Saturation numbers of bipartite graphs in random graphs

Meysam Miralaei[1,a] Ali Mohammadian[2,b,c] Behruz Tayfeh-Rezaie[1,b] Maksim Zhukovskii[3]

[1]School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746, Tehran, Iran

[2]School of Mathematical Sciences, Anhui University,
Hefei 230601, Anhui, China

[3]Department of Computer Science, University of Sheffield,
Sheffield S1 4DP, UK

m.miralaei@ipm.ir ali_m@ahu.edu.cn tayfeh-r@ipm.ir m.zhukovskii@sheffield.ac.uk

Abstract

For a given graph F, the F-saturation number of a graph G, denoted by $\mathrm{sat}(G, F)$, is the minimum number of edges in an edge-maximal F-free subgraph of G. In 2017, Korándi and Sudakov determined $\mathrm{sat}(\mathbb{G}(n, p), K_r)$ asymptotically, where $\mathbb{G}(n, p)$ denotes the Erdős–Rényi random graph and K_r is the complete graph on r vertices. In this paper, among other results, we present an asymptotic upper bound on $\mathrm{sat}(\mathbb{G}(n, p), F)$ for any bipartite graph F and also an asymptotic lower bound on $\mathrm{sat}(\mathbb{G}(n, p), F)$ for any complete bipartite graph F.

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1. Introduction

All graphs in this paper are assumed to be finite, undirected, and without loops or multiple edges. Fix a graph F. We say that a graph G is *F-free* if G has no subgraph isomorphic to F. Turán [17] posed one of the foundational problems in extremal graph theory in 1941 which was about the maximum number of edges in an F-free graph on n vertices. Later, Zykov [18] introduced a dual idea in 1949

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which asks for the minimum number of edges in an edge-maximal F-free graph on n vertices. In this paper, we deal with a random version of this concept. We below define the concept in a more general form.

Let G be a graph. The edge set of G is denoted by $E(G)$. A spanning subgraph H of G is said to be an *F-saturated subgraph of G* if H is F-free and the addition of any edge from $E(G) \setminus E(H)$ to H creates a copy of F. The minimum number of edges in an F-saturated subgraph of G is denoted by $\mathrm{sat}(G, F)$. Let K_r be the complete graph on r vertices and $K_{s,t}$ be the complete bipartite graph with parts of sizes s and t. Usually, $\mathrm{sat}(K_n, F)$ is written as $\mathrm{sat}(n, F)$. Erdős, Hajnal, and Moon [8] proved that

$$\mathrm{sat}(n, K_r) = (r-2)n - \binom{r-1}{2},$$

where $n \geqslant r \geqslant 2$. Also, with the assumption $t \geqslant s$, Bohman, Fonoberova, and Pikhurko [3] proved that

$$\mathrm{sat}(n, K_{s,t}) = \frac{2s+t-3}{2}n + O\left(n^{\frac{3}{4}}\right).$$

We refer the reader to the survey [9] for more known results on saturation in graphs.

Recall that the Erdős–Rényi random graph model $\mathbb{G}(n,p)$ is the probability space of all graphs on a fixed vertex set of size n where every two distinct vertices are adjacent independently with probability p. Throughout this paper, p is assumed to be a fixed real number in $(0,1)$. Recall that the notion 'with high probability', which is written as '*whp*' for brevity, is used whenever an event occurs in $\mathbb{G}(n,p)$ with a probability approaching 1 as $n \to \infty$. The study of saturation numbers in random graphs was initiated in 2017 by Korándi and Sudakov [15]. They proved that *whp*

$$\mathrm{sat}\big(\mathbb{G}(n,p), K_r\big) = \big(1+o(1)\big)n \log_{\frac{1}{1-p}} n$$

for any fixed $r \geqslant 3$. Mohammadian and Tayfeh-Rezaie [16] studied the saturation numbers for stars and found that *whp*

$$\mathrm{sat}\big(\mathbb{G}(n,p), K_{1,t}\big) = \frac{t-1}{2}n - \big(t-1+o(1)\big) \log_{\frac{1}{1-p}} n$$

for any fixed $t \geqslant 2$. Their result was refined by Demyanov and Zhukovskii in [6] where it has been proved that *whp* $\mathrm{sat}\big(\mathbb{G}(n,p), K_{1,t}\big)$ is concentrated in a set of two points. The related classical result had been proved by Kászonyi and Tuza [14] as

$$\mathrm{sat}(n, K_{1,t}) = \begin{cases} \binom{t}{2} + \binom{n-t}{2} & \text{if } t+1 \leqslant n \leqslant \frac{3t}{2}, \\[2mm] \left\lceil \frac{t-1}{2}n - \frac{t^2}{8} \right\rceil & \text{if } n \geqslant \frac{3t}{2}. \end{cases}$$

Demidovich, Skorkin, and Zhukovskii [5] proved that *whp*

$$\mathrm{sat}\big(\mathbb{G}(n,p), C_k\big) = n + \Theta\left(\frac{n}{\log n}\right)$$

for any $k \geqslant 5$, where C_k is a cycle graph on k vertices, while

$$\left(\frac{3}{2}+o(1)\right)n \leqslant \mathrm{sat}\big(\mathbb{G}(n,p), C_4\big) \leqslant \big(c_p + o(1)\big)n$$

for some explicit constant c_p. In particular, $c_{1/2} = 27/14$.

The exact values of both $\mathrm{sat}(n, K_{s,t})$ and $\mathrm{sat}(\mathbb{G}(n,p), K_{s,t})$ are still unknown. Note that, for any connected graph F with no cut edges, both $\mathrm{sat}(n, F)$ and $\mathrm{sat}(\mathbb{G}(n,p), F)$ are at least $n-1$, since each F-saturated subgraph should be connected. Therefore, in particular, *whp* $\mathrm{sat}(\mathbb{G}(n,p), K_{s,t}) \geqslant n-1$ if $t \geqslant s \geqslant 2$. Diskin, Hoshen, and Zhukovskii [7] showed that, for any bipartite graph F, there exists a constant c_F such that $\mathrm{sat}(\mathbb{G}(n,p), F) \leqslant c_F n$ *whp*. However, an explicit value of c_F was not known. In this paper, we prove the following theorem for any arbitrary bipartite graph.

Theorem 1.1. *Let $p \in (0,1)$ be constant and let F be a bipartite graph with no isolated vertices. Let $\{A_1, B_1\}, \ldots, \{A_k, B_k\}$ be the vertex bipartitions of all the connected components of F with $|B_i| \geqslant |A_i|$ for every i. Let $a = \max\{|A_1|, \ldots, |A_k|\}$ and δ be the minimum degree over all vertices from A_i with $|A_i| = a$. Then, whp*

$$\mathrm{sat}\big(\mathbb{G}(n,p), F\big) \leqslant \left(\frac{\delta-1}{p^{a-1}} - \frac{\delta - 2a + 1}{2} + o(1)\right) n.$$

Our proof of Theorem 1.1, which is presented in Section 4, is based on the construction suggested in [7]. Actually, we have tuned the parameters of the construction in order to achieve the optimal bound. For $F = K_{s,t}$ with $t \geqslant s$, Theorem 1.1 shows that *whp*

$$\mathrm{sat}(\mathbb{G}(n,p), K_{s,t}) \leqslant \left(\frac{t-1}{p^{s-1}} - \frac{t-2s+1}{2} + o(1)\right) n.$$

For a lower bound on $\mathrm{sat}(\mathbb{G}(n,p), K_{s,t})$, we prove the following theorem.

Theorem 1.2. *Let $t \geqslant s \geqslant 2$ be fixed integers and let $p \in (0,1)$ be constant. Then, whp*

$$\mathrm{sat}(\mathbb{G}(n,p), K_{s,t}) \geqslant \left(\max\left\{\frac{2s+t-3}{2}, \frac{t-s}{4p^{s-1}} + \frac{s-1}{2}\right\} + o(1)\right) n.$$

The proof of the lower bound in Theorem 1.2 is the most involved part of the paper. It is presented in Section 5. For every fixed $t > s$, our bounds in Theorem 1.2 imply that the $K_{s,t}$-saturation number in $\mathbb{G}(n,p)$ is $\Theta(p^{1-s}n)$. Let us also note that, in the case $s = t = 2$, Theorem 1.2 provides the lower bound obtained in [5], while our upper bound is slightly worse.

As we saw above, *whp* $\mathrm{sat}(\mathbb{G}(n,p), K_r) \gg \mathrm{sat}(n, K_r)$ for any $r \geqslant 3$. For complete bipartite graphs, the saturation number is more stable, that is, $\mathrm{sat}(\mathbb{G}(n,p), K_{s,t})$ is linear in n *whp* as well as $\mathrm{sat}(n, K_{s,t})$. For $t > s \geqslant 2$ and sufficiently small $p \in (0,1)$, there is no asymptotical stability, that is, there exists a constant $c > 1$ such that $\mathrm{sat}(\mathbb{G}(n,p), K_{s,t}) \geqslant c\,\mathrm{sat}(n, K_{s,t})$ *whp*. However, for $s = t$ or $t > s \geqslant 2$ and sufficiently large $p \in (0,1)$, we do not know whether there is an asymptotical stability. Finally, the $K_{1,t}$-saturation number is asymptotically stable, while $\mathrm{sat}(\mathbb{G}(n,p), K_{1,t}) < \mathrm{sat}(n, K_{1,t})$ *whp*. Note that, for cycles, *whp* $\mathrm{sat}(\mathbb{G}(n,p), C_k) \leqslant ((k+2)/(k+3) + o(1))\mathrm{sat}(n, C_k)$ for any $k \geqslant 5$ by a result of Füredi and Kim [11].

For the sake of completeness, we give in Section 3 two simple general lower bounds on $\mathrm{sat}(\mathbb{G}(n,p), F)$ for any arbitrary graph F which are asymptotically tight for certain graph families.

2. Notation and preliminaries

In this section, we introduce notation and formulate several properties of random graphs that will be used in the rest of the paper. First, let us fix some more notation and terminology of graph theory.

Let G be a graph. The vertex set of G is denoted by $V(G)$ and the *order* of G is defined as $|V(G)|$. For a subset X of $V(G)$, we denote the induced subgraph of G on X by $G[X]$. For a subset Y of $E(G)$, we denote by $G-Y$ the graph obtained from G by removing the edges in Y. For a subset Z of $V(G)$, set $N_G(Z)=\{v\in V(G)\,|\,v \text{ is adjacent to all vertices in } Z\}$. For the sake of convenience, we write $N_G(z_1,\ldots,z_k)$ instead of $N_G(\{z_1,\ldots,z_k\})$. For a vertex v of G, we define the *degree* of v as $|N_G(v)|$ and denote by $d_G(v)$. The maximum and the minimum degree of vertices of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For two subsets S and T of $V(G)$, we denote by $E_G(S,T)$ the set of all edges with endpoints in both S and T. We write $E_G(S)$ for $E_G(S,S)$. We drop subscripts if there is no danger of confusion.

In what follows, we recall the probabilistic results that we make use of all in the next sections. The next lemma is well known and can be deduced from the Chernoff bound [12, Theorem 2.1].

Lemma 2.1. *Let $X\sim \mathrm{Bin}(n,p)$ be a binomial random variable with parameters n and p. If $\mathbb{E}[X]\to\infty$ as $n\to\infty$, then $X=\mathbb{E}[X](1+o(1))$ whp.*

The following lemma is a consequence of Proposition 19 in [1].

Lemma 2.2. *For any constant $p\in(0,1)$, there is a constant c, depending on p, such that $\mathbb{G}(n,p)$ has the following property whp. For every subset X of vertices with $|X|\geqslant c\log n$, the number of vertices with no neighbors in X is at most $c\log n$.*

The following Lemma is an immediate consequence of the Chernoff bound [12, Theorem 2.1] and the union bound.

Lemma 2.3. *Let $\lambda>1$ and $p\in(0,1)$ be constants. Then, $\mathbb{G}(n,p)$ has the following property whp. For every two disjoint subsets X,Y of vertices of size at least $\log^\lambda n$, we have $|E(X)|=p\binom{|X|}{2}(1+o(1))$ and $|E(X,Y)|=p|X||Y|(1+o(1))$.*

The following corollary follows from Lemma 2.3 immediately.

Corollary 2.4. *Let $\lambda>1$ and $p\in(0,1)$ be constants. Then, $\mathbb{G}(n,p)$ has the following property whp. For every two subsets X,Y of vertices, $|E(X,Y)|\leqslant 3n\log^\lambda n+p|X||Y|(1+o(1))$.*

Note that, for a positive fixed integer M, the probability that $\mathbb{G}(n,p)$ does not contain a clique of size M is $\exp(-\Theta(n^2))$ due to the Janson bound [12, Theorem 2.14]. Therefore, by the union bound, we get the following.

Lemma 2.5. *Let $\lambda>1$, $p\in(0,1)$ be constants and let M be a positive fixed integer. Then, $\mathbb{G}(n,p)$ has the following property whp. Every subset X of vertices of size at least $\log^\lambda n$ contains a clique of size M.*

3. General lower bounds

In this section, we prove a lower bound on $\mathrm{sat}(G,F)$ for every two graphs G and F which provides a lower bound on $\mathrm{sat}(\mathbb{G}(n,p),F)$. It is trivial that

$$\mathrm{sat}(G,F)\geqslant \frac{\min\{\delta(G),\delta(F)-1\}}{2}n$$

for any two graphs G and F. In order to proceed, we need the following definition.

Let G be a graph and k be a nonnegative integer. A subset S of $V(G)$ is called *k-independent* if the maximum degree of $G[S]$ is at most k. The *k-independence number* of G, denoted by $\alpha_k(G)$, is defined as the maximum cardinality of a k-independent set in G. In particular, $\alpha_0(G) = \alpha(G)$ is the usual independence number of G. Furthermore, define $r(G) = \min_{xy \in E(G)} \max\{d(x), d(y)\}$.

Theorem 3.1. *Let F be a graph and let $r = r(F)$. If $r \geqslant 2$, then, for every graph G on n vertices,*

$$\mathrm{sat}(G, F) \geqslant \frac{(r-1)\big(n - \alpha_{r-2}(G)\big)}{2}.$$

Proof. Let H be an F-saturated subgraph of G. Let $r \geqslant 2$ and let A be the set of vertices of H with degree at most $r-2$ in H. Suppose that there are two vertices $x, y \in A$ with $xy \in E(G) \setminus E(H)$. By definition of r, adding xy to H does not create a copy of F. This is a contradiction, since H is an F-saturated subgraph of G. This implies that $G[A] = H[A]$ and so $|A| \leqslant \alpha_{r-2}(G)$. We hence obtain that

$$|E(H)| \geqslant \frac{\sum\limits_{v \in V(H) \setminus A} d_H(v)}{2} \geqslant \frac{(r-1)\big(n - \alpha_{r-2}(G)\big)}{2}. \qquad \square$$

Theorem 3.2 ([10, 16])**.** *For every constants $p \in (0, 1)$ and $k \geqslant 0$, whp*

$$\alpha_k\big(\mathbb{G}(n,p)\big) = \big(2 + o(1)\big) \log_{\frac{1}{1-p}} n.$$

Actually, we known from [13] that $\alpha_k(\mathbb{G}(n,p))$ is concentrated in a set of two consecutive points *whp*. Using Theorems 3.1 and 3.2, we conclude the following.

Corollary 3.3. *Let F be a graph and let $r = r(F)$. Then, for each fixed real number $p \in (0, 1)$, whp*

$$\mathrm{sat}\big(\mathbb{G}(n,p), F\big) \geqslant \frac{r-1}{2} n - \big(r - 1 + o(1)\big) \log_{\frac{1}{1-p}} n.$$

For $F = K_{1,t}$, the lower bound given in Corollary 3.3 is tight by a result in [16]. However, for graphs F satisfying the property that each edge $uv \in E(F)$ with $\max\{d(u), d(v)\} = r(F)$ is contained in a triangle, the lower bound can be significantly improved.

For any graph G, define $w(G) = \min_{xy \in E(G)} \{\max\{d(x), d(y)\} + |N(x) \cap N(y)|\}$. Cameron and Puleo [4] proved that

$$\mathrm{sat}(n, F) \geqslant \frac{w(F) - 1}{2} n - \frac{w(F)^2 - 4w(F) + 5}{2}$$

for any n. Below, we give a lower bound on $\mathrm{sat}(\mathbb{G}(n,p), F)$ in terms of $w(F)$ which is asymptotically stronger than Corollary 3.3 for many graphs F.

Theorem 3.4. *For any constant $p \in (0, 1)$ and any graph F, whp*

$$\mathrm{sat}\big(\mathbb{G}(n,p), F\big) \geqslant \frac{w(F) - 1}{2} n - O(\log n).$$

Proof. If $w(F) = 1$, then there is nothing to prove. So, assume that $w(F) \geqslant 2$. Let $G \sim \mathbb{G}(n,p)$ and $\ell = c \log n$, where c is given in Lemma 2.2. Assume that H is an arbitrary F-saturated subgraph of G whose vertices are labeled as $u_1, \ldots, u_n$ for which $d_H(u_1) \leqslant \cdots \leqslant d_H(u_n)$. Let $U = \{u_1, \ldots, u_\ell\}$.

For $i=1,\dots,\ell$, let $V_i=N_H(u_i)$ and $V=\bigcup_{i=1}^{\ell}V_i$. Also, for $i=1,\dots,\ell$, define $W_i=N_G(u_i)\setminus(U\cup V\cup W_1\cup\cdots\cup W_{i-1})$ and set $W=\bigcup_{i=1}^{\ell}W_i$. If $d_H(u_\ell)\geqslant w(F)-1$, then

$$|E(H)|\geqslant\frac{\sum\limits_{i=\ell+1}^{n}d_H(u_i)}{2}\geqslant\frac{\big(w(F)-1\big)(n-\ell)}{2}$$

which concludes the assertion. So, we may assume that $d_H(u_\ell)\leqslant w(F)-2$. Then, $|V|\leqslant\sum_{i=1}^{\ell}d_H(u_i)\leqslant\ell(w(F)-2)$. Let $R=V(G)\setminus(U\cup V\cup W)$. Note that R is the set of all vertices in $V(G)\setminus U$ which are not adjacent to any vertex in U and so $|R|\leqslant c\log n$ by Lemma 2.2. Let $x\in W_i$ and let F' be a copy of F in $H+xu_i$. It follows from $d_H(x)\geqslant d_H(u_i)$ that $d_H(x)\geqslant\max\{d_{F'}(x),d_{F'}(u_i)\}-1$. Since $N_H(x)\cap V\supseteq N_{F'}(x)\cap N_{F'}(u_i)$, one concludes that

$$d_H(x)+|N_H(x)\cap V|\geqslant\max\big\{d_{F'}(x),d_{F'}(u_i)\big\}-1+|N_{F'}(x)\cap N_{F'}(u_i)|\geqslant w(F)-1.$$

Now, we may write

$$\begin{aligned}
2|E(H)|&\geqslant\sum_{x\in V}d_H(x)+\sum_{x\in W}d_H(x)\\
&\geqslant\sum_{x\in V}|N_H(x)\cap U|+\sum_{x\in V}|N_H(x)\cap W|+\sum_{x\in W}d_H(x)\\
&\geqslant|V|+\sum_{x\in W}|N_H(x)\cap V|+\sum_{x\in W}d_H(x)\\
&=|V|+\sum_{x\in W}\big(d_H(x)+|N_H(x)\cap V|\big)\\
&\geqslant|V|+\sum_{x\in W}\big(w(F)-1\big)\\
&=|V|+\big(w(F)-1\big)\big(n-|U|-|V|-|R|\big)\\
&=\big(w(F)-1\big)n-\big(w(F)-2\big)|V|-\big(w(F)-1\big)\big(\ell+|R|\big)
\end{aligned}\tag{1}$$

Since $\ell=c\log n$, $|V|\leqslant\ell(w(F)-2)$, and $|R|\leqslant c\log n$, the result follows from (1). □

4. Upper bound for bipartite graphs

In this section, we prove Theorem 1.1. Our proof is based on the construction suggested in [7] which, in turn, resembles the proof strategy of a general linear in n upper bound on $\mathrm{sat}(n,F)$ from [14]. First, we present a useful observation which can be proved straightforwardly.

Observation 4.1. *Let H be an F-free subgraph of G. Then, there is an F-saturated subgraph of G which has H as a subgraph.*

Below, we show how a general linear in n upper bound on $\mathrm{sat}(n,F)$ can be derived from Observation 4.1. While we use the same construction as in [14], we formulate the proof in a different way in order to make the move to random settings smoother.

Theorem 4.2 ([14])**.** *Let F be a graph and S be an independent set in F with maximum possible size. Let $b=|V(F)|-|S|-1$ and $d=\min\{|N_F(x)\cap S|\,|\,x\in V(F)\setminus S\}$. Then,*

$$\mathrm{sat}(n,F)\leqslant\frac{2b+d-1}{2}n-\frac{b(b+d)}{2}.$$

Proof. Let B be a subset of $V(K_n)$ of size b and let $\overline{B} = V(K_n) \setminus B$. Consider the spanning subgraph H_0 of K_n obtained by deleting all edges whose both endpoints are in $\overline{B}$. If there is a copy F' of F in H_0, then $V(F') \cap \overline{B}$ is an independent set of size

$$|V(F') \cap \overline{B}| = |V(F')| - |V(F') \cap B| \geqslant |V(F)| - |B| = |S| + 1,$$

a contradiction. This shows that H_0 is F-free. Using Observation 4.1, there is an F-saturated subgraph of G, say H, with $E(H) \supseteq E(H_0)$. For every $x \in \overline{B}$, we have $|N_H(x) \cap \overline{B}| \leqslant d - 1$, as otherwise the subgraph of H with the edge set $E(H_0) \cup E_H(\{x\}, \overline{B})$ contains a copy of F. Since

$$|E(H)| = |E(H_0)| + \sum_{x \in \overline{B}} |N_H(x) \cap \overline{B}| \leqslant |E(H_0)| + \frac{(d-1)|\overline{B}|}{2},$$

the result follows. $\square$

Let us now prove Theorem 1.1. Note that it is impossible to find a construction as in the proof of Theorem 4.2, since vertex degrees in the random graph equal $np(1+o(1))$. Thus, instead of considering a single clique B with its common neighborhood, we will consider $\Theta(\ln n)$ disjoint sets of constant sizes as well as their common neighborhoods. For the sake of convenience, we handle the case of F being a disjoint union of stars separately. This proves Theorem 1.1 for the case $a = 1$ and generalizes a result given in [16].

Lemma 4.3. *Let $p \in (0,1)$ be constant and let F be the disjoint union of stars $K_{1,t_1}, \dots, K_{1,t_k}$ with $k \geqslant 1$ and $t_1 \geqslant \cdots \geqslant t_k \geqslant 1$. Then, whp*

$$\operatorname{sat}\big(\mathbb{G}(n,p), F\big) = \frac{t_k - 1}{2} n - \big(t_k - 1 + o(1)\big) \log_{\frac{1}{1-p}} n.$$

Proof. In view of Corollary 3.3, it suffices to prove the upper bound. Using Theorem 3.2, $\alpha(\mathbb{G}(n,p)) = (2 + o(1)) \log_{1/(1-p)} n$ *whp*. Let $G \sim \mathbb{G}(n,p)$ and $h = |V(F)| - 1$. Fix an integer-valued function $\ell = \ell(n) = (2 + o(1)) \log_{1/(1-p)} n$ such that $(n - h - \ell)(t_k - 1)$ is even and $\alpha(\mathbb{G}(n,p)) \geqslant \ell$ *whp*. Also, let L be the disjoint union of K_h and an arbitrary regular graph on $n - h - \ell$ vertices with degree $t_k - 1$. We know from a result of Alon and Füredi [2] that, for sufficiently small $\varepsilon > 0$, the graph $\mathbb{G}(n - \ell, n^{-\varepsilon})$ contains a copy of L *whp*. Using the standard multiple-exposure technique, it implies that $\mathbb{G}(n - \ell, p)$ does not contain a copy of L with probability at most $\exp(n^{-\varepsilon + o(1)})$. Thus, by the union bound, *whp* there exists a subset $S \subseteq V(G)$ with $|S| = \ell$ such that S is an independent set in G and $G[V(G) \setminus S]$ has a copy L' of L as a subgraph. Denote by H the spanning subgraph of G with the edge set $E(L')$. It is easily seen that H is an F-saturated subgraph of G and

$$|E(H)| = \frac{(n - h - \ell)(t_k - 1)}{2} + \binom{h}{2} = \frac{t_k - 1}{2} n - \big(t_k - 1 + o(1)\big) \log_{\frac{1}{1-p}} n$$

which concludes the result. $\square$

Remark 4.4. Note that Lemma 4.3 for $t_k = 1$ could be strengthened as follows. If F is a graph with a connected component K_2, then $\operatorname{sat}(\mathbb{G}(n,p), F) \leqslant \binom{|V(F)|-1}{2}$ *whp*. Conversely, if $\operatorname{sat}(\mathbb{G}(n,p), F)$ is bounded from above by a constant, then Corollary 3.3 forces F to have a connected component K_2.

Proof of Theorem 1.1. In view of Lemma 4.3, we may assume that $a \geqslant 2$. Let $G \sim \mathbb{G}(n,p)$, $b = 1 - p^{a-1}$, and $\ell = \lfloor \log_{1/b} n^{2/3} \rfloor$. Without loss of generality, assume that $|A_1| = \cdots = |A_q| > |A_{q+1}| \geqslant \cdots \geqslant |A_k|$ for some q. Fix disjoint arbitrary $(a-1)$-subsets $V_1, \ldots, V_\ell$ and $(a+1)$-subsets $V_{\ell+1}, \ldots, V_{\ell+q-1}$ of $V(G)$. Set $V = \bigcup_{i=1}^{\ell} V_i$ and $V' = \bigcup_{i=\ell+1}^{\ell+q-1} V_i$. Let $M_i = \bigcup_{j=1}^{i} N(V_j)$ for any $i \geqslant 1$. For $i = 1, \ldots, \ell+q-1$, define $W_i = N(V_i) \setminus (V \cup V' \cup M_{i-1})$ and set $W = \bigcup_{i=1}^{\ell} W_i$. Let $R = V(G) \setminus (V \cup W)$. Note that $R = (V(G) \setminus (V \cup M_\ell)) \cup V'$. Set $V'' = V' \cap M_\ell$. A schematic of the structure of $V(G)$ is illustrated in Figure 1.

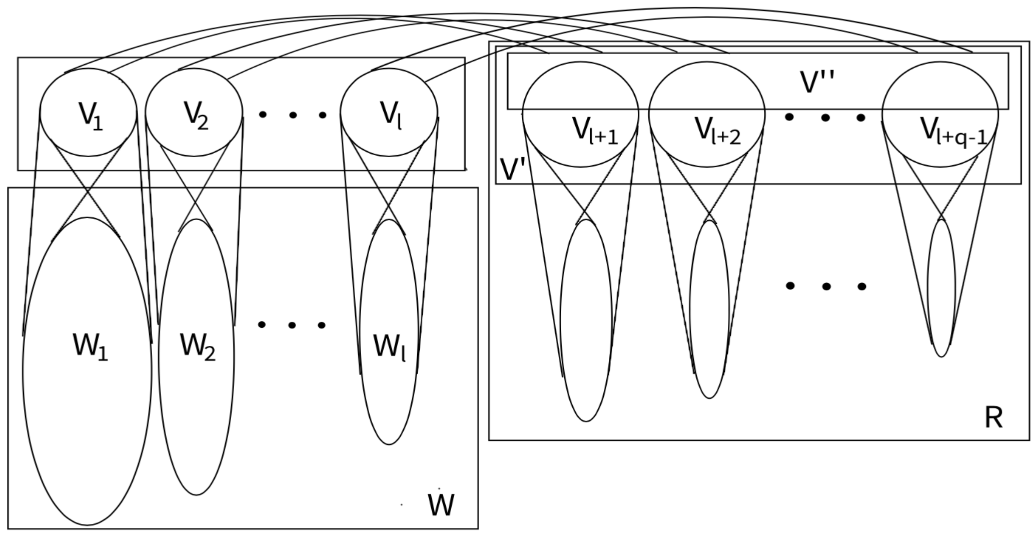

Figure 1. The structure of $V(G)$ described in the proof of Theorem 1.1.

As $|R \setminus V''| \sim \mathrm{Bin}(n - \ell(a-1), b^\ell)$, Lemma 2.1 implies that *whp*

$$|R| = b^\ell\big(n - \ell(a-1)\big)\big(1 + o(1)\big) + |V''|$$

which gives that $|R| = O(n^{1/3})$. Similarly, for all i, *whp*

$$|W_i| = b^{i-1}(1-b)\big(n - \ell(a-1) - (q-1)(a+1)\big)\big(1 + o(1)\big)$$

which yields that $|W_i| = \Omega(n^{1/3})$ for $i = 1, \ldots, \ell+q-1$. In particular, $|W_i| \geqslant \max\{|B_1|, \ldots, |B_k|\} + 1$.

Let H_0 be a spanning subgraph of G with $E(H_0) = \cup_{i=1}^{\ell+q-1} E_G(V_i, W_i)$. By the definition of a, we conclude that H_0 is F-free. Using Observation 4.1, there is an F-saturated subgraph of G, say H, with $E(H) \supseteq E(H_0)$. Now, we bound the number of edges of H. We will use

$$|E(H)| = |E_H(V(G) \setminus W)| + |E_H(V(G) \setminus W, W)| + |E_H(W)|. \tag{2}$$

It follows from $V(G) \setminus W = R \cup V$ that $|V(G) \setminus W| = O(n^{1/3})$ and hence $|E_H(V(G) \setminus W)| = O(n^{2/3})$. For every $i \in \{1, \ldots, \ell\}$ and every $x \in V(G) \setminus V_i$, we have $|N_H(x) \cap W_i| \leqslant \delta - 1$, as otherwise the bipartite subgraph of H with the edge set $E(H_0) \cup E_H(\{x\}, W_i)$ contains a copy of F. Therefore,

$$|E_H(V(G) \setminus W, W)| = \sum_{i=1}^{\ell} |E_H(V(G) \setminus W, W_i)|$$

$$
\begin{aligned}
&= \sum_{i=1}^{\ell} |E_H(V(G) \setminus (V_i \cup W), W_i)| + \sum_{i=1}^{\ell} |E_H(V_i, W_i)| \\
&\leqslant \ell(\delta - 1)|V(G) \setminus W| + \sum_{i=1}^{\ell} |E_H(W_i, V_i)| \\
&\leqslant O\left(n^{\frac{1}{3}} \log n\right) + (a-1)n. \qquad (3)
\end{aligned}
$$

It remains to estimate $|E_H(W)|$. To do this, we write

$$
\begin{aligned}
|E_H(W)| &= \sum_{i=1}^{\ell} \sum_{j=1}^{i-1} |E_H(W_i, W_j)| + \sum_{i=1}^{\ell} |E_H(W_i)| \\
&\leqslant \sum_{i=1}^{\ell} (i-1)(\delta-1)|W_i| + \sum_{i=1}^{\ell} \frac{\delta-1}{2}|W_i| \\
&= \frac{\delta-1}{2} \sum_{i=1}^{\ell} (2i-1)|W_i| \\
&\leqslant \frac{\delta-1}{2} \sum_{i=1}^{\ell} (2i-1) b^{i-1}(1-b)n\big(1+o(1)\big) \\
&\leqslant \frac{\delta-1}{2}(1-b)n\big(1+o(1)\big) \sum_{i=1}^{\ell} (2i-1) b^{i-1} \\
&= \frac{\delta-1}{2}(1-b)n\big(1+o(1)\big) \frac{1+b-(2\ell+1)b^{\ell}+(2\ell-1)b^{\ell+1}}{(1-b)^2} \\
&\leqslant \frac{\delta-1}{2} \left(\frac{1+b}{1-b}\right) n\big(1+o(1)\big). \qquad (4)
\end{aligned}
$$

By (2)–(4), we conclude that

$$
\begin{aligned}
|E(H)| &\leqslant \left(\frac{\delta-1}{2}\left(\frac{1+b}{1-b}\right) + a - 1\right) n\big(1+o(1)\big) \\
&= \left(\frac{\delta-1}{p^{a-1}} - \frac{\delta-2a+1}{2} + o(1)\right) n.
\end{aligned}
$$

Since $\mathrm{sat}(G, F) \leqslant |E(H)|$, the result follows. □

5. Lower bound for $K_{s,t}$

In this section, we prove the two lower bounds in Theorem 1.2. We start from the bound that does not depend on p that is stated separately below. Let us recall that this bound generalizes the lower bound from [5] for $F = K_{2,2}$, that is, $\mathrm{sat}(\mathbb{G}(n,p), K_{2,2}) \geqslant (\frac{3}{2}+o(1))n$ *whp*. However, our argument is simpler and resembles the argument used by Bohman, Fonoberova, and Pikhurko [3] for their asymptotic lower bound on $\mathrm{sat}(n, K_{s,t})$.

Theorem 5.1. *Let $t \geqslant s \geqslant 2$ and $p \in (0,1)$ be constants. Then, whp*

$$
\mathrm{sat}\big(\mathbb{G}(n,p), K_{s,t}\big) \geqslant \left(\frac{2s+t-3}{2} + o(1)\right) n.
$$

Proof. Let $G \sim \mathbb{G}(n,p)$ and let H be a $K_{s,t}$-saturated subgraph of G with minimum possible number of edges. Let $V = V(G)$. By Theorem 1.1, we have that $|E(H)| = O(n)$ *whp*. For the subsets

$$\begin{aligned} A &= \left\{ x \in V \;\middle|\; d_H(x) \geqslant n^{\frac{1}{4}} \right\}, \\ B &= \left\{ x \in V \setminus A \,\middle|\, |N_H(x) \cap A| \leqslant s-2 \right\}, \\ C &= \left\{ x \in V \,\middle|\, d_H(x) \leqslant s+t-3 \right\}, \\ D &= V \setminus (A \cup B \cup C). \end{aligned}$$

of $V(H)$, we prove the following claims.

Claim 5.2. *Whp* $|A| = O(n^{3/4})$.

Proof. Since $|E(H)| \geqslant |A| n^{1/4}/2$, we have $|A| = O(n^{3/4})$. ◇

Claim 5.3. *Whp* $|B| = O(n^{3/4})$.

Proof. Take any two vertices $x, y \in B$ such that $\{x, y\} \in E(G) \setminus E(H)$. The addition of xy to H creates a copy of $K_{s,t}$ with vertex bipartition $\{X, Y\}$ so that $x \in X$ and $y \in Y$. Since $|N_H(x) \cap A| \leqslant s-2$, x has a neighbor $y' \in Y \setminus A$. Similarly, y has a neighbor $x' \in X \setminus A$. This shows that there is a path x, y', x', y of length three which connects x to y in $H - A$. Therefore, every two vertices in B which are adjacent in G are connected in $H - A$ by a path of length one or three. If $|B| \leqslant n^{3/4}$, then we are done. Otherwise, by Lemma 2.3, we have

$$\frac{p}{2}\binom{|B|}{2} \leqslant |E(G[B])| \leqslant |E(H[B])| + |B| \left(n^{\frac{1}{4}} n^{\frac{1}{4}} n^{\frac{1}{4}} \right) \leqslant \frac{|B| n^{\frac{1}{4}}}{2} + |B| n^{\frac{3}{4}}$$

which gives $|B| = O(n^{3/4})$. ◇

Claim 5.4. $|C| \leqslant \log^2 n$.

Proof. By contradiction, assume that $|C| > \log^2 n$. Recall that the Ramsey number $R_{s+t-3}(s+t)$ is the smallest positive integer m such that any coloring of the edges of K_m with $s+t-3$ colors gives a monochromatic copy of K_{s+t}. Using Lemma 2.5, C contains a clique C' of size $M = (s+t-2)R_{s+t-3}(s+t)$ in G. We know that every graph $\varGamma$ contains an independent set of size at least $|V(\varGamma)|/(\varDelta(\varGamma)+1)$. Since each vertex of C' has degree at most $s+t-3$ in H, there is an independent set $C'' \subseteq C'$ with $|C''| \geqslant R_{s+t-3}(s+t)$ in H. For each vertex $x \in C''$, fix an arbitrary ordering of $N_H(x)$ which we encode by a bijection $f_x : N(x) \to \{1, \ldots, d_H(x)\}$. For each pair of distinct vertices $x, y \in C''$ do the following. Fix a copy of $K_{s,t}$ in $H + xy$ with partition $\{X, Y\}$ so that $x \in X$ and $y \in Y$. Since $|(X \setminus \{x\}) \cup (Y \setminus \{y\})| = s+t-2$, there are $x' \in X \setminus \{x\}$ and $y' \in Y \setminus \{y\}$ with $f_x(x') = f_y(y')$. Denote the integer $f_x(x') = f_y(y')$ by a. Clearly, $x'y' \in E(H)$. Now, color the edge xy by a. This defines an edge coloring of $E(G[C''])$ with $s+t-3$ colors. By Ramsey's theorem, there is a $(s+t)$-subset $C''' \subseteq C''$ such that all edges of $G[C''']$ have the same color, say c. For every two distinct vertices $x, y \in C'''$, as $f_x^{-1}(c)$ and $f_y^{-1}(c)$ are adjacent in H, $f_x^{-1}(c) \neq f_y^{-1}(c)$. So $\{f_x^{-1}(c) | x \in C'''\}$ is a clique of order $s+t$ in H which contradicts the $K_{s,t}$-freeness of H, proving the claim. ◇

Using Claims 5.2–5.4, we conclude that $|D| = n - O(n^{3/4})$. Since every vertex in D has at least $s-1$ neighbors in A, we may choose $s-1$ distinct edges for each vertex of D. Put all these edges in a set E_1. Since any vertex in D has at least $s+t-2$ neighbors in H, we conclude that every vertex in D is incident to at least $t-1$ edges in $E(H)\setminus E_1$. Now, we have

$$|E(H)| \geqslant |E(D,V(H))| \geqslant (s-1)|D| + \frac{t-1}{2}|D| \geqslant \left(\frac{2s+t-3}{2} + o(1)\right) n. \qquad \square$$

The second lower bound in Theorem 1.2 is stated below.

Theorem 5.5. *Let $t \geqslant s \geqslant 2$ and $p \in (0,1)$ be constants. Then, whp*

$$\operatorname{sat}\big(\mathbb{G}(n,p), K_{s,t}\big) \geqslant \left(\frac{t-s}{4p^{s-1}} + \frac{s-1}{2} + o(1)\right) n.$$

Proof. If $s=t$, then the assertion follows from Corollary 3.3. So, assume that $t>s$. Let $G \sim \mathbb{G}(n,p)$ and let H be a $K_{s,t}$-saturated subgraph of G with minimum possible number of edges. Let $V = V(G)$. By Theorem 1.1, we have $|E(H)| = O(n)$. Consider the partition $\{A,B,C\}$ of V, where

$$\begin{aligned} A &= \big\{x \in V \,\big|\, d_H(x) < \log n\big\},\\ B &= \left\{x \in V \,\middle|\, \log n \leqslant d_H(x) \leqslant \frac{n}{\log^{s+1} n}\right\},\\ C &= \left\{x \in V \,\middle|\, d_H(x) > \frac{n}{\log^{s+1} n}\right\}. \end{aligned}$$

For any $y \in V$, set $N_y = N_H(y)$ and

$$F_y = \big\{x \in V \,\big|\, |N_H(x,y)| \geqslant t-1\big\}.$$

Moreover, let

$$\mathcal{O} = \big\{Y \subseteq V \,\big|\, |Y| = s-1 \text{ and } |N_H(Y)| \geqslant t\big\}.$$

Further, for any $Y \in \mathcal{O}$, set $N_Y = N_H(Y)$ and

$$F_Y = \big\{x \in V \,\big|\, |N_H(\{x\} \cup Y)| = t-1\big\}.$$

Finally, consider the partition $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ of $\mathcal{O}$, where

$$\begin{aligned} \mathcal{A} &= \{Y \in \mathcal{O} \,|\, Y \cap A \neq \varnothing\},\\ \mathcal{B} &= \{Y \in \mathcal{O} \,|\, Y \cap A = \varnothing \text{ and } Y \cap B \neq \varnothing\},\\ \mathcal{C} &= \{Y \in \mathcal{O} \,|\, Y \subseteq C\}. \end{aligned}$$

Since adding every edge $xx' \in E(G)\setminus E(H)$ to H creates a copy of $K_{s,t}$ in H, we conclude that $E(G)\setminus E(H) \subseteq \bigcup_{Y\in\mathcal{O}} E_G(N_Y, F_Y)$. Therefore, using Lemma 2.3, we find that *whp*

$$\left|\bigcup_{Y\in\mathcal{O}} E_G(N_Y,F_Y)\right| \geqslant |E(G)\setminus E(H)| = \frac{n^2 p}{2}\big(1+o(1)\big). \tag{5}$$

Note that $E_G(N_Y,F_Y) \subseteq E_G(N_y,F_y)$ for every $y \in Y$, since $N_Y \subseteq N_y$ and $F_Y \subseteq F_y$. For every vertex $y \in V$, by a double counting of the set $\{(x,S) \,|\, x \in F_y, S \subseteq N_H(x,y), \text{ and } |S| = s\}$, we derive that

$$|F_y|\binom{t-1}{s} \leqslant \binom{|N_y|}{s}(t-1).$$

It follows from $t>s$ that $|F_y| \leqslant |N_y|^s$. Hence, $|F_y| \leqslant \log^s n$ for every $y \in A$. This gives

$$\left|\bigcup_{Y\in\mathcal{A}} E_G(N_Y,F_Y)\right| \leqslant \left|\bigcup_{y\in A} E_G(N_y,F_y)\right| \leqslant \sum_{y\in A} |N_y||F_y| \leqslant n(\log n)\log^s n = n\log^{s+1} n. \tag{6}$$

Since $|E(H)| = O(n)$ *whp*, we get that $|F_y \setminus A| = O(n/\log n)$ for each $y \in V$ *whp*. Using this, we may write *whp*

$$\begin{aligned}
\left|\bigcup_{Y\in\mathcal{B}} E_G(N_Y,F_Y)\right| &\leqslant \left|\bigcup_{y\in B} E_G(N_y,F_y)\right| \\
&\leqslant \sum_{y\in B} |E_G(N_y,F_y)| \\
&\leqslant \sum_{y\in B} |E_G(N_y,F_y\setminus A)| + \sum_{y\in B} |E_G(N_y,F_y\cap A)| \\
&\leqslant \sum_{y\in B} |N_y||F_y\setminus A| + \sum_{y\in B} |N_y||F_y\cap A| \\
&\leqslant O\left(\frac{n}{\log n}\right)\sum_{y\in B}|N_y| + \frac{n}{\log^{s+1} n}\sum_{x\in A}|F_x\cap B| \\
&\leqslant O\left(\frac{n}{\log n}\right)|E(H)| + \frac{n}{\log^{s+1} n}|A|\log^s n \\
&= O\left(\frac{n^2}{\log n}\right)
\end{aligned} \tag{7}$$

Since $|E(H)| = O(n)$ *whp*, we deduce that $|C| = O(\log^{s+1} n)$ *whp* and so $|\mathcal{C}| \leqslant |C|^{s-1} = O(\log^{s^2-1} n)$ *whp*. Now, by setting $\lambda = 2$ in Corollary 2.4, we obtain that *whp*

$$\begin{aligned}
\left|\bigcup_{Y\in\mathcal{C}} E_G(N_Y,F_Y)\right| &\leqslant \sum_{Y\in\mathcal{C}} |E_G(N_Y,F_Y)| \\
&\leqslant \sum_{Y\in\mathcal{C}} \Big(3n\log^2 n + p|N_Y||F_Y|\big(1+o(1)\big)\Big) \\
&\leqslant 3|\mathcal{C}|n\log^2 n + \sum_{Y\in\mathcal{C}} p^s n|F_Y|\big(1+o(1)\big) \\
&\leqslant O\left(n\log^{s^2+1} n\right) + p^s n\left(\sum_{Y\in\mathcal{C}}|F_Y|\right)\big(1+o(1)\big).
\end{aligned} \tag{8}$$

Therefore, by (5)–(8), we find that *whp*

$$\sum_{Y\in\mathcal{C}} |F_Y| \geqslant \frac{n}{2p^{s-1}}\big(1+o(1)\big). \tag{9}$$

Set

$$S = \bigcup_{\substack{X,Y\in\mathcal{C}\\ X\neq Y}} N_X \cap N_Y.$$

Note that $|S| \leqslant \binom{|\mathcal{C}|}{2}(t-1) = O(\log^{2s^2-2} n)$ *whp*. For every $Y \in \mathcal{C}$, set $M_Y = N_Y \setminus S$. Let

$$F' = \left\{ x \in \bigcup_{Y \in \mathcal{C}} F_Y \;\middle|\; |N_x \cap S| \geqslant s \right\}.$$

We claim that $|F'| \leqslant \binom{|S|}{s}(t-1)$. To see this, suppose otherwise. By the pigeonhole principle, there is a t-subset T of F' such that $|N_H(T) \cap S| \geqslant s$ which gives a copy of $K_{s,t}$ in H, a contradiction. This proves the claim which in turn implies that $|F'| = O(\log^{2s^3-2s} n)$. For every $Y \in \mathcal{C}$, set $F'_Y = F_Y \setminus F'$. Noting that the sets M_Y are mutually disjoint and $F_Y \cap Y = \varnothing$ for every $Y \in \mathcal{O}$, we may write *whp*

$$\begin{aligned}
2|E(H)| &= \sum_{Y \in \mathcal{C}} \sum_{x \in M_Y} d_H(x) + \sum_{x \notin \bigcup_{Y \in \mathcal{C}} M_Y} d_H(x) \\
&\geqslant \sum_{Y \in \mathcal{C}} \big(|E_H(M_Y, F'_Y \setminus M_Y)| + 2|E_H(M_Y, F'_Y \cap M_Y)| + |E_H(M_Y, Y)|\big) + (s-1)\left|V \setminus \bigcup_{Y \in \mathcal{C}} M_Y\right| \\
&\geqslant \sum_{Y \in \mathcal{C}} \big((t-s)|F'_Y \setminus M_Y| + (t-s)|F'_Y \cap M_Y| + (s-1)|M_Y|\big) + (s-1)\left(n - \sum_{Y \in \mathcal{C}} |M_Y|\right) \\
&= (s-1)n + \sum_{Y \in \mathcal{C}} (t-s)|F'_Y| \\
&= (s-1)n + (t-s)\left(\left(\sum_{Y \in \mathcal{C}} |F_Y|\right) - |\mathcal{C}||F'|\right) \\
&\geqslant (s-1)n + (t-s)\left(\frac{n}{2p^{s-1}}\big(1+o(1)\big) - O\left(\log^{2s^3+s^2-2s-1} n\right)\right) \\
&= \left(\frac{t-s}{2p^{s-1}} + s - 1 + o(1)\right) n,
\end{aligned}$$

where the last inequality follows from (9), completing the proof. □

We point out here that Theorem 1.2 is concluded from Theorems 5.1 and 5.5.

Remark 5.6. It is worth noting that using the proof of Theorem 5.1, one may improve the estimate on the number of edges of H in the last paragraph of proof of Theorem 5.5 to obtain

$$\operatorname{sat}(\mathbb{G}(n,p), K_{s,t}) \geqslant \left(\frac{t-s}{4p^{s-1}} + s - 1 + o(1)\right) n.$$

For the sake of clarity of presentation we disregarded this improvement in the proof of Theorem 5.5.

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