Percolating sets in bootstrap percolation on the Hamming graphs and triangular graphs

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Abstract

The r-neighbor bootstrap percolation on a graph is an activation process of the vertices. The process starts with some initially activated vertices and then, in each round, any inactive vertex with at least r active neighbors becomes activated. A set of initially activated vertices leading to the activation of all vertices is said to be a percolating set. Denote the minimum size of a percolating set in the r-neighbor bootstrap percolation process on a graph G by $m(G, r)$. In this paper, we present upper and lower bounds on $m(K_n^d, r)$, where K_n^d is the Cartesian product of d copies of the complete graph K_n which is referred as the Hamming graph. Among other results, when d goes to infinity, we show that $m(K_n^d, r) = \frac{1+o(1)}{(d+1)!} r^d$ if $r \gg d^2$ and $n \geqslant r+1$. Furthermore, we explicitly determine $m(L(K_n), r)$, where $L(K_n)$ is the line graph of K_n also known as triangular graph.

Keywords: Bootstrap percolation, Hamming graph, Percolating set, Triangular graph.

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1 Introduction

Bootstrap percolation process on graphs can be interpreted as a cellular automaton, a concept introduced by von Neumann [16]. It has been extensively investigated in several diverse fields such as combinatorics, probability theory, statistical physics and social sciences. The r-neighbor model is the most studied version of this process in the literature. It was introduced in 1979 by Chalupa, Leath and Reich [9]. In the r-neighbor bootstrap percolation process on a graph, first some vertices are initially activated and then, in each phase, any inactive vertex with at least r active neighbors becomes activated. Once a vertex becomes activated, it remains active forever. This process has also been treated in the literature under other names like irreversible threshold, influence propagation and dynamic monopoly.

Throughout this paper, all graphs are assumed to be finite, undirected, without loops and multiple edges. For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. For a vertex v of G, we set $N(v) = \{x \in V(G) \,|\, x \text{ is adjacent to } v\}$. The *degree* of v is defined to be $|N(v)|$. Given a nonnegative integer r and a graph G, the *r-neighbor bootstrap percolation process* on G begins with a subset A_0 of $V(G)$ whose elements are initially activated and then, at step i of the process, the set A_i of active vertices is

$$A_i = A_{i-1} \cup \left\{ v \in V(G) \,\middle|\, |N(v) \cap A_{i-1}| \geqslant r \right\}$$

for each $i \geqslant 1$. We say A_0 is a *percolating set* of G if $\bigcup_{i \geqslant 0} A_i = V(G)$. The main extremal problem here is to determine the minimum size of a percolating set which is denoted by $m(G, r)$. The size of percolating sets has been studied for various families of graphs such as hypercubes [15], grids [4, 6, 12], tori [12], trees [17] and random graphs [10, 13].

Let us fix some notation and terminology. The *Cartesian product* of two graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either $g_1 = g_2$ and h_1 is adjacent to h_2 or $h_1 = h_2$ and g_1 is adjacent to g_2. For any integer $n \geqslant 1$, we let $[\![n]\!] = \{0, 1, \ldots, n-1\}$ and we use the convention that $[\![0]\!] = \varnothing$. We denote the complete graph on n vertices by K_n and we consider $[\![n]\!]$ as the vertex set of K_n. Denote by K_n^d the Cartesian product of d vertex disjoint copies of K_n, that is, the Hamming graph of dimension d. The *line graph* of a graph G, written $L(G)$, is the graph whose vertex set is $E(G)$ and in which two vertices of $L(G)$ are adjacent if they share an endpoint. The line graph of a complete graph is known as a *triangular graph*.

Balister, Bollobás, Lee and Narayanan [1] gave the lower bound $(r/d)^d$ and the upper bound $r^d/(2d!)$ on $m(K_n^d, r)$. The lower bound follows from Theorem 2.3 of [1] and the upper bound is stated in [1] as a remark without proof. In this paper, we improve their lower bound by utilizing a polynomial technique introduced by Hambardzumyan, Hatami and Qian [12]. In order to improve the upper bound, we present a percolating set of K_n^d in the r-neighbor bootstrap percolation process. Letting $\delta = (d-2)/(d-1)$, we establish that

$$\frac{1}{r}\binom{d+r}{d+1} \leqslant m\big(K_n^d, r\big) \leqslant \frac{(r+2d-1)^d - \delta^2(r-2)^d}{2d!},$$

for any positive integers n, r, d with $n \geqslant r+1$ and $d \geqslant 2$. This in particular implies that, when d goes to infinity, $m(K_n^d, r) = \frac{1+o(1)}{(d+1)!} r^d$ if $r \gg d^2$ and $n \geqslant r+1$. It is worth to mention that a random version of the r-neighbor bootstrap percolation process on the Hamming graphs has been investigated in [11]. Among other results, we will present an exact formula for $m(L(K_n), r)$.

The paper is organized as follows. In Section 2, we determine $m(K_n^2, r)$. In Section 3, we apply a polynomial technique to find an auxiliary quantity which will be used to get a lower bound for $m(K_n^d, r)$. In Section 4, we present our upper and lower bounds for $m(K_n^d, r)$ which will result in an asymptotic formula for $m(K_n^d, r)$. An exact formula for $m(K_n^d, 2)$ and a tighter upper bound on $m(K_n^d, 3)$ will be given in Section 5. Finally, we present an exact formula for $m(L(K_n), r)$ in Section 6 as the last result of the paper.

2 Two-dimensional Hamming graphs

For any integers $n \geqslant 1$ and $r \geqslant 0$, it is clear that $m(K_n, r) = \min\{n, r\}$. In this section, we deal with the first nontrivial case, that is, the Hamming graph of dimension 2. More precisely, we derive an exact formula for $m(K_n^2, r)$. However, it is not easy to provide an exact formula for the same problem for higher dimensions which will be investigated in the subsequent sections. The result of this section will be the basis of an inductive proof for higher dimensions in Section 4.

If $n \leqslant \lceil r/2 \rceil$, then the degree of any vertex of K_n^2 is $2n-2 \leqslant r-1$. This implies that no vertex of K_n^2 can be activated by other vertices in the r-neighbor bootstrap percolation process. Therefore, any percolating set of K_n^2 consists of all vertices, meaning that $m(K_n^2, r) = n^2$. The following theorem resolves the remaining cases.

Theorem 2.1 *For any nonnegative integers n and r with $n \geqslant \lceil r/2 \rceil + 1$,*

$$m\big(K_n^2, r\big) = \left\lfloor \frac{(r+1)^2}{4} \right\rfloor.$$

Proof. We first present a percolating set of K_n^2 in the r-neighbor bootstrap percolation process. Let

$$V_{n,r} = \left\{(x, y) \in [\![n]\!]^2 \,\middle|\, x + (n-1-y) \leqslant \left\lceil \tfrac{r}{2} \right\rceil - 1 \text{ or } (n-1-x) + y \leqslant \left\lfloor \tfrac{r}{2} \right\rfloor - 1\right\}. \quad (1)$$

As an example, $V_{6,5}$ is shown in Figure 1.

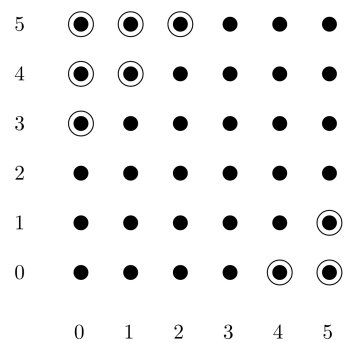

Figure 1. The set $V_{6,5}$ is outlined with circles drawn around its elements. The points' coordinates can be found using the numbers written on the left and bottom margins.

Since $n \geqslant \lceil r/2 \rceil + 1$, we have

$$|V_{n,r}| = \sum_{i=1}^{\lceil \frac{r}{2} \rceil} i + \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} i = \binom{\lceil \frac{r}{2} \rceil + 1}{2} + \binom{\lfloor \frac{r}{2} \rfloor + 1}{2} = \left\lfloor \frac{(r+1)^2}{4} \right\rfloor .$$

Note that $V_{n,r} \cap [\![n-1]\!]^2 = V_{n-1,r-2}$. We prove by induction on r that $V_{n,r}$ is a percolating set in the r-neighbor bootstrap percolation process on K_n^2. The statement is trivial for $r = 0, 1$. Let $r \geqslant 2$ and assume that the vertices in $V_{n,r}$ are initially activated. The points on the lines $x = n-1$ and $y = n-1$ become consecutively activated from top to bottom and from right to left, respectively. Remove from K_n^2 all the vertices in the set

$$L = \left\{ (x, y) \in [\![n]\!]^2 \,\middle|\, x = n-1 \text{ or } y = n-1 \right\}$$

to get K_{n-1}^2. By the induction hypothesis, $V_{n-1,r-2} = V_{n,r} \cap [\![n-1]\!]^2$ is a percolating set of K_{n-1}^2 in the $(r-2)$-neighbor bootstrap percolation process. Since each vertex in $[\![n-1]\!]^2$ has two additional active neighbors in L, we conclude that $V_{n-1,r-2} \cup L$ is a percolating set of K_n^2 in the r-neighbor bootstrap percolation process. This proves the assertion.

We next use induction on r to establish that any percolating set of K_n^2 in the r-neighbor bootstrap percolation process has at least $\lfloor (r+1)^2/4 \rfloor$ elements. The statement is trivially true for $r = 0, 1$. Let $r \geqslant 2$ and consider a percolating set A in the r-neighbor bootstrap percolation process on K_n^2. Without loss of generality, one may assume that $(n-1, n-1)$ is the first vertex in $[\![n]\!]^2 \setminus A$ that becomes activated. So, $(n-1, n-1)$ must have at least r initially activated neighbors in L, meaning that $|A \cap L| \geqslant r$. Remove from K_n^2 all vertices in L to get K_{n-1}^2. Since $A \cup L$ is a percolating set in the r-neighbor bootstrap percolation process on K_n^2 and each vertex in $[\![n-1]\!]^2$ has exactly two neighbors in L, we deduce that $A \cap [\![n-1]\!]^2$ is a percolating set of K_{n-1}^2 in the

$(r-2)$-neighbor bootstrap percolation process. It follows from the induction hypothesis that $|A \cap [\![n-1]\!]^2| \geqslant \lfloor (r-1)^2/4 \rfloor$. Therefore,

$$|A| \geqslant |A \cap L| + \left|A \cap [\![n-1]\!]^2\right| \geqslant r + \left\lfloor \frac{(r-1)^2}{4} \right\rfloor = \left\lfloor \frac{(r+1)^2}{4} \right\rfloor. \qquad \square$$

3 Polynomials and bootstrap percolation

The main result of this section is an exact formula for a quantity that gives rise to a lower bound for $m(K_n^d, r)$. We use a polynomial technique introduced by Hambardzumyan, Hatami and Qian [12] to determine $m(K_2^d, r)$. This quantity is the minimum size of a percolating set in another version of bootstrap percolation which is called 'graph bootstrap percolation' and is closely related to the r-neighbor bootstrap percolation. The notion of graph bootstrap percolation was introduced by Bollobás in 1968 under the name of 'weak saturation' [8] and was later studied in 2012 by Balogh, Bollobás and Morris [3]. We recall the formal definition here. Given two graphs G and H, the *H-bootstrap percolation process* on G begins with a subset E_0 of $E(G)$ whose elements are initially activated and then, at step i of the process, the set of active edges is

$$E_i = E_{i-1} \cup \left\{ e \in E(G) \;\middle|\; \begin{array}{l} \text{There exists a subgraph } H_e \text{ of } G \text{ such} \\ \text{that } H_e \text{ is isomorphic to } H,\ e \in E(H_e) \\ \text{and } E(H_e) \setminus \{e\} \subseteq E_{i-1}. \end{array} \right\}$$

for each $i \geqslant 1$. The set E_0 is called a *percolating set* of G provided $\bigcup_{i \geqslant 0} E_i = E(G)$. The minimum size of a percolating set in the H-bootstrap percolation process on G is said to be the *weak saturation number* of H in G and is denoted by $wsat(G, H)$. We refer to the S_{r+1}-bootstrap percolation as the *r-edge bootstrap percolation*, where S_{r+1} denotes the star graph with $r+1$ edges. For simplicity and following [12], we let $m_e(G, r) = wsat(G, S_{r+1})$. Roughly speaking, the r-edge bootstrap percolation can be considered as an edge analogue of the r-neighbor bootstrap percolation. It is worth to mention that 2-edge bootstrap percolation had been studied in 1984 by Lenormand and Zarcone under the name of 'bond percolation' [14]. We know from [15] that

$$m_e(G, r) \leqslant r\, m(G, r). \tag{2}$$

Using this inequality and by computing $m_e(K_n^d, r)$ in the current section, we will present a lower bound on $m(K_n^d, r)$ in the next section. We first state the following definition which slightly differs from the original definition in [12].

Definition 3.1 Let r be a nonnegative integer and let G be a graph equipped with a proper edge coloring $c : E(G) \longrightarrow \mathbb{R}$. Let $W_c(G, r)$ be the vector space over $\mathbb{R}$ consisting of all functions $\phi : E(G) \longrightarrow \mathbb{R}$ for which there exist polynomials $\{P_v(x)\}_{v \in V(G)}$ satisfying

(i) $\deg P_v(x) \leqslant r-1$ for any vertex $v \in V(G)$;

(ii) $P_u(c(uv)) = P_v(c(uv)) = \phi(uv)$ for each edge $uv \in E(G)$.

It is said that the polynomials $\{P_v(x)\}_{v\in V(G)}$ *recognize* ϕ. Notice that we adopt the convention that the degree of the zero polynomial is -1.

The following theorem provides an interesting linear algebraic lower bound on $m_e(G,r)$. Other surprising applications of vector spaces and polynomials for bootstrap percolation can be found in [1, 5, 15]. The method presented in the following theorem can be regarded as a special case of a general framework introduced in [5].

Theorem 3.2 (Hambardzumyan, Hatami, Qian [12]) *Let r be a nonnegative integer and let $c : E(G) \longrightarrow \mathbb{R}$ be a proper edge coloring of a graph G. Then $m_e(G,r) \geqslant \dim W_c(G,r)$.*

To have a sense on how the dimension of the vector space $W_c(G,r)$ is related to $m_e(G,r)$, one may consider a percolating set $E_0 \subseteq E(G)$ and observe the fact that if a function $\phi \in W_c(G,r)$ vanishes on E_0, then $\phi = 0$. This fact forces that $\dim W_c(G,r) \leqslant |E_0|$ by considering $W_c(G,r)$ as a subspace of the space of all arbitrary functions from $E(G)$ to $\mathbb{R}$. To observe the fact, note that throughout the process, for the newly activated edge $e \notin E_0$, at least one of the endpoints of e, say v, is incident to at least r already activated edges. This shows that the polynomial P_v has at least r distinct roots, implying $P_v = 0$ and thus $\phi(e) = 0$. Since E_0 is a percolating set of G, ϕ eventually vanishes on $E(G)$ which means that $\phi = 0$.

The rest of this section is dedicated to first find an appropriate proper edge coloring of K_n for which $\dim W_c(K_n,r)$ attains its maximum value which turns out to be equal to $m_e(K_n,r) = \binom{r+1}{2}$. Although the latter equality can be proved by a simple combinatorial argument, our approach using the above mentioned polynomial technique has the advantage that we can extend the argument to higher dimensions afterward. So, the next step will be introducing a proper edge coloring of K_n^d and calculating $\dim W_c(K_n,r)$. Doing this, we will have a lower bound on $m_e(K_n^d,r)$ which we will show that it is an upper bound as well.

Before proceeding, let us present here a direct combinatorial proof for the equality $m_e(K_n,r) = \binom{r+1}{2}$ for $n \geqslant r+1$. As the edges of a clique of size $r+1$ clearly forms a percolating set in the r-edge bootstrap percolation process on K_n, it is enough to prove that this number of edges is necessary. Suppose that $v_1,\ldots,v_n$ is an order of vertices of K_n that will be incident to at least r active edges during the process. Obviously, v_1 is incident to at least r initially active edges. The vertex v_2 needs to be incident to at least $r-1$ new initially active edges. With the same argument, we conclude that any percolating set is necessarily of size at least $r + (r-1) + \cdots + 1 = \binom{r+1}{2}$, as we aimed for.

Lemma 3.3 *For any positive integers n and r with $n \geqslant r+1$, there exists a proper edge coloring $c : E(K_n) \longrightarrow \mathbb{R}$ such that* $\dim W_c(K_n, r) \geqslant \binom{r+1}{2}$.

Proof. We introduce an edge coloring c and $\binom{r+1}{2}$ independent vectors in $W_c(K_n, r)$. Fix arbitrary distinct nonzero real numbers $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ and let $c(ij) = \gamma_i\gamma_j$ for any edge $ij \in E(K_n)$. Obviously, $c : E(K_n) \longrightarrow \mathbb{R}$ is a proper edge coloring of K_n. For each edge $uv \in E(K_n)$ with $u, v \in [\![r+1]\!]$, we define polynomials $P_0^{uv}(x), P_1^{uv}(x), \ldots, P_{n-1}^{uv}(x)$ as follows. For any $i \in [\![n]\!]$, let

$$P_i^{uv}(x) = \begin{cases} 0, & \text{if } i \in [\![r+1]\!] \setminus \{u, v\}; \\ \displaystyle\prod_{\substack{k \in [\![r+1]\!] \\ k \notin \{u,v\}}} \frac{x - \gamma_i\gamma_k}{\gamma_u\gamma_v - \gamma_i\gamma_k}, & \text{if } i \in \{u, v\}; \\ \displaystyle\prod_{\substack{k \in [\![r+1]\!] \\ k \notin \{u,v\}}} \frac{(x - \gamma_i\gamma_k)(\gamma_i - \gamma_k)}{\gamma_i(\gamma_u - \gamma_k)(\gamma_v - \gamma_k)}, & \text{if } i \in \{r+1, \ldots, n-1\}. \end{cases}$$

We have $\deg P_i^{uv}(x) \leqslant r - 1$ and $P_i^{uv}(c(ij)) = P_j^{uv}(c(ij))$. To see the latter equality, note that

$$P_i^{uv}(\gamma_i\gamma_j) = \begin{cases} 0, & \text{if } i \in [\![r+1]\!] \setminus \{u, v\}; \\ \displaystyle\prod_{\substack{k \in [\![r+1]\!] \\ k \notin \{u,v\}}} \frac{\gamma_j - \gamma_k}{\frac{\gamma_u\gamma_v}{\gamma_i} - \gamma_k}, & \text{if } i \in \{u, v\}; \\ \displaystyle\prod_{\substack{k \in [\![r+1]\!] \\ k \notin \{u,v\}}} \frac{(\gamma_i - \gamma_k)(\gamma_j - \gamma_k)}{(\gamma_u - \gamma_k)(\gamma_v - \gamma_k)}, & \text{if } i \in \{r+1, \ldots, n-1\}. \end{cases}$$

Now, using the symmetry between i and j, the equality $P_i^{uv}(\gamma_i\gamma_j) = P_j^{uv}(\gamma_i\gamma_j)$ is easily verified by just considering the following cases:

(i) $i \in [\![r+1]\!] \setminus \{u, v\}$. In this case, $P_i^{uv}(\gamma_i\gamma_j) = P_j^{uv}(\gamma_i\gamma_j) = 0$.

(ii) $\{i, j\} = \{u, v\}$. In this case, $P_i^{uv}(\gamma_i\gamma_j) = P_j^{uv}(\gamma_i\gamma_j) = 1$.

(iii) $i \in \{u, v\}$ and $j \in \{r+1, \ldots, n-1\}$.

(iv) $i, j \in \{r+1, \ldots, n-1\}$.

Define $\phi_{uv} : E(K_n) \longrightarrow \mathbb{R}$ as $\phi_{uv}(ij) = P_i^{uv}(c(ij))$. We saw above that the polynomials $\left\{P_i^{uv}(x)\right\}_{i\in[\![n]\!]}$ recognize ϕ_{uv}. Note that ϕ_{uv} vanishes on each edge ij with $i, j \in [\![r+1]\!]$ except on uv. From this, it follows that $\{\phi_{uv}\}_{u,v\in[\![r+1]\!]}$ is a linearly independent subset of $W_c(K_n, r)$. This completes the proof. □

Lemma 3.4 *Let n, r be positive integers and let $c : E(G) \longrightarrow \mathbb{R}$ be a proper edge coloring of a graph G. Then, there is a proper edge coloring $\widehat{c} : E(G\Box K_n) \longrightarrow \mathbb{R}$ such that*

$$\dim W_{\widehat{c}}(G\Box K_n, r) \geqslant \sum_{t=0}^{n-1} \dim W_c(G, r-t),$$

where $W_c(G, i)$ is interpreted as $\{0\}$ if $i < 0$.

Proof. Consider arbitrary distinct nonzero real numbers $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ such that none of the numbers $\gamma_i\gamma_j$ is in the image of c. For any two adjacent vertices $u = (g, i)$ and $v = (h, j)$ of $G\Box K_n$, define

$$\widehat{c}(uv) = \begin{cases} c(gh), & \text{if } i = j; \\ \gamma_i\gamma_j, & \text{if } g = h. \end{cases}$$

Fix $t \in [\![n]\!]$, a basis $\mathscr{B}_t$ for $W_c(G, r-t)$ and a function $\phi \in \mathscr{B}_t$. According to Definition 3.1, there exist polynomials $\{P_g^{\phi}(x)\}_{g\in V(G)}$ recognizing ϕ. Define polynomial $Q_u^{t,\phi}$ for any vertex $u = (g, i) \in V(G\Box K_n)$ as $Q_u^{t,\phi}(x) = P_g^{\phi}(x)\mathit{\Gamma}_i^t(x)$, where

$$\mathit{\Gamma}_i^t(x) = \prod_{\ell=0}^{t-1}(\gamma_i - \gamma_\ell)\left(\frac{x}{\gamma_i} - \gamma_\ell\right). \tag{3}$$

Note that $\mathit{\Gamma}_i^t(\gamma_i\gamma_j) = \mathit{\Gamma}_j^t(\gamma_i\gamma_j)$ for all i and j. Also, we know from Definition 3.1 that $P_g^{\phi}(c(gh)) = P_h^{\phi}(c(gh))$ for each edge $gh \in E(G)$. Hence, $Q_u^{t,\phi}$ and $Q_v^{t,\phi}$ have the same value on $\widehat{c}(uv)$ for any edge $uv \in E(G\Box K_n)$. This implies that $\{Q_u^{t,\phi}\}_{u\in V(G\Box K_n)}$ recognize a function $\mathit{\Psi}_{t,\phi} \in W_{\widehat{c}}(G\Box K_n, r)$.

Since we may choose the pair (t, ϕ) in $\sum_{t=0}^{n-1} \dim W_c(G, r-t)$ different ways, it remains to show that all functions $\mathit{\Psi}_{t,\phi}$ are linearly independent. Suppose that $\sum_{t,\phi} \lambda_{t,\phi}\mathit{\Psi}_{t,\phi} = 0$ for some scalars $\lambda_{t,\phi} \in \mathbb{R}$. Towards a contradiction, assume that τ is the smallest number from $[\![n]\!]$ such that $\lambda_{\tau,\phi} \neq 0$ for some ϕ. If $i < t$, then $\gamma_i - \gamma_i$ appears in the expression of $\mathit{\Gamma}_i^t$ given in (3) and so $\mathit{\Gamma}_i^t = 0$. This yields that $Q_{(g,\tau)}^{t,\phi} = 0$ for any integer $t > \tau$ and vertex $g \in V(G)$. Thus, for any two adjacent vertices $u = (g, \tau)$ and $v = (h, \tau)$ in $G\Box K_n$, we have

$$\sum_{t,\phi} \lambda_{t,\phi}\mathit{\Psi}_{t,\phi}(uv) = \sum_{t,\phi} \lambda_{t,\phi}Q_u^{t,\phi}\big(\widehat{c}(uv)\big) = \sum_{\phi\in\mathscr{B}_\tau} \lambda_{\tau,\phi}P_g^{\phi}\big(c(gh)\big)\mathit{\Gamma}_\tau^\tau\big(c(gh)\big) = 0.$$

Our assumption on $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ implies that $\Gamma_\tau^\tau(c(gh)) \neq 0$. Therefore,

$$\left(\sum_{\phi \in \mathscr{B}_\tau} \lambda_{\tau,\phi}\phi\right)(gh) = \sum_{\phi \in \mathscr{B}_\tau} \lambda_{\tau,\phi} P_g^\phi\big(c(gh)\big) = 0$$

for each edge $gh \in E(G)$. This is a contradiction, since $\mathscr{B}_\tau$ is a basis for $W_c(G, r-\tau)$. $\square$

Lemma 3.5 *Let n, r be positive integers and let G be a graph with all vertices of degree at least r. Then*

$$m_e(G \square K_n, r) \leqslant \sum_{t=0}^{n-1} m_e(G, r-t),$$

where $m_e(G, i)$ is interpreted as 0 if $i < 0$.

Proof. For any t with $0 \leqslant t \leqslant \min\{r, n-1\}$, consider the subgraph G_t of $G \square K_n$ induced by $\{(v,t) \in V(G \square K_n) \,|\, v \in V(G)\}$ which is clearly isomorphic to G. Also, consider a percolating set U_t of the minimum possible size in the $(r-t)$-edge bootstrap percolation process on G_t and activate its elements. We show that the edges of $G_0, \ldots, G_{n-1}$ become activated in the r-edge bootstrap percolation process consecutively. At first, the edges of G_0 become activated in the r-edge bootstrap percolation process, according to the definition of U_0. Let $t \geqslant 1$ and assume that the edges of $G_0, \ldots, G_{t-1}$ are activated. Since any vertex $(v,t) \in V(G_t)$ is incident to t active edges with endpoints in $\{(v,i) \,|\, 0 \leqslant i \leqslant t-1\}$, we conclude that the edges of G_t become activated in the r-edge bootstrap percolation process on G_t by considering U_t as the set of initially activated vertices. Hence, $\bigcup_{t \geqslant 0} U_t$ is a percolating set of size $\sum_{t=0}^{n-1} m_e(G, r-t)$ in the r-edge bootstrap percolation process on $G \square H$. $\square$

Theorem 3.6 *Let n, r, d be positive integers with $n \geqslant r+1$. Then $m_e(K_n^d, r) = \binom{d+r}{d+1}$.*

Proof. First, we prove by induction on d that there exists a proper edge coloring $c_d : E(G) \longrightarrow \mathbb{R}$ such that $\dim W_{c_d}(K_n^d, r) \geqslant \binom{d+r}{d+1}$. The case $d = 1$ is exactly Lemma 3.3. By Lemma 3.4 and the induction hypothesis, there is a proper edge coloring $c_d : E(K_n^d) \longrightarrow \mathbb{R}$ such that

$$\begin{aligned}
\dim W_{c_d}\Big(K_n^d, r\Big) &\geqslant \sum_{t=0}^{n-1} \dim W_{c_{d-1}}\Big(K_n^{d-1}, r-t\Big) \\
&\geqslant \sum_{t=0}^{r-1} \binom{d-1+r-t}{d} \\
&= \binom{d+r}{d+1}.
\end{aligned}$$

It follows from Theorem 3.2 that $m_e(K_n^d, r) \geqslant \binom{d+r}{d+1}$. Now, we establish by induction on d that $m_e(K_n^d, r) \leqslant \binom{d+r}{d+1}$. The edges of K_n with two endpoints in $[\![r+1]\!]$ clearly form a percolating set in the r-edge bootstrap percolation process on K_n and so there is nothing to prove for $d = 1$. By applying Lemma 3.5 and the induction hypothesis, we obtain that

$$\begin{aligned} m_e\Big(K_n^d, r\Big) &\leqslant \sum_{t=0}^{n-1} m_e\Big(K_n^{d-1}, r-t\Big) \\ &\leqslant \sum_{t=0}^{r-1} \binom{d-1+r-t}{d} \\ &= \binom{d+r}{d+1}. \end{aligned}$$

□

4 Multi-dimensional Hamming graphs

Balister, Bollobás, Lee and Narayanan [1] gave the lower bound $(r/d)^d$ and the upper bound $r^d/(2d!)$ on $m(K_n^d, r)$. Although the lower bound follows from Theorem 2.3 of [1], the upper bound is stated in [1] without a proof. In this section, we improve both bounds which result in an asymptotic formula for $m(K_n^d, r)$. Indeed, the previous section paved the way to reach a drastic improvement in the lower bound. We use a generalization of the construction given in Theorem 2.1 to attain the upper bound $r^d/(2d!)$, and then, in order to improve it, we carefully chop the corners of the construction in such a way that it will still remain a percolating set.

To begin with, let us fix the notation we shall use throughout this section. We set $d \geqslant 2$ and $\delta = (d-2)/(d-1)$. For a point $t = (t_1, \ldots, t_d) \in \{0,1\}^d$ and a subset $P \subseteq [\![n]\!]^d$, we define

$$P(t) = \left\{ (x_1, \ldots, x_d) \in [\![n]\!]^d \;\middle|\; \begin{array}{l} \text{There exists } (p_1, \ldots, p_d) \in P \text{ such that} \\ x_i = t_i(n-1-p_i) + (1-t_i)p_i \text{ for all } i. \end{array} \right\}. \tag{4}$$

Roughly speaking, $P(t)$ is a region in $[\![n]\!]^d$ congruent to P around the point $(n-1)t$ instead of the origin. Let

$$A_{n,r}^d = \left\{ (x_1, \ldots, x_d) \in [\![n]\!]^d \;\middle|\; \sum_{i=1}^d x_i \leqslant \left\lceil \tfrac{r}{2} \right\rceil - 1 \right\} \tag{5}$$

and

$$B_{n,r}^d = \left\{ (x_1, \ldots, x_d) \in [\![n]\!]^d \;\middle|\; x_1 + x_2 + \delta \sum_{i=3}^d x_i < \delta\left(\left\lceil \tfrac{r}{2} \right\rceil - 1\right) \right\}. \tag{6}$$

Note that $B^d_{n,r} \subseteq A^d_{n,r}$ as $\delta < 1$. Set

$$C^d_{n,r} = A^d_{n,r} \setminus B^d_{n,r}. \tag{7}$$

For the above sets, we define

$$\mathcal{A}^d_{n,r} = \bigcup_{t \in T^d} A^d_{n,r}(t), \tag{8}$$

$$\mathcal{B}^d_{n,r} = \bigcup_{t \in T^d} B^d_{n,r}(t)$$

and

$$\mathcal{C}^d_{n,r} = \bigcup_{t \in T^d} C^d_{n,r}(t),$$

where

$$T^d = \left\{ (t_1, \ldots, t_d) \in \{0,1\}^d \,\middle|\, t_1 = t_2 \right\}.$$

As an instance, $\mathcal{C}^3_{5,4}$ is depicted in Figure 2.

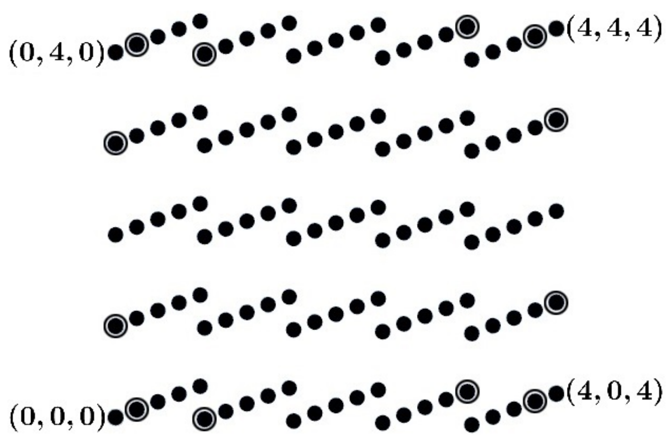

Figure 2. The set $\mathcal{C}^3_{5,4}$ is outlined with circles drawn around its elements. The points $(0,0,0)$, $(0,4,0)$, $(4,0,4)$ and $(4,4,4)$ form the set $\mathcal{B}^3_{5,4}$. The other points' coordinates can be found using these points' coordinates.

Notice that all $A^d_{n,r}(t)$ are pairwise disjoint under the condition $n \geqslant r+1$.

Lemma 4.1 *Let n, r, d be positive integers with $n \geqslant r+1$ and $d \geqslant 2$. Then $\mathcal{A}^d_{n,r}$ is a percolating set of K^d_n in the r-neighbor bootstrap percolation process.*

Proof. Let $s = \lceil r/2 \rceil$. We use an induction argument on d. We know from the proof of Theorem 2.1 that the set $V_{n,r}$, given in (1), is a percolating set of K_n^2 in the r-neighbor bootstrap percolation process. So, the set

$$\widehat{V}_{n,r} = \left\{ (x_1, x_2) \in [\![n]\!]^2 \,\middle|\, x_1 + (n-1-x_2) \leqslant \lceil \tfrac{r}{2} \rceil - 1 \text{ or } (n-1-x_1) + x_2 \leqslant \lceil \tfrac{r}{2} \rceil - 1 \right\}$$

is also a percolating set of K_n^2 in the r-neighbor bootstrap percolation process. Using (4), it is easy to check that

$$\widehat{V}_{n,r}(0,1) = \left\{ (x_1, x_2) \in [\![n]\!]^2 \,\middle|\, \begin{array}{l} x_1 + x_2 \leqslant \lceil \frac{r}{2} \rceil - 1 \text{ or} \\ (n-1-x_1) + (n-1-x_2) \leqslant \lceil \frac{r}{2} \rceil - 1 \end{array} \right\}$$

which is in turn equal to $\mathcal{A}^2_{n,r}$ by (8). This shows that the assertion is true for $d = 2$. Let $d \geqslant 3$ and assume that the assertion holds for $d-1$. Set $P_i = \{(x_1, \ldots, x_d) \in [\![n]\!]^d \mid x_d = i\}$ and $Q_i = P_i \cap \mathcal{A}^d_{n,r}$. We claim that, after ignoring the last coordinate, both Q_i and Q_{n-1-i} are exactly $\mathcal{A}^{d-1}_{n,r-2i}$ for every $i \in [\![s]\!]$.

To prove the claim, consider an arbitrary element $\alpha = (\alpha_1, \ldots, \alpha_d) \in P_i \cap A^d_{n,r}(t)$ for some $t = (t_1, \ldots, t_d) \in T^d$. By (4) and (5), there exists $(a_1, \ldots, a_d)$ with $\sum_{\ell=1}^d a_\ell \leqslant s-1$ such that $\alpha_\ell = t_\ell(n-1-a_\ell) + (1-t_\ell)a_\ell$ for $\ell = 1, \ldots, d$. Furthermore, $\alpha_d = i$, since $\alpha \in P_i$. If $t_d = 1$, then $i = n-1-a_d \geqslant n-1-(s-1) \geqslant s > i$, a contradiction. Therefore, $t_d = 0$ and so $a_d = i$. This means that $(\alpha_1, \ldots, \alpha_{d-1})$ is belong to

$$\left\{ (x_1, \ldots, x_{d-1}) \in [\![n]\!]^{d-1} \,\middle|\, \begin{array}{l} \text{There is } (a_1, \ldots, a_{d-1}) \text{ with } \sum_{\ell=1}^{d-1} a_\ell \leqslant s-1-i \\ \text{such that } x_\ell = t_\ell(n-1-a_\ell) + (1-t_\ell)a_i \text{ for all } \ell. \end{array} \right\}$$

which is equal to $A^{d-1}_{n,r-2i}(t_1, \ldots, t_{d-1})$. Conversely, if $(\alpha_1, \ldots, \alpha_{d-1}) \in A^{d-1}_{n,r-2i}(t)$ for some $t = (t_1, \ldots, t_{d-1}) \in T^{d-1}$, then $(\alpha_1, \ldots, \alpha_{d-1}, i) \in P_i \cap A^d_{n,r}(\hat{t})$, where $\hat{t} = (t_1, \ldots, t_{d-1}, 0) \in T^d$. This proves the claim on Q_i. A similar argument works for Q_{n-1-i}.

We consider the following iterative procedure for any $i \in [\![s]\!]$. At step i, we show that the vertices in $P_i \cup P_{n-1-i}$ become activated. The induction hypothesis implies that all vertices in P_0 and P_{n-1} are activated by Q_0 and Q_{n-1}, respectively. Hence, there is nothing to prove for $i = 0$. Assume that $i \geqslant 1$. Each vertex in $P_i \cup P_{n-1-i}$ has already $2i$ active neighbors from the previous steps. So, in order to activate the vertices in $P_i \cup P_{n-1-i}$, it is enough to consider the $(r-2i)$-neighbor bootstrap percolation process on $P_i \cup P_{n-1-i}$. This is done by the induction hypothesis and by considering $Q_i \cup Q_{n-1-i}$ as the initially activated set, since both Q_i and Q_{n-1-i} are copies of $\mathcal{A}^{d-1}_{n,r-2i}$.

Finally, we observe that any vertex in $\bigcup_{i=s}^{n-s-1} P_i$ has at least r neighbors in $\bigcup_{i=0}^{s-1}(P_i \cup P_{n-1-i})$ and so it becomes activated. This completes the proof, since $\bigcup_{i=0}^{n-1} P_i = [\![n]\!]^d$ and $\bigcup_{i=0}^{n-1} Q_i = \mathcal{A}^d_{n,r}$. □

Lemma 4.2 *Let n, r, d be positive integers with $n \geqslant r+1$ and $d \geqslant 2$. Then $\mathcal{C}_{n,r}^d$ is a percolating set of K_n^d in the r-neighbor bootstrap percolation process.*

Proof. By Lemma 4.1, it suffices to prove that all vertices in $\mathcal{B}_{n,r}^d$ become activated in the r-neighbor bootstrap percolation process on K_n^d. Note that once a vertex in $B_{n,r}^d$ becomes activated, the corresponding vertices in all other $B_{n,r}^d(t)$ become simultaneously activated, due to symmetry. So, it is sufficient to show that any vertex in $B_{n,r}^d$ becomes activated in the r-neighbor bootstrap percolation process on K_n^d. Since $B_{n,r}^2 = \varnothing$, we may assume that $d \geqslant 3$. Fix an arbitrary vertex $x = (x_1, \ldots, x_d) \in B_{n,r}^d$ and denote by η_x^i the number of neighbors of x in $\mathcal{C}_{n,r}^d$ differing from x in the coordinate i. Letting $\sigma_x = x_3 + \cdots + x_d$ and $s = \lceil r/2 \rceil$, it follows from (5)–(7) that

$$\begin{aligned}
\eta_x^1 &= \left| \left\{ (y, x_2, \ldots, x_d) \in [\![n]\!]^d \;\middle|\; \delta(s-1-\sigma_x) - x_2 \leqslant y \leqslant s-1-\sigma_x - x_2 \right\} \right| \\
&= s - 1 - \sigma_x - x_2 - \left\lceil \delta\left(s-1-\sigma_x\right) - x_2 \right\rceil + 1 \\
&= s - \sigma_x - \left\lceil \delta\left(s-1-\sigma_x\right) \right\rceil.
\end{aligned}$$

By symmetry, $\eta_x^2 = s - \sigma_x - \lceil \delta(s-1-\sigma_x) \rceil$. Also, by (5)–(7), η_x^3 is equal to the total number of points of the form $(x_1, x_2, y, x_4 \ldots, x_d)$ or $(x_1, x_2, n-1-y, x_4 \ldots, x_d)$ with

$$s - 1 - (\sigma_x - x_3) - \frac{x_1 + x_2}{\delta} \leqslant y \leqslant s - 1 - (\sigma_x - x_3) - x_1 - x_2.$$

So, it follows from $\delta = (d-2)/(d-1)$ that

$$\begin{aligned}
\eta_x^3 &= 2\left(s - 1 - (\sigma_x - x_3) - x_1 - x_2 - \left\lceil s - 1 - (\sigma_x - x_3) - \frac{x_1 + x_2}{\delta} \right\rceil + 1 \right) \\
&= 2\left(\left\lfloor \frac{x_1 + x_2}{d-2} \right\rfloor + 1 \right).
\end{aligned}$$

Again, by symmetry,

$$\eta_x^4 = \cdots = \eta_x^d = 2\left(\left\lfloor \frac{x_1 + x_2}{d-2} \right\rfloor + 1 \right)$$

whenever $d \geqslant 4$. Therefore, by letting $\eta_x = \eta_x^1 + \cdots + \eta_x^d$, we derive that

$$\eta_x = 2\Big(s - \sigma_x - \left\lceil \delta\left(s-1-\sigma_x\right) \right\rceil \Big) + 2(d-2)\left(\left\lfloor \frac{x_1 + x_2}{d-2} \right\rfloor + 1 \right).$$

Since $s \geqslant r/2$ and

$$\left\lfloor \frac{x_1 + x_2}{d-2} \right\rfloor \geqslant \frac{x_1 + x_2 - (d-3)}{d-2},$$

we obtain that $\eta_x \geqslant r - 2(\rho_x + \sigma_x)$, where $\rho_x = \lceil \delta(s-1-\sigma_x) \rceil - (x_1 + x_2 + 1)$. Note that $\rho_x \geqslant 0$ in view of the definition of $B^d_{n,r}$.

We now prove by induction on $\tau_x = \rho_x + 2\sigma_x$ that any vertex $x \in B^d_{n,r}$ becomes activated in the r-neighbor bootstrap percolation process on K^d_n. If $\tau_x = 0$, then $\rho_x = \sigma_x = 0$ and it follows from $\eta_x \geqslant r - 2(\rho_x + \sigma_x)$ that x has at least r active neighbors, we are done. So, we may assume that $\tau_x \geqslant 1$. In view of the inequality $\eta_x \geqslant r - 2(\rho_x + \sigma_x)$, it is sufficient to show that at least $2(\rho_x + \sigma_x)$ neighbors of x in $\mathcal{B}^d_{n,r}$ have been activated during the previous induction steps. For this, consider the sets

$$P_x = \bigcup_{i=1}^{2} \left\{ w \in [\![n]\!]^d \;\middle|\; \begin{array}{l} x \text{ and } w \text{ coincide in all coordinates except the} \\ i\text{th coordinate and } w_i \in \{x_i + 1, \ldots, x_i + \rho_x\}. \end{array} \right\},$$

$$Q_x = \bigcup_{i=3}^{d} \left\{ w \in [\![n]\!]^d \;\middle|\; \begin{array}{l} x \text{ and } w \text{ coincide in all coordinates except} \\ \text{the } i\text{th coordinate and } w_i \in [\![x_i]\!]. \end{array} \right\}$$

and

$$Q'_x = \bigcup_{i=3}^{d} \left\{ w \in [\![n]\!]^d \;\middle|\; \begin{array}{l} x \text{ and } w \text{ coincide in all coordinates except} \\ \text{the } i\text{th coordinate and } n-1-w_i \in [\![x_i]\!]. \end{array} \right\},$$

where $w = (w_1, \ldots, w_d)$. We have $P_x \cup Q_x \cup Q'_x \subseteq N(x) \cap \mathcal{B}^d_{n,r}$. Further, $\tau_w < \tau_x$ for each vertex $w \in P_x \cup Q_x$. Therefore, by the induction hypothesis and the symmetry of $\mathcal{B}^d_{n,r}$, we deduce that $P_x \cup Q_x \cup Q'_x$ is a set of active vertices of size $2(\rho_x + \sigma_x)$. Thus, x becomes activated, as required. $\square$

We need the following theorem in order to prove our result about the upper bound on $m(K^d_n, r)$.

Theorem 4.3 (Beged-Dov [7]) *Let $a_1, \ldots, a_k, b$ be positive numbers satisfying $b \geqslant \min\{a_1, \ldots, a_k\}$ and let N be the number of solutions of $a_1x_1 + \cdots + a_kx_k \leqslant b$ for the nonnegative integers $x_1, \ldots, x_k$. Then*

$$\frac{b^k}{k!a_1 \cdots a_k} \leqslant N \leqslant \frac{(a_1 + \cdots + a_k + b)^k}{k!a_1 \cdots a_k}.$$

Theorem 4.4 *Let n, r, d be positive integers with $n \geqslant r + 1$ and $d \geqslant 2$. Then*

$$\frac{1}{r}\binom{d+r}{d+1} \leqslant m\big(K^d_n, r\big) \leqslant \frac{(r+2d-1)^d - \delta^2(r-2)^d}{2d!},$$

where $\delta = (d-2)/(d-1)$.

Proof. The lower bound is obtained from (2) and Theorem 3.6. For the upper bound, note that $\mathcal{C}^d_{n,r}$ is a percolating set in the r-neighbor bootstrap percolation process on K^d_n by Lemma 4.2. It follows from $B^d_{n,r} \subseteq A^d_{n,r}$ and Theorem 4.3 that

$$\begin{aligned}\left|C^d_{n,r}\right| &= \left|A^d_{n,r}\right| - \left|B^d_{n,r}\right| \\ &\leqslant \frac{\left(d + \lceil \frac{r}{2} \rceil - 1\right)^d}{d!} - \frac{\left(\delta\left(\lceil \frac{r}{2} \rceil - 1\right)\right)^d}{d!\delta^{d-2}} \\ &\leqslant \frac{(r+2d-1)^d - \delta^2 (r-2)^d}{2^d d!}.\end{aligned}$$

As $|T^d| = 2^{d-1}$, we have

$$\left|\mathcal{C}^d_{n,r}\right| \leqslant \sum_{t \in T^d} \left|C^d_{n,r}(t)\right| \leqslant \frac{(r+2d-1)^d - \delta^2 (r-2)^d}{2d!}.$$

This proves the upper bound. □

Corollary 4.5 *Let $r \to \infty$, $n \geqslant r+1$ and $d = o(\sqrt{r})$. Then*

$$\frac{r^d}{(d+1)!}\big(1+o(1)\big) \leqslant m\big(K^d_n, r\big) \leqslant \frac{r^d (2d-3)}{2d!(d-1)^2}\big(1+o(1)\big).$$

In particular, if in addition $d \to \infty$, then $m(K^d_n, r) = \frac{1+o(1)}{(d+1)!} r^d$.

5 2-, 3-neighbor bootstrap percolation on Hamming graphs

In Section 2, we provided an exact formula for the special case $m(K^2_n, r)$. Here, we focus on another type of interesting special cases and try to find the exact formula for $m(K^d_n, r)$ when r is small. It is trivial that $m(K^d_n, 1) = 1$. In the next theorems, we precisely determine $m(K^d_n, 2)$ as the first nontrivial case, and moreover, we use a result given in [15] to derive a tighter upper bound on $m(K^d_n, 3)$. Finding an exact formula for $m(K^d_n, 3)$ remains an interesting open problem.

Before proceeding, for $i = 1, \ldots, d$, let e_i be the point in $\{0,1\}^d$ whose ith coordinate is 1 and whose other coordinates are 0. Recall that the *weight* of a tuple is defined to be the number of its nonzero entries.

Theorem 5.1 *Let n, d be positive integers with $n \geqslant 2$. Then $m(K^d_n, 2) = \lceil d/2 \rceil + 1$.*

Proof. We may assume that $d \geqslant 2$, since the assertion is obviously valid for $d = 1$. The inequality $m(K^d_n, 2) \geqslant \lceil d/2 \rceil + 1$ holds by Lemma 2.4 of [4] for $n = 2$ and by Theorem

4.4 for $n \geqslant 3$. In order to finish the proof, we present a percolating set of K_n^d of size $\lceil d/2 \rceil + 1$ in 2-neighbor bootstrap percolation process. For this, let $o = (0, \ldots, 0)$ and define

$$S = \begin{cases} T \cup \{o\}, & \text{if } d \text{ is even;} \\ T \cup \{o, e_d\}, & \text{if } d \text{ is odd,} \end{cases}$$

where

$$T = \left\{ e_{2i+1} + e_{2i+2} \,\middle|\, i \in [\![\lfloor \tfrac{d}{2} \rfloor]\!] \right\}.$$

Assume that the vertices in S are initially activated. As the first step, the vertices in $\{e_1, \ldots, e_d\} \setminus S$ become activated, since each of them has exactly two neighbors in S. Then, by the percolation rule, all remaining vertices of K_n^d of weight 1 become activated. After that, the remaining vertices of K_n^d of weights $2, \ldots, d$ become consecutively activated, since each vertex of the weight w is adjacent to w vertices of the weight $w - 1$. Therefore, S is a percolating set of K_n^d in 2-neighbor bootstrap percolation process. This completes the proof, since $|S| = \lceil d/2 \rceil + 1$. □

It is proven in [15] that $m(K_2^d, 3) = \lceil d(d+3)/6 \rceil + 1$. Using this and Theorem 4.4, we give lower and upper bounds on $m(K_n^d, 3)$ below.

Theorem 5.2 *Let n, d be positive integers with $n \geqslant 4$. Then*

$$\left\lceil \frac{d(d+5)}{6} \right\rceil + 1 \leqslant m(K_n^d, 3) \leqslant \left\lceil \frac{d(d+6)}{6} \right\rceil + 1.$$

Proof. The lower bound obviously holds for $d = 1$ and comes from Theorem 4.4 for $d \geqslant 2$. In order to establish the upper bound, it suffices to present a percolating set of K_n^d of size $\lceil d(d+6)/6 \rceil + 1$ in 3-neighbor bootstrap percolation process. By Theorem 1.4 of [15], there is a subset $S \subseteq \{0,1\}^d$ of size $\lceil d(d+3)/6 \rceil + 1$ such that all vertices in $\{0,1\}^d$ become activated in 3-neighbor bootstrap percolation process if the vertices in S have been initially activated. Let $T = \{(n-1)(e_{2i+1} + e_{2i+2}) \mid i \in [\![\lceil d/2 \rceil]\!]\}$, where it is assumed that $e_{d+1} = e_1$. We have

$$|S \cup T| = \left(\left\lceil \frac{d(d+3)}{6} \right\rceil + 1 \right) + \left\lceil \frac{d}{2} \right\rceil = \left\lceil \frac{d(d+6)}{6} \right\rceil + 1.$$

To end the proof, we show that $S \cup T$ is a percolating set of K_n^d in 3-neighbor bootstrap percolation process. For this, assume that the vertices in $S \cup T$ are initially activated. According to the selection of S, all vertices in $\{0,1\}^d$ become activated. We know from the proof of Theorem 2.1 that the set $V_{n,3}$, given in (1), is a percolating set of K_n^2 in the 3-neighbor bootstrap percolation process. Therefore, for any $i \in [\![\lceil d/2 \rceil]\!]$, by suitable translation and rotation, it is obtained that

$$W_i = \left\{ x e_{2i+1} + y e_{2i+2} \in [\![n]\!]^d \,\middle|\, x + y \leqslant 1 \text{ or } (n-1-x) + (n-1-y) = 0 \right\}$$

is a percolating set of the subgraph of K_n^d induced by $\{xe_{2i+1} + ye_{2i+2} \,|\, x, y \in [\![n]\!]\}$ in 3-neighbor bootstrap percolation process. Since all W_i are subsets of $S \cup T$, all vertices of K_n^d of weight 1 have become activated. As a next step, all remaining vertices of K_n^d of the form $xe_i + e_j$ become activated, since $xe_i + e_j$ is adjacent to the active vertices xe_i, e_j and $e_i + e_j$ if $x \notin \{0, 1\}$. After that, all remaining vertices of K_n^d of weight 2 become activated, since $xe_i + ye_j$ is adjacent to the active vertices xe_i, ye_j, $xe_i + e_j$ and $e_i + ye_j$ provided $x, y \notin \{0, 1\}$. Finally, all remaining vertices of K_n^d of weights $3, \ldots, d$ become consecutively activated, since each vertex of the weight w is adjacent to w vertices of the weight $w - 1$. This shows that $S \cup T$ is a percolating set of K_n^d in 3-neighbor bootstrap percolation process, completing the proof. □

6 Triangular graph

It is well known that K_n^2 is isomorphic to the line graph of $K_{n,n}$, the complete bipartite graph with equal parts of size n. Since $m(K_n^2, r)$ is determined in Section 2, it is natural to look for the same parameter for the line graph of K_n. In this section, we aim to compute $m(L(K_n), r)$. Note that one may view the activation of vertices in the r-neighbor bootstrap percolation process on $L(K_n)$ as the activation of edges of K_n. Therefore, by this correspondence, in each round of the r-neighbor bootstrap percolation process on $L(K_n)$, an edge $e \in E(K_n)$ becomes activated if the number of active edges incident to either of the endpoints of e is at least r.

Since the vertex degrees of $L(K_n)$ are all equal to $2n - 4$, they are at most $r - 1$ if $n \leqslant \lceil r/2 \rceil + 1$ and so in this case no percolation occurs in $L(K_n)$, forcing $m(L(K_n), r) = \binom{n}{2}$. The following theorem determines $m(L(K_n), r)$ for the remaining cases.

Theorem 6.1 *Let n, r be nonnegative integers with $n \geqslant \lceil r/2 \rceil + 2$. Then*

$$m\big(L(K_n), r\big) = \left\lfloor \frac{(r+2)^2}{8} \right\rfloor.$$

Proof. We consider the edge set of K_n as $\{(i, j) \,|\, i, j \in [\![n]\!] \text{ and } i < j\}$. First, we show that $m(L(K_n), r) \leqslant \lfloor (r+2)^2/8 \rfloor$. For this, define

$$E_{n,r} = \begin{cases} E'_{n,r} \cup E''_{n,r}, & \text{if } r \text{ is even;} \\ \\ E'_{n,r}, & \text{if } r \text{ is odd,} \end{cases}$$

where

$$E'_{n,r} = \Big\{ (i, j) \,\Big|\, i \in [\![\lceil \tfrac{r}{2} \rceil]\!] \text{ and } j \in \big\{ n - \lceil \tfrac{r}{2} \rceil + i, \ldots, n - 1 \big\} \Big\}$$

and

$$E''_{n,r} = \Big\{ \big(n - \tfrac{r}{2} + 2k - 1, n - \tfrac{r}{2} + 2k\big) \,\Big|\, k \in [\![\lceil \tfrac{r}{4} \rceil]\!] \Big\}.$$

The condition $n \geqslant \lceil r/2 \rceil + 2$ ensures that

$$|E_{n,r}| = |E'_{n,r}| + \varepsilon |E''_{n,r}| = \sum_{i=0}^{\lceil \frac{r}{2} \rceil - 1} \left(\left\lceil \frac{r}{2} \right\rceil - i \right) + \varepsilon \left\lceil \frac{r}{4} \right\rceil = \left\lfloor \frac{(r+2)^2}{8} \right\rfloor,$$

where

$$\varepsilon = \begin{cases} 1, & \text{if } r \text{ is even;} \\ 0, & \text{if } r \text{ is odd.} \end{cases}$$

For instance, $E_{8,6}$, $E_{8,7}$ and $E_{8,8}$ are depicted in Figure 3.

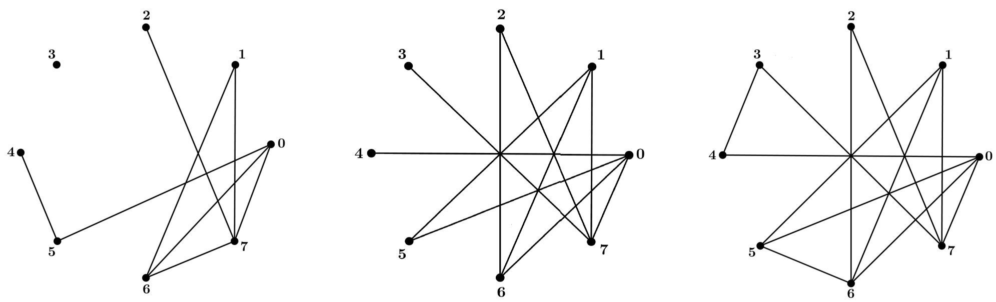

Figure 3. From left to right, the edges of K_8 which are contained in $E_{8,6}$, $E_{8,7}$ or $E_{8,8}$.

We claim that the activation of $E_{n,r}$ leads to the activation of $E(K_n)$ in the r-neighbor bootstrap percolation process on $L(K_n)$. To prove the claim, we first show that $E_{n,r} \cap [\![n-1]\!]^2 = E_{n-1,r-2}$. For this, we note that

$$\begin{aligned} E'_{n,r} \cap [\![n-1]\!]^2 &= \left\{ (i,j) \,\middle|\, i \in [\![\lceil \tfrac{r}{2} \rceil - 1]\!] \text{ and } j \in \{n - \lceil \tfrac{r}{2} \rceil + i, \dots, n-2\} \right\} \\ &= \left\{ (i,j) \,\middle|\, i \in [\![\lceil \tfrac{r-2}{2} \rceil]\!] \text{ and } j \in \{n - 1 - \lceil \tfrac{r-2}{2} \rceil + i, \dots, n-2\} \right\} \\ &= E'_{n-1,r-2} \end{aligned}$$

Moreover, if $r \equiv 0 \pmod 4$, then

$$\begin{aligned} E''_{n,r} \cap [\![n-1]\!]^2 &= \left\{ \left(n - \tfrac{r}{2} + 2k - 1, n - \tfrac{r}{2} + 2k\right) \,\middle|\, k \in [\![\lceil \tfrac{r}{4} \rceil]\!] \right\} \\ &= \left\{ \left(n - \tfrac{r-2}{2} + 2k - 2, n - 1 - \tfrac{r-2}{2} + 2k\right) \,\middle|\, k \in [\![\lceil \tfrac{r-2}{4} \rceil]\!] \right\} \\ &= E''_{n-1,r-2}, \end{aligned}$$

and, if $r \equiv 2 \pmod 4$, then

$$\begin{aligned} E''_{n,r} \cap [\![n-1]\!]^2 &= \left\{ \left(n - \tfrac{r}{2} + 2k - 1, n - \tfrac{r}{2} + 2k\right) \,\middle|\, k \in [\![\lceil \tfrac{r}{4} \rceil - 1]\!] \right\} \\ &= \left\{ \left(n - \tfrac{r-2}{2} + 2k - 2, n - 1 - \tfrac{r-2}{2} + 2k\right) \,\middle|\, k \in [\![\lceil \tfrac{r-2}{4} \rceil]\!] \right\} \\ &= E''_{n-1,r-2}. \end{aligned}$$

The above relations confirm that $E_{n,r} \cap [\![n-1]\!]^2 = E_{n-1,r-2}$. Now, we use an induction argument on r to prove the claim. Since $E_{n,0} = \varnothing$ and $E_{n,1} = \{(0, n-1)\}$, the claim is valid for $r = 0, 1$. So, assume that $r \geqslant 2$. For $i = 1, \ldots, n - \lceil r/2 \rceil$, we show that the number of edges in $E_{n,r}$ incident to the vertex $n - i$ is equal to

$$\begin{cases} \lceil \frac{r}{2} \rceil + \epsilon, & \text{if } i = 1; \\ \max\left\{ \lfloor \frac{r}{2} \rfloor - i + 2, 0 \right\}, & \text{if } 2 \leqslant i \leqslant n - \lceil \frac{r}{2} \rceil, \end{cases} \tag{9}$$

where

$$\epsilon = \begin{cases} 1, & \text{if } r \equiv 2 \pmod 4; \\ 0, & \text{otherwise.} \end{cases}$$

To see this, we note that, for $i = 1, \ldots, n - \lceil r/2 \rceil$, the set of edges in $E_{n,r}$ which are incident to the vertex $n - i$ is

$$\begin{cases} T_i, & \text{if either } r \text{ is odd or } i = 1 \text{ and } r \equiv 0 \pmod 4; \\ T_i \cup \{(n-i-1, n-i)\}, & \text{if } r \text{ is even and } \frac{r}{2} \equiv i \pmod 2; \\ T_i \cup \{(n-i, n-i+1)\}, & \text{otherwise,} \end{cases}$$

where

$$T_i = \left\{ (k, n-i) \,\middle|\, k \in [\![\lceil \tfrac{r}{2} \rceil - i + 1]\!] \right\}.$$

It follows from (9) that the number of edges in $E_{n,r}$ incident to one of the vertices $n-1$ and $n-2-\epsilon$ is equal to $(\lceil r/2 \rceil + \epsilon) + (\lfloor r/2 \rfloor - \epsilon) = r$. Therefore, as the first step, the edge with the endpoints $n-1$ and $n-2-\epsilon$ becomes activated. After that, the number of active edges incident to the vertex $n-1$ increases by 1 and thus, as the second step, the edge with the endpoints $n-1$ and $n-3-\epsilon$ becomes activated by a similar reasoning. Repeating this argument, all edges incident to $n-1$ become activated and so we may remove the vertex $n-1$ from K_n to use the induction hypothesis. Since each edge $(i, j) \in E(K_n)$ with $i, j \in [\![n-1]\!]$ is adjacent to the edges $(i, n-1)$ and $(j, n-1)$ in $L(K_n)$, we may consider the $(r-2)$-neighbor bootstrap percolation process on $L(K_{n-1})$ with the initially activated set $E_{n,r} \cap [\![n-1]\!]^2 = E_{n-1,r-2}$. Hence, the induction hypothesis implies that the activation of $E_{n-1,r-2}$ leads to the activation of $E(K_{n-1})$, proving the claim. So, we showed that $m(L(K_n),r) \leqslant \lfloor (r+2)^2/8 \rfloor$.

In order to end the proof, we need to establish that any percolating set in the r-neighbor bootstrap percolation process on $L(K_n)$ is of size at least $\lfloor (r+2)^2/8 \rfloor$. Fix a percolating set A of size $m(L(K_n),r)$ in the r-neighbor bootstrap percolation process on $L(K_n)$. We consider A as a set of edges of K_n. As we saw above, $|A| \leqslant \lfloor (r+2)^2/8 \rfloor$. Assume that $f : e_0, \ldots, e_{k-1}$ is an order in which the edges in $E(K_n) \setminus A$ become activated, where $k = \binom{n}{2} - |A|$. We find a maximal subsequence $g : e_{i_0}, \ldots, e_{i_{\ell-1}}$ of the sequence f as follows. Set $e_{i_0} = e_0$ and, after choosing $e_{i_0}, \ldots, e_{i_{t-1}}$, let e_{i_t} be the first edge in the sequence $e_{i_{t-1}+1}, \ldots, e_{k-1}$ which does not share an endpoint with any of $e_{i_0}, \ldots, e_{i_{t-1}}$.

We want to establish that $\ell \geqslant \lfloor r/4 \rfloor + 1$. Let $e_{i_t} = u_t v_t$ for any $t \in [\![\ell]\!]$. Since e_{i_t} becomes activated after the activation of $e_0, \ldots, e_{i_t-1}$, by the percolation rule, there must be at least r edges in $A \cup \{e_0, \ldots, e_{i_t-1}\}$ incident to one of u_t and v_t. As u_t and v_t are in total incident to $2n-3$ edges in K_n, the set X_t consists of the edges among $e_{i_t}, \ldots, e_k$ with an endpoint in $\{u_t, v_t\}$ is of size at most $2n-r-3$. By the definition of g, one concludes that $\{e_0, \ldots, e_{k-1}\} = \bigcup_{t=0}^{\ell-1} X_t$ and thus $k \leqslant \ell(2n-r-3)$. The assumption $n \geqslant \lceil r/2 \rceil + 2$ concludes that $2n-r-3 \geqslant 1$ and thus

$$\ell \geqslant \frac{k}{2n-r-3} \geqslant \frac{\binom{n}{2} - \frac{(r+2)^2}{8}}{2n-r-3} = \frac{1}{8}\left(2n+r+1-\frac{1}{2n-r-3}\right) \geqslant \frac{2n+r}{8} \geqslant \frac{r+2}{4}$$

which means that $\ell > \lfloor r/4 \rfloor$, as desired.

For every $t \in [\![\ell]\!]$, we show that each of u_t and v_t is incident to at most $2t$ edges among $e_0, \ldots, e_{i_t-1}$. To see this, let $e_h \in \{e_0, \ldots, e_{i_t-1}\}$ and $e_h = xy$ with $x \in \{u_t, v_t\}$. Then, $y \in \{u_0, \ldots, u_{t-1}, v_0, \ldots, v_{t-1}\}$, since, otherwise, in the process of determination of the subsequence g, we should have selected e_h which contradicts to the selection of e_{i_t} later in the process. So, for the activation of e_{i_t}, we need that the set Y_t consists of the edges in A with one endpoint in $\{u_t, v_t\}$ and the other endpoint not in $\{u_0, \ldots, u_{t-1}, v_0, \ldots, v_{t-1}\}$ to be of size at least $r-4t$. Since $Y_0, \ldots, Y_{\ell-1}$ are pairwise disjoint,

$$|A| \geqslant \sum_{t=0}^{\ell-1} |Y_t| \geqslant \sum_{t=0}^{\lfloor \frac{r}{4} \rfloor} (r-4t) = \left\lfloor \frac{(r+2)^2}{8} \right\rfloor,$$

completing the proof. □

7 Concluding remarks

In this paper, we computed the exact value of $m(K_n^2, r)$. We also determined $m_e(K_n^d, r)$ whenever $n \geqslant r+1$ and $d \geqslant 1$. Using this formula, an asymptotic formula for $m(K_n^d, r)$

was derived when both r and d go to infinity with $d = o(\sqrt{r})$ and $n \geqslant r+1$. However, finding an asymptotic formula for $m(K_n^d, r)$ when one of r and d is fixed and the other one goes to infinity is challenging to find. An asymptotic formula for $m(K_2^d, r)$ when r is fixed and d goes to infinity has been given in [15], settling a conjecture raised by Balogh and Bollobás in [2]. One may also think of finding the exact value of $m(K_n^d, r)$ for small r. Trivially, $m(K_n^d, 1) = 1$ and we know from Theorem 5.1 that $m(K_n^d, 2) = \lceil d/2 \rceil + 1$ for any $n \geqslant 2$. As we mentioned before, it is established in [15] that $m(K_2^d, 3) = \lceil d(d+3)/6 \rceil + 1$. In addition, we proved in Theorem 5.2 that $\lceil d(d+5)/6 \rceil + 1 \leqslant m(K_n^d, 3) \leqslant \lceil d(d+6)/6 \rceil + 1$ for any $n \geqslant 4$. It seems a challenging problem to find the exact value of $m(K_n^d, 3)$. As the last result, we obtained the exact value of $m(L(K_n), r)$. It would be an interesting problem to apply the polynomial technique raised in Section 3 to determine $m_e(L(K_n), r)$.

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