseteq Z(G)$ and $|G'| = p$ be a prime. If $g \in G'$, then we have
Pr

\text{g}^n(G) = \begin{cases} 
\frac{1}{p} \left(1 + \frac{p - 1}{p^{nk}}\right) & \text{if } g = 1, \\
\frac{1}{p} \left(1 - \frac{1}{p^{nk}}\right) & \text{if } g \neq 1,
\end{cases}

where \( k = \frac{1}{2} \log_p |G : Z(G)|. \)

\textbf{Theorem 4.17 (}[29], Proposition 6.5). Let \( G \) be a finite nonabelian group and \( g \in G' \).

If \( G' \cap Z(G) = \{1\} \), \(|G'| = p\), where \( p \) is a prime, and \( \text{inv}(G) = r \), then we have

\[ \Pr^g_n(G) = \begin{cases} 
\frac{r^n + p - 1}{pr^n} & \text{if } g = 1, \\
\frac{r^n - 1}{pr^n} & \text{if } g \neq 1.
\end{cases} \]

\textbf{Theorem 4.18 (}[29], Proposition 6.7). For any \( \varepsilon \in \mathbb{R} \) with \( \varepsilon > 0 \) and for any prime number \( p \), there exists a finite group \( G \) such that the inequality

\[ \left| \Pr^g_n(G) - \frac{1}{p} \right| < \varepsilon \]

holds true for all \( g \in G' \).

5. COMMUTATORS OF TWO SUBGROUPS WHICH ARE EQUAL TO A GIVEN ELEMENT

In 2007, A. Erfanian, R. Rezaei, and P. Lescot [14] studied the probability \( \Pr(H, G) \) that an element of a given subgroup \( H \) of a finite group \( G \) commutes with an element of \( G \) (see also [26]). Note that \( \Pr(G, G) = \Pr(G) \). This notion has been further generalized as follows. Let \( G \) be a finite group and \( g \in G' \). Let \( H \) and \( K \) be two subgroups of \( G \). Consider the ratio

\[ \Pr_g(H, K) = \frac{|\{(x, y) \in H \times K \mid [x, y] = g\}|}{|H||K|}. \]

If \( g = 1 \), then for brevity we write \( \Pr_1(H, K) = \Pr(H, K) \). Note that for \( H = K = G \), we have \( \Pr_g(H, K) = \Pr_g(G) \).

The following theorem says that \( \Pr_g(H, K) \) is not very far from being symmetric with respect to \( H \) and \( K \).

\textbf{Theorem 5.1 (}[14], Proposition 2.1). Let \( G \) be a finite group and let \( g \in G' \). If \( H \) and \( K \) are two subgroups of \( G \), then we have

\[ \Pr_g(H, K) = \Pr_{g^{-1}}(K, H). \]
However, if \( g^2 = 1 \), or, if \( g \in H \cup K \) (for example, when \( H \) or \( K \) is normal in \( G \)), then we have

\[
\Pr_g(H, K) = \Pr_g(K, H) = \Pr_g^{-1}(H, K).
\]

The quantity \( \Pr_g(H, K) \) respects the cartesian product in the following sense.

**Theorem 5.2** ([4], Proposition 2.2). Let \( G_1 \) and \( G_2 \) be two finite groups with subgroups \( H_1, K_1 \subseteq G_1 \) and \( H_2, K_2 \subseteq G_2 \). Let \( g_1 \in G'_1 \) and \( g_2 \in G'_2 \). Then we have

\[
\Pr_{(g_1, g_2)}(H_1 \times H_2, K_1 \times K_2) = \Pr_{g_1}(H_1, K_1)\Pr_{g_2}(H_2, K_2).
\]

Now, we have the following computing formula which plays a key role in the study of \( \Pr_g(H, K) \).

**Theorem 5.3** ([4], Theorem 2.3). Let \( G \) be a finite group and let \( g \in G' \). If \( H \) and \( K \) are two subgroups of \( G \), then we have

\[
\Pr_g(H, K) = \frac{1}{|H||K|} \sum_{x \in H} |C_K(x)| = \frac{1}{|H|} \sum_{x \in H} \frac{1}{|\text{Cl}_K(x)|},
\]

where \( C_K(x) = \{ y \in K \mid xy = yx \} \) and \( \text{Cl}_K(x) = \{ yxy^{-1} \mid y \in K \} \), the \( K \)-conjugacy class of \( x \).

The following result generalizes the well known formula \( \Pr(G) = k(G)/|G| \).

**Theorem 5.4** ([4], Corollary 2.4). Let \( G \) be a finite group and let \( H \) and \( K \) be two subgroups of \( G \). If \( H \trianglelefteq G \), then we have

\[
\Pr(H, K) = \frac{k_K(H)}{|H|},
\]

where \( k_K(H) \) is the number of \( K \)-conjugacy classes that constitute \( H \).

Let \( G \) be a finite group. If \( H \trianglelefteq G \) with \( C_G(x) \subseteq H \) for all \( x \in H \setminus \{1\} \), then using Sylow’s theorems and the fact that nontrivial \( p \)-groups have nontrivial centers, we have \( \gcd(|H|, |G : H|) = 1 \). Therefore, by the Schur-Zassenhaus theorem, \( H \) has a complement in \( G \). Such groups belong to a well known class of groups called the Frobenius groups; for example, the alternating group \( A_4 \), the dihedral groups of order \( 2n \) with \( n \) odd, the nonabelian groups of order \( pq \), where \( p \) and \( q \) are primes with \( q|(p-1) \), etc.

**Theorem 5.5** ([4], Proposition 2.5). Let \( G \) be a finite group. If \( H \) is an abelian normal subgroup of \( G \) with a complement \( K \) in \( G \) and \( g \in G' \), then we have

\[
\Pr_g(H, G) = \Pr_g(H, K).
\]

As a consequence, we conclude the following theorem.

**Theorem 5.6** ([4], Corollary 2.6). Let \( G \) be a finite group and let \( g \in G' \). If \( H \trianglelefteq G \) with \( C_G(x) = H \) for all \( x \in H \setminus \{1\} \), then we have

\[
\Pr_g(H, G) = \Pr_g(H, K),
\]
where $K$ is a complement of $H$ in $G$. In particular,
\[
\Pr(H, G) = \frac{1}{|H|} + \frac{|H| - 1}{|G|}.
\]

The following result gives us some conditional lower bounds for $\Pr_g(H, K)$.

**Theorem 5.7** ([4], Proposition 3.1). Let $G$ be a finite group and $g \in G'$. Let $H$ and $K$ be any two subgroups of $G$. If $g \neq 1$, then the following statements hold:

1. If $\Pr_g(H, K) \neq 0$ then $\Pr_g(H, K) \geq \frac{|C_H(K)||C_K(H)|}{|H||K|}$.

2. If $\Pr_g(H, G) \neq 0$ then $\Pr_g(H, G) \geq \frac{2|H \cap Z(G)||C_G(H)|}{|H||G|}$.

3. If $\Pr_g(G) \neq 0$ then $\Pr_g(G) \geq \frac{3}{|G : Z(G)|^2}$.

The next two results are generalizations of Theorem 3.7 and Theorem 3.8.

**Theorem 5.8** ([4], Proposition 3.2). Let $G$ be a finite group and let $g \in G'$. If $H$ and $K$ are any two subgroups of $G$, then
\[
\Pr_g(H, K) \leq \Pr(H, K).
\]
Moreover, the equality holds if and only if $g = 1$.

**Theorem 5.9** ([4], Proposition 3.3). Let $G$ be a finite group and $g \in G'$, $g \neq 1$. Let $H$ and $K$ be any two subgroups of $G$. If $p$ is the smallest prime divisor of $|G|$, then we have
\[
\Pr_g(H, K) \leq \frac{|H| - |C_H(K)|}{p|H|} < \frac{1}{p}.
\]

The quantity $\Pr_g(H, K)$ is monotonic in the following sense.

**Theorem 5.10** ([4], Proposition 3.4). Let $G$ be a finite group. Let $H$, $K_1$, and $K_2$ be three subgroups of $G$ with $K_1 \subseteq K_2$. Then we have
\[
\Pr(H, K_1) \geq \Pr(H, K_2).
\]
Moreover, the equality holds if and only if $C_{\ell K_1}(x) = C_{\ell K_2}(x)$ for all $x \in H$.

**Theorem 5.11** ([4], Proposition 3.5). Let $G$ be a finite group. Let $H$, $K_1$, and $K_2$ be three subgroups of $G$ with $K_1 \subseteq K_2$. Then we have
\[
\Pr(H, K_2) \geq \frac{1}{|K_2 : K_1|} \left( \Pr(H, K_1) + \frac{|K_2| - |K_1|}{|H||K_1|} \right).
\]
Moreover, the equality holds if and only if $C_H(x) = \{1\}$ for all $x \in K_2 \setminus K_1$. 
Theorem 5.12 ([9], Proposition 3.6). Let $G$ be a finite group. Let $H_1 \subseteq H_2$ and $K_1 \subseteq K_2$ be subgroups of $G$ and $g \in G'$. Then we have
\[
\Pr_g(H_1, K_1) \leq |H_2 : H_1||K_2 : K_1|\Pr_g(H_2, K_2).
\]
Moreover, the equality holds if and only if
\[
g^{-1}x \notin \mathcal{C}_{K_2}(x) \text{ for all } x \in H_2 \setminus H_1,
\]
\[
g^{-1}x \notin \mathcal{C}_{K_2}(x) \setminus \mathcal{C}_{K_1}(x) \text{ for all } x \in H_1,
\]
and \(C_{K_1}(x) = C_{K_2}(x)\) for all \(x \in H_1\) with \(g^{-1}x \in \mathcal{C}_{K_1}(x)\).

In particular, for \(g = 1\), the condition for equality reduces to \(H_1 = H_2\) and \(K_1 = K_2\).

The following result also generalizes Theorem 3.7 in some sense.

Theorem 5.13 ([9], Corollary 3.7). Let $G$ be a finite group, let $H$ be a subgroup of $G$ and $g \in G'$. Then we have
\[
\Pr_g(H, G) \leq |G : H|\Pr(G).
\]
Moreover, the equality holds if and only if \(g = 1\) and $H = G$.

We continue the survey by mentioning a few generalizations of some results obtained in [14].

Theorem 5.14 ([9], Theorem 3.8). Let $G$ be a finite group and let $p$ be the smallest prime divisor of $|G|$. If $H$ and $K$ are any two subgroups of $G$, then we have
\[
\Pr(H, K) \geq \frac{|C_H(K)|}{|H|} + \frac{p(|H| - |X_H| - |C_H(K)|) + |X_H|}{|H||K|}
\]
and
\[
\Pr(H, K) \leq \frac{(p - 1)|C_H(K)| + |H|}{p|H|} - \frac{|X_H|(|K| - p)}{p|H||K|},
\]
where \(X_H = \{x \in H \mid C_K(x) = \{1\}\}\). Moreover, in each of these bounds, $H$ and $K$ can be interchanged.

Theorem 5.15 ([9], Corollary 3.9). Let $G$ be a finite group and let $p$ be the smallest prime divisor of $|G|$. If $H$ and $K$ are two subgroups of $G$ such that $[H, K] \neq \{1\}$, then we have
\[
\Pr(H, K) \leq \frac{2p - 1}{p^2}.
\]
In particular, we conclude that $\Pr(H, K) \leq 3/4$.

Theorem 5.16 ([9], Proposition 3.10). Let $G$ be a finite group and let $H$ and $K$ be any two subgroups of $G$. If $\Pr(H, K) = (2p - 1)/p^2$ for some prime $p$, then $p$ divides $|G|$. If $p$ happens to be the smallest prime divisor of $|G|$, then we have
\[
\frac{H}{C_H(K)} \cong C_p \cong \frac{K}{C_K(H)},
\]
and hence, $H \neq K$. 
In particular, we conclude that
\[
\frac{H}{C_H(K)} \cong C_2 \cong \frac{K}{C_K(H)} \text{ provided } \Pr(H, K) = \frac{3}{4}.
\]

We conclude the survey with the following result, which gives us a character theoretical formula.

**Theorem 5.17** ([4], Theorem 4.2 and Proposition 4.4). Let \( G \) be a finite group. If \( H \) is a normal subgroup of \( G \) and \( g \in G' \), then we have
\[
\Pr_g(H, G) = \frac{1}{|G|} \sum_{\chi \in \text{Irr}(G)} \left[ \chi_H, \chi_H \right] \chi(g).
\]

Consequently, we conclude that
\[
\left| \Pr_g(H, G) - \frac{1}{|G'|} \right| \leq |G : H| \left( \Pr(G) - \frac{1}{|G'|} \right).
\]

The above result yields, in particular, that if \( G \) is a finite group with \( |G'| \leq p^2 \), where \( p \) is the smallest prime divisor of \( |G| \), then every element of \( G' \) is a commutator.

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