Weak saturation numbers in random graphs

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Abstract

For two given graphs G and F, a graph H is said to be weakly (G, F)-saturated if H is a spanning subgraph of G which has no copy of F as a subgraph and one can add all edges in $E(G) \setminus E(H)$ to H in some order so that a new copy of F is created at each step. The weak saturation number $\mathrm{wsat}(G, F)$ is the minimum number of edges of a weakly (G, F)-saturated graph. In this paper, we deal with the relation between $\mathrm{wsat}(\mathbb{G}(n, p), F)$ and $\mathrm{wsat}(K_n, F)$, where $\mathbb{G}(n, p)$ denotes the Erdős–Rényi random graph and K_n denotes the complete graph on n vertices. For every graph F and constant p, we prove that $\mathrm{wsat}(\mathbb{G}(n, p), F) = \mathrm{wsat}(K_n, F)(1 + o(1))$ with high probability. Also, for some graphs F including complete graphs, complete bipartite graphs, and connected graphs with minimum degree 1 or 2, it is shown that there exists an $\varepsilon(F) > 0$ such that, for any $p \geqslant n^{-\varepsilon(F)} \log n$, $\mathrm{wsat}(\mathbb{G}(n, p), F) = \mathrm{wsat}(K_n, F)$ with high probability.

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1. Introduction

All graphs throughout this paper are assumed to be finite, undirected, and without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$ and the edge set of G is denoted by $E(G)$. For two given graphs G and F, a graph H is said to be *weakly* (G, F)*-saturated* if

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H is a spanning subgraph of G which has no copy of F as a subgraph and there is an ordering $e_1, e_2, \ldots$ of edges in $E(G) \setminus E(H)$ such that for $i = 1, 2, \ldots$ the spanning subgraph of G with the edge set $E(H) \cup \{e_1, \ldots, e_i\}$ has a copy of F containing e_i. The minimum number of edges in a weakly (G, F)-saturated graph is called the *weak saturation number* of F in G and is denoted by $\mathrm{wsat}(G, F)$. Let K_n be the complete graph on n vertices and $K_{s,t}$ be the complete bipartite graph with parts of sizes s and t. For the purpose of simplification, a weakly (K_n, F)-saturated graph is called weakly F-saturated if there is no danger of ambiguity and moreover, $\mathrm{wsat}(K_n, F)$ is written as $\mathrm{wsat}(n, F)$.

The notion of weak saturation was initially introduced by Bollobás [6] in 1968. Weak saturation is closely related to the so-called 'graph bootstrap percolation' which was introduced for the first time in [3]. It is worth mentioning that the study of any extremal parameter is an important task in graph theory and usually receives a great deal of attention. Determining the exact value of $\mathrm{wsat}(n, F)$ for a given graph F is often quite difficult. Although the weak saturation number has been studied for a long time, related literature is still poor. Lovász [16] established that

$$\mathrm{wsat}(n, K_s) = (s-2)n - \binom{s-1}{2}$$

when $n \geqslant s \geqslant 2$, settling a conjecture of Bollobás [6]. Kalai [12] proved that

$$\mathrm{wsat}(n, K_{t,t}) = (t-1)n - \binom{t-1}{2}$$

if $n \geqslant 4t - 4$. An alternative proof for the result is given by Kronenberg, Martins, and Morrison [15] for every $n \geqslant 3t - 3$. They also established that, for every $t > s$ and sufficiently large n,

$$(s-1)(n-t+1) + \binom{t}{2} \leqslant \mathrm{wsat}(n, K_{s,t}) \leqslant (s-1)(n-s) + \binom{t}{2}.$$

Miralaei, Mohammadian, and Tayfeh-Rezaie [17] determined the exact value of $\mathrm{wsat}(n, K_{2,t})$. They found that, for every $t \geqslant 3$ and $n \geqslant t + 2$,

$$\mathrm{wsat}(n, K_{2,t}) = \begin{cases} n - 1 + \binom{t}{2} & \text{if } t \text{ is even and } n \leqslant 2t - 2, \\ n - 2 + \binom{t}{2} & \text{otherwise.} \end{cases}$$

For more results on weak saturation and related topics, we refer to the survey [8].

Random analogues of different parameters in extremal graph theory have been extensively studied in the literature. These studies often reveal the behavior of extremal parameters for a typical graph. Recall that the Erdős–Rényi random graph model $\mathbb{G}(n, p)$ is the probability space of all graphs on a fixed vertex set of size n where every two distinct vertices are adjacent independently with probability p. Also, recall that the notion 'with high probability' is used whenever an event occurs in $\mathbb{G}(n, p)$ with a probability approaching 1 as n goes to infinity.

The study of the weak saturation problem in random graphs was initiated by Korándi and Sudakov [14]. They proved that, for every constant $p \in (0, 1)$ and integer $s \geqslant 3$,

$$\mathrm{wsat}\big(\mathbb{G}(n, p), K_s\big) = \mathrm{wsat}(n, K_s)$$

with high probability. We will sometimes use the notion 'stability' for the graph F if $\mathrm{wsat}(\mathbb{G}(n,p),F) = \mathrm{wsat}(n,F)$ with high probability. Bidgoli, Mohammadian, Tayfeh-Rezaie, and Zhukovskii [5] established the existence of a threshold function for the property $\mathrm{wsat}(\mathbb{G}(n,p),K_s) = \mathrm{wsat}(n,K_s)$ and provided the following upper and lower bounds on the function for any $s \geqslant 3$.

- There exists a constant c_s such that, if $p \leqslant c_s n^{-2/(s+1)} \log^{2/((s-2)(s+1))} n$, then $\mathrm{wsat}(\mathbb{G}(n,p),K_s) \neq \mathrm{wsat}(n,K_s)$ with high probability.
- If $p \geqslant n^{-1/(2s-3)} \log^{(s-1)/(2s-3)} n$, then $\mathrm{wsat}(\mathbb{G}(n,p),K_s) = \mathrm{wsat}(n,K_s)$ with high probability.

Borowiecki and Sidorowicz [7] proved that

$$\mathrm{wsat}(n,K_{1,t}) = \binom{t}{2}$$

provided $n \geqslant t+1$. A short proof of the result is given in [9]. Kalinichenko and Zhukovskii [13] investigated $\mathrm{wsat}(\mathbb{G}(n,p),K_{1,t})$ and provided the following bounds for $t \geqslant 3$.

- There exists a constant c_t such that, if $n^{-2} \ll p \leqslant c_t n^{-1/(t-1)} \log^{-(t-2)/(t-1)} n$, then $\mathrm{wsat}(\mathbb{G}(n,p),K_{1,t}) \neq \mathrm{wsat}(n,K_{1,t})$ with high probability.
- There exists a constant d_t such that, if $p \geqslant d_t n^{-1/(t-1)} \log^{-(t-2)/(t-1)} n$, then $\mathrm{wsat}(\mathbb{G}(n,p),K_{1,t}) = \mathrm{wsat}(n,K_{1,t})$ with high probability.

For a graph G and a subset X of $V(G)$, denote by $N_G(X)$ the set of vertices of G which are adjacent to all vertices in X and set $N_G[X] = X \cup N_G(X)$. For the sake of convenience, $N_G(v_1,\ldots,v_k)$ is written instead of $N_G(\{v_1,\ldots,v_k\})$. For a vertex v of G, we define the *degree* of v as $|N_G(v)|$ and denote it by $d_G(v)$. The maximum and minimum degrees of the vertices of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Furthermore, denote by $e(\mathbb{G}(n,p))$ the random variable counting the edges in $\mathbb{G}(n,p)$.

Kalinichenko and Zhukovskii [13] studied sufficient conditions on weakly (K_n,F)-saturated graphs such that the equality $\mathrm{wsat}(\mathbb{G}(n,p),F) = \mathrm{wsat}(n,F)$ holds with high probability. They proved the following theorem.

Theorem 1.1 ([13]). *Let F be a graph with $\delta(F) \geqslant 1$. Also, let $p \in (0,1)$ and $c \geqslant \delta(F)-1$ be constants. For every positive integer n, assume that there exists a weakly (K_n,F)-saturated graph H_n containing a set of vertices S_n with $|S_n| \leqslant c$ such that each vertex from $V(K_n)\setminus S_n$ is adjacent to at least $\delta(F)-1$ vertices in S_n. Then, there exists a weakly $(\mathbb{G}(n,p),F)$-saturated graph with $\min\{e(\mathbb{G}(n,p)),|E(H_n)|\}$ edges with high probability.*

The following corollary immediately follows from Theorem 1.1 and a result due to Spencer [19].

Corollary 1.2 ([13]). *Let F be a graph with $\delta(F) \geqslant 1$ and let $p \in (0,1)$ be constant. For every positive integer n, assume that there exists a weakly (K_n,F)-saturated graph H_n with $|E(H_n)| =$ $\mathrm{wsat}(n,F)$ satisfying the property described in Theorem 1.1. Then, $\mathrm{wsat}(\mathbb{G}(n,p),F) = \mathrm{wsat}(n,F)$ with high probability.*

It has been verified in [13] that Corollary 1.2 implies the stability for some graphs F such as complete graphs and complete bipartite graphs.

In this paper, we continue to explore the relation between $\mathrm{wsat}(\mathbb{G}(n,p),F)$ and $\mathrm{wsat}(n,F)$. We find the asymptotic behavior of $\mathrm{wsat}(\mathbb{G}(n,p),F)$ and sometimes its exact value compared to $\mathrm{wsat}(n,F)$. Regarding the asymptotic behavior, we prove the following theorem in Section 2.

Theorem 1.3. *Let F be a graph with $\delta(F) \geqslant 1$ and let $p \in (0,1)$ be constant. Then,* $\mathrm{wsat}(\mathbb{G}(n,p),F) = \mathrm{wsat}(n,F)(1+o(1))$ *with high probability.*

Notice that addition of isolated vertices to the graph F does not change the weak saturation number. Subsequently, we only consider graphs F with $\delta(F) \geqslant 1$. Especially, Theorem 1.3 holds for all graphs F. In Section 3, we present a sufficient condition on $\mathrm{wsat}(n,F)$ for which $\mathrm{wsat}(\mathbb{G}(n,p),F) = \mathrm{wsat}(n,F)$ with high probability.

Theorem 1.4. *Let F be a graph with $\delta(F) \geqslant 1$ and let $\mathrm{wsat}(n,F) \geqslant (\delta(F)-1)n + D$ for all n, where D is a constant depending on F. Then, there exist a positive integer k and a constant d_F such that $\mathrm{wsat}(n,F) = (\delta(F)-1)n + d_F$ and, for any $p \geqslant n^{-1/(2k+3)}\log n$, $\mathrm{wsat}(\mathbb{G}(n,p),F) = \mathrm{wsat}(n,F)$ with high probability.*

For some graphs F including complete graphs, complete bipartite graphs, and connected graphs with minimum degree 1 or 2, Theorem 1.4 shows that there is an $\varepsilon(F) > 0$ such that, for any $p \geqslant n^{-\varepsilon(F)}\log n$, $\mathrm{wsat}(\mathbb{G}(n,p),F) = \mathrm{wsat}(n,F)$ with high probability, see Remark 3.11. Note that Theorem 1.4 is a generalization of Theorem 1.1 and Corollary 1.2 with simpler arguments. Actually, a weakly (K_n,F)-saturated graph having the structure described in Theorem 1.1 has $(\delta(F)-1)n+D$ edges for some constant D depending on F. Our proof does not require a specific structure of the weakly (K_n,F)-saturated graph, only the number of edges in it is needed. This makes it easier to construct arguments and apply the result not only to complete graphs and complete bipartite graphs which can be also easily done with Theorem 1.1, but also to some other types of graphs.

Finally, we consider the case $p = o(1)$ in Section 4 where we present a condition which implies that $\mathrm{wsat}(\mathbb{G}(n,p),F) = e(\mathbb{G}(n,p))(1+o(1))$ with high probability.

Let us fix here more notation and terminology that we use in the rest of the paper. Let G be a graph. For a vertex v of G, we denote by $G-v$ the graph obtained from G by removing v and all of its incident edges. For two nonadjacent vertices u, v of G, let $G+uv$ denote the graph obtained from G by adding an edge between u and v. For a subset X of $V(G)$, we denote the induced subgraph of G on X by $G[X]$. For two disjoint subsets S and T of $V(G)$, denote by $E_G(S,T)$ the set of all edges with an endpoint in both S and T. For the purpose of simplicity, $E_G(v,T)$ is written instead of $E_G(\{v\},T)$. For a positive integer d, the *d-th power* of G, denoted by G^d, is the graph with vertex set $V(G)$ and two vertices x, y are adjacent in G^d if and only if the distance between x, y in G is at most d.

2. Asymptotic stability

In this section, we prove that $\mathrm{wsat}(\mathbb{G}(n,p),F) = \mathrm{wsat}(n,F)(1+o(1))$ with high probability for any constant p and any graph F. To proceed, we need to recall the following result on the appearance of the powers of a Hamiltonian cycle in $\mathbb{G}(n,p)$.

Theorem 2.1 ([11, 18])**.** *If $k \geqslant 2$ and $p \gg n^{-1/k}$, then $\mathbb{G}(n,p)$ contains the k-th power of a Hamiltonian cycle with high probability.*

Lemma 2.2. *Let $p = n^{-o(1)}$. Then, there is a function $w(n)$ tending to infinity as $n \to \infty$ such that the vertex set of $\mathbb{G}(n,p)$ can be partitioned into cliques of size at least $w(n)$ with high probability.*

Proof. Let $\mathcal{H}_d$ be the event that $\mathbb{G}(n,p)$ contains the d-th power of a Hamiltonian cycle. Since $p = n^{-o(1)}$, Theorem 2.1 implies that, for any $\varepsilon > 0$ and any integer $k \geqslant 2$, there is a minimum integer $N(k,\varepsilon)$ such that $\mathbb{P}[\mathcal{H}_k] \geqslant 1-\varepsilon$ for any $n \geqslant N(k,\varepsilon)$. Note that $N(k,\varepsilon) \geqslant k+1$ and $N(k_1,\varepsilon_1) \geqslant N(k_2,\varepsilon_2)$ if $k_1 \geqslant k_2$ and $\varepsilon_1 \leqslant \varepsilon_2$. Let $m_1 = 2$ and for any $k \geqslant 2$, define

$$m_k = \min\left\{\ell \,\middle|\, \ell > m_{k-1} \text{ and } N\left(\ell, \frac{1}{k}\right) > N\left(m_{k-1}, \frac{1}{k-1}\right)\right\}.$$

Clearly, $\{N(m_k, \frac{1}{k})\}_{k=1}^{\infty}$ is strictly increasing. Now, define the function w as

$$w(n) = m_k \text{ for any } k \geqslant 1 \text{ and } n \in \left[N\left(m_k, \frac{1}{k}\right), N\left(m_{k+1}, \frac{1}{k+1}\right)\right).$$

Obviously, $w(n) \to \infty$ as $n \to \infty$. We show that $\mathbb{P}[\mathcal{H}_{w(n)}] \geqslant 1 - \frac{1}{k}$ for every k and $n \geqslant N(m_k, \frac{1}{k})$. To see this, fix k and assume that $N(m_\ell, \frac{1}{\ell}) \leqslant n < N(m_{\ell+1}, \frac{1}{\ell+1})$ for some $\ell \geqslant k$. Then,

$$\mathbb{P}\left[\mathcal{H}_{w(n)}\right] = \mathbb{P}\left[\mathcal{H}_{m_\ell}\right] \geqslant 1 - \frac{1}{\ell} \geqslant 1 - \frac{1}{k}.$$

Thus, $\mathbb{G}(n,p)$ contains the $w(n)$-th power of a Hamiltonian cycle with high probability, say C. Note that every $w(n)+1$ consecutive vertices of C form a clique in $\mathbb{G}(n,p)$. Now, by partitioning the vertices of $\mathbb{G}(n,p)$ so that each part consists of at least $w(n)/2$ consecutive vertices of C, the result follows. □

The following lemma is obtained from Theorem 2 in [19].

Lemma 2.3 ([19]). *For any integer $s \geqslant 3$, there exists a constant c such that if $p \geqslant cn^{-2/(s+1)} \log^{2/((s-2)(s+1))} n$, then every two vertices of $\mathbb{G}(n,p)$ have a clique of size $s-2$ in their common neighbors with high probability.*

The following corollary immediately follows from Lemma 2.3.

Corollary 2.4. *Let F be a graph and let $s = |V(F)| \geqslant 3$ and $\delta(F) \geqslant 1$. There is a constant c such that if $p \geqslant cn^{-2/(s+1)} \log^{2/((s-2)(s+1))} n$, then $\mathrm{wsat}(\mathbb{G}(n,p), F) \geqslant \mathrm{wsat}(n,F)$ with high probability.*

Proof. Let H be a weakly $(\mathbb{G}(n,p), F)$-saturated graph with minimum possible number of edges. Using Lemma 2.3, $\mathbb{G}(n,p)$ is weakly F-saturated graph with high probability. This shows that H is also a weakly F-saturated graph, the result follows. □

Remark 2.5. A graph G is called *balanced* if $(|E(H)|-1)/(|V(H)|-2) \leqslant \lambda_G$ for all proper subgraphs H of G with $|V(H)| \geqslant 3$, where $\lambda_G = (|E(G)|-2)/(|V(G)|-2)$. It was proved in [4] that if F is a balanced graph and $p \gg n^{-1/\lambda_F + o(1)}$, then $\mathbb{G}(n,p)$ is weakly F-saturated with high probability, implying that $\mathrm{wsat}(\mathbb{G}(n,p), F) \geqslant \mathrm{wsat}(n,F)$ with high probability. In general, threshold probability functions for the property that $\mathbb{G}(n,p)$ is weakly F-saturated are still unknown.

The following lemma immediately follows from the Chernoff bound [2, Corollary A.1.2] and the union bound.

Lemma 2.6. *For any positive integer k, there exists a constant c such that if $p \geqslant c((\log n)/n)^{1/k}$, then every k-subset of the vertex set of $\mathbb{G}(n,p)$ has at least $p^k n/2$ common neighbors with high probability.*

The following interesting result, due to Alon [1], will be used in the next theorem and the subsequent sections.

Theorem 2.7 ([1])**.** *Let F be a graph with $\delta(F) \geqslant 1$. Then, there exists a constant c_F such that* $\mathrm{wsat}(n, F) = (c_F + o(1))n$.

We are now in a position to prove the main result of this section. Recall Theorem 1.3.

Theorem 1.3. *Let F be a graph with $\delta(F) \geqslant 1$ and let $p \in (0,1)$ be constant. Then,* $\mathrm{wsat}(\mathbb{G}(n,p), F) = \mathrm{wsat}(n, F)(1 + o(1))$ *with high probability.*

Proof. It follows from Theorem 2.7 that there exists a constant c_F such that $\mathrm{wsat}(n, F) = (c_F + o(1))n$. By Corollary 2.4, it remains to show that $\mathrm{wsat}(\mathbb{G}(n,p), F) \leqslant (c_F + o(1))n$ with high probability.

Note that $\mathrm{wsat}(G, K_2) = 0$ for every graph G and so there is nothing to prove when $F = K_2$. Therefore, we may assume that F has $s \geqslant 3$ vertices. By Lemma 2.2, there is a function $w(n)$ such that $w(n)$ goes to infinity when $n \to \infty$ and, with high probability, the vertex set of $G \sim \mathbb{G}(n,p)$ admits a partition into $V_1, \ldots, V_m$ such that each V_i is a clique of size at least $w(n)$.

Fix an $(s-2)$-subset S in V_1 and let $i \in \{1, \ldots, m\}$. If $|N_G(S) \cap V_i| \geqslant s - 1$, then let S_i be an arbitrary $(s-1)$-subset of $N_G(S) \cap V_i$ and otherwise, let S_i be an arbitrary $(s-1)$-subset of $V_i \setminus S$ containing $N_G(S) \cap V_i$. Consider the set $N_G(S \cup S_i)$. Using Lemma 2.6, we have $|N_G(S \cup S_i)| \geqslant p^{2s-3}n/2$. Since p is constant and $m \leqslant n/w(n)$, there exists an index i' such that $|N_G(S \cup S_i) \cap V_{i'}| \geqslant s - 2$. Let R_i be an arbitrary $(s-2)$-subset of $N_G(S \cup S_i) \cap V_{i'}$.

We are going to introduce a weakly (G, F)-saturated graph H with $|E(H)| \leqslant (c_F + o(1))n$. For $i = 1, \ldots, m$, let H_i be a weakly $(G[V_i], F)$-saturated graph with $\mathrm{wsat}(|V_i|, F)$ edges. Define H to be a spanning subgraph of G with $E(H) = \bigcup_{i=1}^{m}(E(H_i) \cup E_G(S_i, S \cup R_i))$. We have

$$|E(H)| \leqslant \sum_{i=1}^{m} \Big(\big(c_F + o(1)\big)|V_i| + 2(s-1)(s-2) \Big) = \big(c_F + o(1)\big)n.$$

So, it remains to prove that H is weakly (G, F)-saturated. To do this, we show that the edges in $E(G) \setminus E(H)$ can be added to H in some order through a weakly (G, F)-saturated process. We use H to denote the resulting graph in any step of the process as well. Note that, in each step of the process, every edge $xy \in E(G) \setminus E(H)$ can be added to H whenever $N_H(x, y)$ contains a clique of size $s - 2$ in H. For $i = 1, \ldots, m$, as H_i is weakly $(G[V_i], F)$-saturated, we may add the edges in $E(G[V_i]) \setminus E(H_i)$ in an appropriate order. By doing this, each V_i becomes a clique in H.

Then, we add all edges in $E_G(S, N_G(S)) \setminus E(H)$ to H as follows. Let $x \in N_G(S) \cap V_i$ for some i. If $x \in S_i$, then x is already joined to all vertices of S in H and there is nothing to prove. So, we may assume that $x \notin S_i$. In this case, according to the choice of S_i, we have $S_i \subseteq N_G(S)$ and therefore $S_i \subseteq N_H(S)$. Hence, for every $y \in S$, it follows from $S_i \subseteq N_H(x, y)$ that the edge xy can be added to H.

Now, we add all edges in $E_G(V_i, V_j) \setminus E(H)$ to H for all distinct indices $i, j \in \{1, \ldots, m\}$ as follows. Let $x \in V_i$ and $y \in V_j$ with $1 \leqslant i < j \leqslant m$ such that x, y are not adjacent in H. Consider the set $U = N_G(S \cup S_i \cup S_j \cup R_i \cup R_j \cup \{x, y\})$. Using Lemma 2.6, we have $|U| \geqslant p^{5s-6}n/2$. Since p is constant and $m \leqslant n/w(n)$, there exists an index k such that $|U \cap V_k| \geqslant s - 2$. Let T be an arbitrary $(s-2)$-subset of $U \cap V_k$.

We claim that x can be joined to all vertices in T in H. First, we add all remaining edges between R_i and T to H. This is possible since both R_i and T are subsets of $N_G(S)$ and are already joined to S in H. Next, we add all remaining edges between S_i and T to H. This is possible since both S_i and T are already joined to R_i in H. Finally, we add all remaining edges between x and T to H. This is possible since both x and T are already joined to $S_i \setminus \{x\}$ in H. Therefore, the claim is proved. Similarly, y can be joined to all vertices in T. Now, we may add the edge xy to H, since $T \subseteq N_H(x, y)$. The proof is completed. □

3. Exact stability

In this section, we examine the property $\text{wsat}(\mathbb{G}(n,p), F) = \text{wsat}(n, F)$. We present a sufficient condition on $\text{wsat}(n, F)$ such that the latter equality holds. The class of graphs satisfying the presented condition includes complete graphs, complete bipartite graphs, graphs with minimum degree 1, and graphs with minimum degree 2 which either are connected or have no cut edge. First, we recall the following general lower bound on $\text{wsat}(n, F)$ which we will later extend it to $\text{wsat}(\mathbb{G}(n,p), F)$

Theorem 3.1 ([9]). *Let F be a graph with s vertices, t edges, and minimum degree $\delta \geqslant 1$. Then, for any $n \geqslant s$,*

$$\text{wsat}(n, F) \geqslant t - 1 + \frac{(\delta - 1)(n - s)}{2}.$$

Theorem 3.2. *Let G and F be two given graphs with $|V(G)| \geqslant |V(F)|$ and $\delta(F) \geqslant 1$. Then,*

$$\text{wsat}(G, F) \geqslant \min\left\{|E(G)|, |E(F)| - 1 + \frac{\min\left\{\delta(G), \delta(F) - 1\right\}\big(|V(G)| - |V(F)|\big)}{2}\right\}.$$

Proof. Let H be a weakly (G, F)-saturated graph with minimum possible number of edges. If $H = G$, then there is nothing to prove. So, assume that $H \neq G$. When the first edge in $E(G) \setminus E(H)$ is added to H a copy F_1 of F is created. Letting $H_1 = H[V(F_1)]$, we have $|E(H_1)| \geqslant |E(F)| - 1$. Also, every vertex $v \in V(H) \setminus V(H_1)$ must have degree at least $\min\{\delta(G), \delta(F) - 1\}$ in H. To see this, if all edges in G incident to v appear in H, then $d_H(v) = d_G(v) \geqslant \delta(G)$. Otherwise, when the first edge incident to v from $E(G) \setminus E(H)$ is added to H, the degree of v in the resulting graph must be at least $\delta(F)$, implying $d_H(v) \geqslant \delta(F) - 1$. Therefore,

$$\begin{aligned} |E(H)| &\geqslant |E(H_1)| + |E(H) \setminus E(H_1)| \\ &\geqslant |E(F)| - 1 + \frac{\min\left\{\delta(G), \delta(F) - 1\right\}\big(|V(G)| - |V(F)|\big)}{2}. \end{aligned}$$

□

We know from [10, Theorem 3.4] that if $p \gg \frac{\log n}{n}$, then $\delta(\mathbb{G}(n,p)) = np(1+o(1))$ with high probability. From this fact, we obtain the following consequence.

Corollary 3.3. *Let F be a graph with s vertices, t edges, and minimum degree $\delta \geqslant 1$. If $p \gg \frac{\log n}{n}$, then*

$$\mathrm{wsat}\big(\mathbb{G}(n,p), F\big) \geqslant t - 1 + \frac{(\delta-1)(n-s)}{2},$$

with high probability.

Now, we switch to an upper bound on $\mathrm{wsat}(n, F)$.

Theorem 3.4 ([9])**.** *Let F be a graph with s vertices and minimum degree $\delta \geqslant 1$. Then, for every $n \geqslant m \geqslant s-1$,*

$$\mathrm{wsat}(n,F) \leqslant (\delta-1)(n-m) + \mathrm{wsat}(m,F). \tag{1}$$

In particular, for every $n \geqslant s-1$,

$$\mathrm{wsat}(n,F) \leqslant (\delta-1)n + \frac{(s-1)(s-2\delta)}{2}. \tag{2}$$

The relation (1) in Theorem 3.4 was proved in [9] for $m \in \{|V(F)|-1, |V(F)|\}$. Above, we have formulated a more general statement which can be proved using a similar argument. The inequality (1) for $m = |V(F)|-1$ results in an upper bound on $\mathrm{wsat}(n,F)$ which is given in (2).

We will also need some knowledge about the presence of small subgraphs in a random graph. First, we recall the following definition. For any graph G, define

$$m(G) = \max\left\{ \frac{|E(H)|}{|V(H)|} \;\middle|\; H \text{ is a subgraph of } G \right\}.$$

To use later, we recall the following result which appears in [10] as Theorem 5.3.

Theorem 3.5 ([10])**.** *Let G be a graph with $|E(G)| \geqslant 1$. Then, $n^{-1/m(G)}$ is a threshold probability for the property that $\mathbb{G}(n,p)$ has a copy of G as a subgraph.*

The following lemma was proved in [5]. We apply it in the next theorem.

Lemma 3.6 ([5])**.** *Let $k \geqslant 2$ be a fixed integer and let $p \geqslant n^{-\frac{1}{2k-1}} \log n$. Then, with high probability, $\mathbb{G}(n,p)$ has the property that, for every k-subset S of vertices, the induced subgraph on $N_G(S)$ contains the $(k-1)$-th power of a Hamiltonian path.*

The following theorem gives an upper bound on $\mathrm{wsat}(\mathbb{G}(n,p), F)$ which is linear in terms of n. The idea of proof comes from [5].

Theorem 3.7. *Let F be a graph with s vertices and minimum degree $\delta \geqslant 1$. Also, let $m \geqslant s-1$ and $p \geqslant n^{-1/(2m+3)} \log n$. Then,*

$$\mathrm{wsat}\big(\mathbb{G}(n,p), F\big) \leqslant (\delta-1)(n-m) + \mathrm{wsat}(m,F)$$

with high probability. In particular, for $p \geqslant n^{-1/(2s+1)} \log n$,

$$\mathrm{wsat}\big(\mathbb{G}(n,p), F\big) \leqslant (\delta-1)n + \frac{(s-1)(s-2\delta)}{2}$$

with high probability.

Proof. As the assertion trivially holds for $F = K_2$, we assume that $s \geqslant 3$. By Theorem 3.5, we may consider a clique Ω of size m in $G \sim \mathbb{G}(n,p)$. We define a spanning subgraph H of G as follows. Let H_0 be a weakly $(G[\Omega], F)$-saturated graph with $\mathrm{wsat}(m,F)$ edges. The graph H contains all edges of H_0 and moreover, for every $v \in N_G(\Omega)$, we add $\delta - 1$ arbitrary edges of $E_G(v,\Omega)$ to H. Also, for every $v \in V(G) \setminus N_G[\Omega]$, we add $\delta - 1$ edges of $E_G(v, N_G(v))$ to H as described below. Using Lemma 3.6, the graph $H_v = G[N_G(\{v\} \cup \Omega)]$ contains the $(s-2)$-th power of a Hamiltonian path. Starting from a beginning vertex, denote the vertices of H_v going in the natural order induced by the Hamiltonian path by $x_1^v, \ldots, x_{h_v}^v$, where $h_v = |V(H_v)|$. We add the edges $vx_1^v, \ldots, vx_{\delta-1}^v$ to H for any $v \in V(G) \setminus N_G[\Omega]$. Since $|E(H)| = (\delta-1)(n-m) + \mathrm{wsat}(m,F)$, it suffices to prove that H is a weakly (G,F)-saturated graph. To do this, we show that the edges in $E(G) \setminus E(H)$ can be added to H in some order through a weakly (G,F)-saturated process.

As H_0 is weakly $(G[\Omega], F)$-saturated, we may add the edges in $E(G[\Omega]) \setminus E(H_0)$ to H in an appropriate order. By doing this, Ω becomes a clique in H. Let $u \in \Omega$ and $v \in N_G(\Omega)$ such that $uv \in E(G) \setminus E(H)$. Let $w \in V(F)$ such that $d_F(w) = \delta(F)$. Since Ω is a clique in H, it contains a copy F_0 of $F - w$ with $u \in V(F_0)$ and so $H[V(F_0) \cup \{v\}] + uv$ is a copy of F. This shows that all edges in $E_G(\Omega, N_G(\Omega)) \setminus E(H)$ may be added to H simultaneously. Now, all edges in $E(G) \setminus E(H)$ with both endpoints in $N_G(\Omega)$ can be added to H, since they belong to a copy of F containing a $(s-2)$-subset of Ω. Next, for every $v \in V(G) \setminus N_G[\Omega]$, we may add to H the edges $vx_\delta^v, \ldots, vx_{h_v}^v$ one by one, since every such edge belongs to a copy of F containing the previous $s-2$ vertices of the $(s-2)$-th power of the Hamiltonian path. Finally, for each edge $xy \in E(G) \setminus E(H)$ with endpoints in $V(G) \setminus N_G[\Omega]$, by Lemma 3.6, $N_G(\{x,y\} \cup \Omega)$ contains a clique Ω_{xy} of size at least $s-2$ with $\Omega_{xy} \subseteq N_G(\Omega)$, so xy can be added to H as well. □

The following consequence is obtained from Corollary 3.3 and Theorem 3.7.

Corollary 3.8. *Let F be a graph with s vertices and $\delta(F) \geqslant 1$ and let $p \geqslant n^{-1/(2s+1)} \log n$. Then,* $\mathrm{wsat}(\mathbb{G}(n,p), F)$ *is bounded from above by a constant with high probability if and only if $\delta(F) = 1$.*

We are now ready to prove the main result of this section. Recall Theorem 1.4.

Theorem 1.4. *Let F be a graph with $\delta(F) \geqslant 1$ and let $\mathrm{wsat}(n,F) \geqslant (\delta(F)-1)n + D$ for all n, where D is a constant depending on F. Then, there exist a positive integer k and a constant d_F such that $\mathrm{wsat}(n,F) = (\delta(F)-1)n + d_F$ and, for any $p \geqslant n^{-1/(2k+3)} \log n$, $\mathrm{wsat}(\mathbb{G}(n,p), F) = \mathrm{wsat}(n,F)$ with high probability.*

Proof. For simplicity, let $\delta = \delta(F)$. By Theorem 2.7, there exist a constant c_F and a function $\varphi = o(n)$ such that $\mathrm{wsat}(n,F) = c_F n + \varphi(n)$ for all n. By the assumption and Theorem 3.4, we have

$$(\delta-1)n + D \leqslant \mathrm{wsat}(n,F) \leqslant (\delta-1)n + \frac{(s-1)(s-2\delta)}{2},$$

where $s = |V(F)|$. Since $\varphi(n) = o(n)$, the inequality above implies that $c_F = \delta - 1$ and $\varphi(n) \geqslant D$ for any n. By Theorem 3.4, for any $n \geqslant m \geqslant s-1$,

$$\begin{aligned}(\delta-1)n + \varphi(n) &= \mathrm{wsat}(n,F)\\ &\leqslant (\delta-1)(n-m) + \mathrm{wsat}(m,F)\\ &= (\delta-1)(n-m) + \big((\delta-1)m + \varphi(m)\big)\\ &= (\delta-1)n + \varphi(m)\end{aligned}$$

and so $\varphi(n) \leqslant \varphi(m)$. This shows that φ is decreasing. As φ is an integer valued function and bounded from below, there exists a constant d_F such that $\varphi(n) = d_F$ for any sufficiently large n. Assume that k is the smallest n satisfying $n \geqslant s-1$ and $\varphi(n) = d_F$.

Let $p \geqslant n^{-1/(2k+3)} \log n$. Corollary 2.4 yields that $\mathrm{wsat}(\mathbb{G}(n,p), F) \geqslant \mathrm{wsat}(n,F)$ with high probability. Moreover, Theorem 3.7 implies that

$$\mathrm{wsat}\big(\mathbb{G}(n,p), F\big) \leqslant (\delta-1)(n-k) + \mathrm{wsat}(k,F),$$

with high probability. Therefore,

$$\begin{aligned}(\delta-1)n + d_F &= \mathrm{wsat}(n,F)\\ &\leqslant \mathrm{wsat}\big(\mathbb{G}(n,p), F\big)\\ &\leqslant (\delta-1)(n-k) + \mathrm{wsat}(k,F)\\ &= (\delta-1)(n-k) + \big((\delta-1)k + d_F\big)\\ &= (\delta-1)n + d_F,\end{aligned}$$

and hence $\mathrm{wsat}(\mathbb{G}(n,p), F) = (\delta-1)n + d_F = \mathrm{wsat}(n,F)$ with high probability, as required. □

The following result shows that some graphs with minimum degree 2 satisfy the assumption of Theorem 1.4.

Lemma 3.9. *Let F be a graph with $\delta(F) = 2$. If F either is connected or has no cut edge, then* $\mathrm{wsat}(n,F) \geqslant n-1$ *for any positive integer n.*

Proof. Let H be a weakly (K_n, F)-saturated graph with minimum possible number of edges. If H is connected, then $\mathrm{wsat}(n,F) = |E(H)| \geqslant n-1$, as required. So, let H be disconnected. Then, the first edge in $E(K_n) \setminus E(H)$ joining two connected components of H in a weakly F-saturated process is a cut edge of a copy of F. Hence, in order to prove the assertion, we may assume that F is a connected graph with a cut edge.

We prove that $\mathrm{wsat}(n,F) \geqslant n$ for any $n \geqslant 3$. To do this, we show that none of the connected components of H is a tree. By contradiction, suppose that H has a connected component T that is a tree. Assume that $uv \in E(K_n) \setminus E(H)$ is the first edge in the order of weakly F-saturated process which has an endpoint in $V(T)$. First, suppose that $u \in V(T)$ and $v \notin V(T)$. Let F' be a copy of F created by adding uv during the process. Since the induced subgraph of F' on $V(F') \cap V(T)$ is a forest, either it is K_1 or it has at least two vertices of degree at most 1. This implies that $\delta(F) \leqslant 1$, a contradiction. Next, suppose that $u, v \in V(T)$. Since T is a tree and F is connected, F has exactly one cycle. As $\delta(F) = 2$, F must be a cycle graph, a contradiction. □

Remark 3.10. We give an example of a graph F with minimum degree 2 such that $\mathrm{wsat}(n,F) < n-1$. Let $F = K_3 \cup D_3$, where D_3 is a graph consisting of two vertex disjoint triangles which are joined by a single edge. Letting $n \geqslant 15$ and $n \equiv 0 \pmod 3$, it is easy to check that $D_3 \cup \frac{n-6}{3} P_3$ is a weakly F-saturated graph which implies that $\mathrm{wsat}(n,F) \leqslant \frac{2n}{3} + 3$.

Remark 3.11. It is worth mentioning that the assumption $\mathrm{wsat}(n,F) \geqslant (\delta-1)n+D$ for a constant D, given in Theorem 1.4, holds for several graph families. These include complete graphs [16], complete bipartite graphs [15, Proposition 15], graphs with minimum degree 1 (Theorem 3.1), and graphs with minimum degree 2 which either are connected or have no cut edge (Lemma 3.9). So, Theorem 1.4 can be applied for these graph families.

4. Small p

For any graph F, if p is small enough, then $\mathbb{G}(n,p)$ has no copy of F as a subgraph with high probability. Hence, for such p, with high probability, $\mathbb{G}(n,p)$ is the unique weakly $(\mathbb{G}(n,p),F)$-saturated graph which means that $\mathrm{wsat}(\mathbb{G}(n,p),F) = e(\mathbb{G}(n,p))$. In this section, we derive a similar result for slightly bigger p.

For any graph G, define

$$\mu(G) = \max\left\{m(G), \frac{|E(G)|-1}{|V(G)|-2}\right\}$$

if $|V(G)| \neq 2$ and otherwise, define $\mu(G)$ to be equal to $m(G)$. Below, we state a useful observation which can be proved straightforwardly.

Observation 4.1. *For every two graphs G and F, let $X_F(G)$ denote the number of copies of F in G. Then,*

$$|E(G)| - X_F(G) \leqslant \mathrm{wsat}(G,F) \leqslant |E(G)|.$$

Theorem 4.2. *Let F be a graph with $\delta(F) \geqslant 1$. Then, for any $p \ll n^{-1/\mu(F)}$,*

$$\mathrm{wsat}\big(\mathbb{G}(n,p),F\big) = e\big(\mathbb{G}(n,p)\big)\big(1+o(1)\big)$$

with high probability.

Proof. For simplicity, let $s = |V(F)|$, $t = |E(F)|$, $m = m(F)$, and $\mu = \mu(F)$. If $\Delta(F) = 1$, then $F = tK_2$ and thus $e(\mathbb{G}(n,p)) = 0$ with high probability using Theorem 3.5, so there is nothing to prove. Hence, we assume that $\Delta(F) \geqslant 2$ which gives $m(F) \geqslant \frac{2}{3}$. Denote by X_F the random variable that counts the number of copies of F in $\mathbb{G}(n,p)$. Let $p \leqslant n^{-1/\mu}\omega(n)$, where $\omega(n) \to 0$ when $n \to \infty$. From Observation 4.1,

$$e\big(\mathbb{G}(n,p)\big) - X_F \leqslant \mathrm{wsat}\big(\mathbb{G}(n,p),F\big) \leqslant e\big(\mathbb{G}(n,p)\big)$$

and therefore it is enough to show that $X_F = o(e(\mathbb{G}(n,p)))$. If $\mu \leqslant m$, then it follows from Theorem 3.5 that $X_F = 0$, we are done. If $p \leqslant n^{-7/4}$, then $p \ll n^{-3/2} \leqslant n^{-1/m}$ and hence $X_F = 0$ using Theorem 3.5, again we are done. So, we may assume that $\mu = \frac{t-1}{s-2}$ and $p \geqslant n^{-7/4}$. Since $p \gg n^{-2}$, it follows from [2, Theorem 4.4.4] that $e(\mathbb{G}(n,p)) = n^2p/2(1+o(1))$ with high probability. We know from [10, Lemma 5.1] that

$$\mathbb{E}(X_F) = \frac{s!}{|Aut(F)|}\binom{n}{s}p^t,$$

where $Aut(F)$ is the automorphism group of F. Also, using the Markov bound [10, Lemma 22.1],

$$\mathbb{P}\left[X_F \geqslant \frac{\mathbb{E}(X_F)}{\sqrt{\omega(n)}}\right] \leqslant \sqrt{\omega(n)} \longrightarrow 0$$

which implies that $X_F \leqslant \mathbb{E}(X_F)/\sqrt{\omega(n)}$ with high probability. Now, since $\mathbb{E}(X_F) \leqslant n^s p^t$, $e(\mathbb{G}(n,p)) \geqslant n^2p/3$ with high probability, and $\mu = \frac{t-1}{s-2}$, we obtain that with high probability

$$X_F \leqslant \frac{\mathbb{E}(X_F)}{\sqrt{\omega(n)}}$$

$$
\begin{aligned}
&\leqslant \frac{n^s p^t}{\sqrt{\omega(n)}} \\
&\leqslant \frac{3n^{s-2}p^{t-1}}{\sqrt{\omega(n)}} e\big(\mathbb{G}(n,p)\big) \\
&\leqslant \frac{3n^{s-2}}{\sqrt{\omega(n)}} \left(n^{-\frac{1}{\mu}}\omega(n)\right)^{t-1} e\big(\mathbb{G}(n,p)\big) \\
&\leqslant 3\big(\omega(n)\big)^{t-\frac{3}{2}} e\big(\mathbb{G}(n,p)\big)
\end{aligned}
$$

which means that $X_F = o(e(\mathbb{G}(n,p)))$ as $t \geqslant 2$. The proof is complete. □

Corollary 4.3. *Let F be a graph with $\mu(F) \geqslant 2$ and let $p = \frac{2c_F + o(1)}{n}$, where c_F is introduced in Theorem 2.7. Then,* $\mathrm{wsat}(\mathbb{G}(n,p),F) = \mathrm{wsat}(n,F)(1+o(1))$ *with high probability.*

Proof. As $p \gg n^{-2}$, it follows from [2, Theorem 4.4.4], Theorem 2.7, and Theorem 4.2 that

$$
\begin{aligned}
\mathrm{wsat}\big(\mathbb{G}(n,p),F\big) &= e\big(\mathbb{G}(n,p)\big)\big(1+o(1)\big) \\
&= \frac{n^2 p}{2}\big(1+o(1)\big) \\
&= \big(c_F + o(1)\big)n \\
&= \mathrm{wsat}(n,F)\big(1+o(1)\big)
\end{aligned}
$$

with high probability. □

Remark 4.4. It has been proved in [5] that for any fixed integer $s \geqslant 3$, there exists a positive constant λ such that

$$
\mathrm{wsat}\big(\mathbb{G}(n,p),K_s\big) \leqslant (s-2)n + \frac{\log^{\lambda} n}{p^{2s-3}} \tag{3}
$$

with high probability. Let F be a graph on s vertices with $\delta(F) \geqslant 1$. Since $\mathrm{wsat}(\mathbb{G}(n,p),F) \leqslant \mathrm{wsat}(\mathbb{G}(n,p),K_s)$, we find that the upper bound given in (3) also holds for $\mathrm{wsat}(\mathbb{G}(n,p),F)$. Using this and our results in the current paper, we deduce that the following holds for $\mathrm{wsat}(\mathbb{G}(n,p),F)$ with high probability.

- If $p \ll n^{-1/\mu(F)}$, then $\mathrm{wsat}(\mathbb{G}(n,p),F) = e(\mathbb{G}(n,p))(1+o(1))$.
- If $n^{-1/\mu(F)} \leqslant p \leqslant n^{-1/(s-1)}$, then $\mathrm{wsat}(\mathbb{G}(n,p),F) \leqslant e(\mathbb{G}(n,p))$.
- If $n^{-1/(s-1)} \ll p \leqslant n^{-1/(2s+1)}$, then $\mathrm{wsat}(\mathbb{G}(n,p),F) \leqslant (s-2)n + p^{-(2s-3)}\log^{\lambda} n$.
- If $p \geqslant n^{-1/(2s+1)}\log n$, then $\mathrm{wsat}(\mathbb{G}(n,p),F) \leqslant (\delta(F)-1)n + O(1)$.
- If p is constant, then $\mathrm{wsat}(\mathbb{G}(n,p),F) = \mathrm{wsat}(n,F)(1+o(1))$.

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