# Hadamard matrices with few distinct types

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**Abstract**

The notion of type of quadruples of rows is proven to be useful in the classification of Hadamard matrices. In this paper, we investigate Hadamard matrices with few distinct types. Among other results, the Sylvester Hadamard matrices are shown to be characterized by their spectrum of types.

**Keywords:** Hadamard matrix, Sylvester Hadamard matrix, type.

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## 1 Introduction

A *Hadamard matrix* of order $n$ is an $n \times n$ matrix $H$ with entries in $\{-1, 1\}$ such that $HH^\top = nI$, where $H^\top$ is the transpose of $H$ and $I$ is the $n \times n$ identity matrix. It is well known that the order of a Hadamard matrix is 1, 2, or a multiple of 4 [10]. It is a longstanding open question whether Hadamard matrices of order $n$ exist for every $n$ divisible by 4. The order 668 is the smallest for which the existence of a Hadamard matrix is open [7]. Hadamard matrices were first investigated in [11] by Sylvester who gave an explicit construction for Hadamard matrices of any order that is a power of 2. Such matrices were later considered by Hadamard as solutions to the problem of finding the maximum determinant of an $n \times n$ matrix with entries from the complex unit disk [2]. Since then, Hadamard matrices have been widely studied and have found many applications in combinatorics and other scientific areas [4].

Two Hadamard matrices are said to be *equivalent* if one can be obtained from the other by a sequence of row negations, row permutations, column negations, and column permutations. The complete classification of Hadamard matrices up to order 32, with respect to the equivalence relation, has been achieved by several authors. For references we refer to [5]. The resulting classification is shown in Table 1. As it can be seen from Table 1, a combinatorial explosion in the number of Hadamard matrices occurs in the order 32. Full classification in order 36 or more seems to be difficult and perhaps inaccessible.

| $n$ | 1 | 2 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 |
|---|---|---|---|---|---|---|---|---|---|---|
| # | 1 | 1 | 1 | 1 | 1 | 5 | 3 | 60 | 487 | 13710027 |

**Table 1.** The number of equivalence classes of Hadamard matrices of order $n \leqslant 32$.

In the above mentioned classifications, the authors associated an integer number, called the type, to any quadruple of rows of a Hadamard matrix. We give the definition of type in the next section. It seems that the notion of type deserves to be investigated to a greater extent. Apparently, Hadamard matrices with few distinct types are very rare and have nice combinatorial properties. For instance, the Sylvester Hadamard matrices have only two distinct types for quadruples of rows. Furthermore, there are five Hadamard matrices obtained from strongly regular graphs on 36 vertices with exactly two distinct types [9]. In this paper, we show that there exists no Hadamard matrix of order larger than 12 whose quadruples of rows are all of the same type. We then focus on Hadamard matrices with two distinct types. Among other results, it is established that the Sylvester Hadamard matrices are characterized by their spectrum of types.

## 2 Preliminaries

In this section, we fix our notation and present some preliminary results. We denote the zero vector and the all one vector of length $k$ by $\mathtt{0}_k$ and $\mathtt{1}_k$, respectively. A zero matrix is denoted by $\mathbf{0}$. For convenience, we respectively use the notation

$$\overset{r}{+} \text{ and } \overset{s}{-}$$

instead of

$$\underbrace{1 \cdots 1}_{r} \text{ and } \underbrace{-1 \cdots -1}_{s}\cdot$$

We drop the superscripts whenever there is no danger of confusion.

Let $H$ be a Hadamard matrix of order $n$. We know from [1] that, by a sequence of row negations, column negations, and column permutations, every four distinct rows $i, j, k, \ell$ of $H$ may be transformed to the form

$$\begin{array}{ccccccccccc} & & s & t & t & s & t & s & s & t \\ i & : & + & + & + & + & + & + & + & + \\ j & : & + & + & + & + & - & - & - & - \\ k & : & + & + & - & - & + & + & - & - \\ \ell & : & + & - & + & - & + & - & + & - \end{array} \qquad (1)$$

for some uniquely determined $s, t$ with $s + t = n/4$ and $0 \leqslant t \leqslant \lfloor n/8 \rfloor$. Following [8], we define the *type* of the four rows $i, j, k, \ell$ as $T_{ijk\ell} = t$. It is straightforward to check that $T_{ijk\ell} = \frac{n-P_{ijk\ell}}{8}$, where

$$P_{ijk\ell} = \left| \sum_{r=1}^{n} h_{ir}h_{jr}h_{kr}h_{\ell r} \right|$$

assuming that $h_{uv}$ is the $(u, v)$-entry of $H$. This in particular shows that 'type' is an equivalence invariant, meaning that any negation of rows and columns and any permutation of columns leaves the type unchanged.

The following lemma plays a key role in the sequel of paper.

**Lemma 1.** *Let $H$ be a Hadamard matrix of order $4m$. Fix three rows of $H$ and let $\kappa_t$ be the number of other rows which are of type $t$ with theses three rows. Then*

$$\sum_{t=0}^{\lfloor \frac{m}{2} \rfloor} \kappa_t (m - 2t)^2 = m^2.$$

*Proof.* Let $n = 4m$. Without loss of generality, assume that the fixed three rows of $H$ take the form

$$\begin{array}{cccc} m & m & m & m \\ + & + & + & + \\ + & + & - & - \\ + & - & + & - \end{array}$$

and have been put as the first three rows of $H$. Let $x^\top = (\mathsf{1}_m, \mathsf{0}_m, \mathsf{0}_m, \mathsf{0}_m)$. We deduce from (1) that $(Hx)^\top$ is of the form

$$\Big(m, m, m, \pm(m - 2T_{1234}), \pm(m - 2T_{1235}), \dots, \pm(m - 2T_{123n})\Big).$$

Since $(Hx)^\top(Hx) = x^\top H^\top Hx = nx^\top x = nm$, we obtain that

$$nm = (Hx)^\top(Hx) = 3m^2 + \sum_{t=0}^{\lfloor \frac{m}{2} \rfloor} \kappa_t (m - 2t)^2,$$

as desired. □

The following result originally proven in Proposition 2.1 of [8] is an easy consequence of Lemma 1.

**Corollary 2.** *Let $n \geqslant 8$ and $H$ be a Hadamard matrix of order $n$. If there exists a quadruple $\{i, j, k, \ell\}$ of rows of $H$ with $T_{ijk\ell} = 0$, then $n \equiv 0 \pmod 8$.*

The following result is a generalization of Lemma 2 of [6].

**Corollary 3.** *Let $n \geqslant 4$ and $H$ be a Hadamard matrix of order $n$. If there exist three distinct rows $i, j, k$ of $H$ such that all quadruples $\{i, j, k, \ell\}$ of rows are of the same type, then $n = 4$ or $n = 12$.*

*Proof.* Let $n = 4m$. Assume that for three distinct rows $i, j, k$ of $H$, four rows $i, j, k, \ell$ are of type $t$ for any $\ell \notin \{i, j, k\}$. By Lemma 1, we have $(n-3)(m-2t)^2 = m^2$. This means that $n^2$ is divisible by $n - 3$. Therefore, $9 = n^2 - (n^2 - 9)$ is divisible by $n - 3$ and we conclude that $n = 4$ or $n = 12$. □

**Corollary 4.** *Let $n \geqslant 4$ and $H$ be a Hadamard matrix of order $n$. If all quadruples of rows are of the same type, then $n = 4$ or $n = 12$.*

## 3 Hadamard matrices with two distinct types

In this section, we investigate Hadamard matrices whose types of quadruples of rows take few distinct values. By Corollary 4, any Hadamard matrix of order larger than 12 has at least two distinct types. Thus, it is natural to ask about Hadamard matrices with exactly two distinct types. We expect such matrices to be very rare and structurally nice. The complete classification of theses objects seems difficult. We here obtain some partial results. In particular, we examine the Hadamard matrices of order $n$ having types $\alpha$ and $\beta$ for any quadruple of rows with $(\alpha,\beta) \in \{(0,\frac{n}{8}),(1,\frac{n-4}{8}),(\frac{n}{16},\frac{n}{8})\}$. Note that these pairs of types satisfy the equation given in Lemma 1.

The following lemma is useful in eliminating some possible solutions of the equation stated in Lemma 1.

**Lemma 5.** *Let $i,j,k,p,q$ be five distinct rows of a Hadamard matrix of order $4m$. Then $T_{ijkp}+T_{ijkq} \geqslant m/2$. Moreover, if the equality occurs, then these five rows can be written as*

$$\begin{array}{cccccccccccccc}
 & & \frac{m}{2} & t' & s' & t & s & \frac{m}{2} & t & s & \frac{m}{2} & \frac{m}{2} & t' & s' \\
i & : & + & + & + & + & + & + & + & + & + & + & + & + \\
j & : & + & + & + & + & + & + & - & - & - & - & - & - \\
k & : & + & + & + & - & - & - & + & + & + & - & - & - \\
p & : & + & + & - & + & - & - & + & - & - & + & + & - \\
q & : & + & - & + & - & + & - & - & + & - & + & - & +
\end{array} \qquad (2)$$

*where $t = m/2 - t' = T_{ijkp}$ and $s = m/2 - s' = T_{ijkq}$.*

*Proof.* Without loss of generality, we may assume that

$$\begin{array}{cccccccccccccccccc}
 & & a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & b_4 & c_1 & c_2 & c_3 & c_4 & d_1 & d_2 & d_3 & d_4 \\
i & : & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
j & : & + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\
k & : & + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\
p & : & + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
q & : & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & -,
\end{array}$$

where $T_{ijkp} = a_3 + a_4$ and $T_{ijkq} = a_2 + a_4$. Let $t = T_{ijkp}$ and $s = T_{ijkq}$. It follows from (1) that

$$a_3 + a_4 = b_1 + b_2 = c_1 + c_2 = d_3 + d_4 = t,$$

$$a_2 + a_4 = b_1 + b_3 = c_1 + c_3 = d_2 + d_4 = s,$$

and

$$a_1+a_2+a_3+a_4 = b_1+b_2+b_3+b_4 = c_1+c_2+c_3+c_4 = d_1+d_2+d_3+d_4 = m.$$

Solving the equations above, we obtain that

$$\left\{\begin{array}{l} a_2 = m - t - a_1 \\ a_3 = m - s - a_1 \\ a_4 = t + s - m + a_1, \end{array}\right. \qquad \left\{\begin{array}{l} b_2 = t - b_1 \\ b_3 = s - b_1 \\ b_4 = m - t - s + b_1, \end{array}\right.$$
$$\left\{\begin{array}{l} c_2 = t - c_1 \\ c_3 = s - c_1 \\ c_4 = m - t - s + c_1, \end{array}\right. \qquad \left\{\begin{array}{l} d_2 = m - t - d_1 \\ d_3 = m - s - d_1 \\ d_4 = t + s - m + d_1. \end{array}\right. \qquad (3)$$

The inner product of two rows $p$ and $q$ is equal to $4(a_1+b_1+c_1+d_1-m)$. So the orthogonality of rows $p$ and $q$ implies that $a_1+b_1+c_1+d_1=m$. Since $a_4, d_4 \geqslant 0$, we deduce that both $a_1$ and $d_1$ are at least $m-t-s$. Therefore, $m \geqslant a_1+d_1 \geqslant 2(m-t-s)$ and so $t+s \geqslant m/2$, as desired.

If $t+s=m/2$, then $a_1+d_1=m$. As mentioned above, since $a_1$ and $d_1$ are at least $m-t-s$, we conclude that $a_1=d_1=m/2$. By $a_1+b_1+c_1+d_1=m$, we find that $b_1=c_1=0$. Now, the result follows from (3). □

**Theorem 6.** *There exists no Hadamard matrix of order* $16t$ *whose quadruples of rows are all of type* $t$ *or* $2t$.

*Proof.* Assume, in contradiction, that there exists a Hadamard matrix $H$ of order $n=16t$ whose quadruples of rows are all of type $t$ or $2t$. Let $\kappa_t$ and $\kappa_{2t}$ be the number of rows which respectively are of type $t$ and $2t$ with the first three rows. By applying Lemma 1, we find that $\kappa_t=4$ and $\kappa_{2t}=n-7$. Without loss of generality, we may assume that $T_{1234}=T_{1235}=T_{1236}=T_{1237}=t$. For any pair $p,q \in \{4,5,6,7\}$, since the equality holds in Lemma 5, five rows $1,2,3,p,q$ can be written as (2). Thus, it is straightforward to check that we necessarily have the following configuration:

$$\begin{array}{ccccccccccccccccccc}
 & & t & t & t & t & t & t & t & t & t & t & t & t & t & t & t & t \\
1 & : & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
2 & : & + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\
3 & : & + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\
4 & : & + & + & + & - & + & - & - & - & + & - & - & - & + & + & + & - \\
5 & : & + & + & - & + & - & + & - & - & - & + & - & - & + & + & - & + \\
6 & : & + & - & + & + & - & - & + & - & - & - & + & - & + & - & + & + \\
7 & : & - & + & + & + & - & - & - & + & - & - & - & + & - & + & + & +.
\end{array}$$

It turns out that $P_{4567}=n$ and so $T_{4567}=0$, a contradiction. □

It has been shown in [3] that there are exactly five equivalence classes of Hadamard matrices of order 16. We prove the following result without any reference to these equivalence classes.

**Corollary 7.** *Every Hadamard matrix of order* 16 *has four rows of type* 0.

*Proof.* By Lemma 1, for each triple $\{i,j,k\}$ of rows of a Hadamard matrix of order 16, one of the following holds:

(i) There are one row $\ell$ with $T_{ijk\ell}=0$ and twelve rows $\ell$ with $T_{ijk\ell}=2$.

(ii) There are four rows $\ell$ with $T_{ijk\ell}=1$ and nine rows $\ell$ with $T_{ijk\ell}=2$.

Now, the result follows from Theorem 6. $\square$

Recall that the *Hadamard product* of two $(-1,1)$-vectors $a=(a_1,\ldots,a_n)$ and $b=(b_1,\ldots,b_n)$ is defined as $a\circ b=(a_1b_1,\ldots,a_nb_n)$. We also define $\sigma(a)=|a_1+\cdots+a_n|$. It is not hard to check that

$$\sigma(a\circ b)\geqslant \sigma(a)+\sigma(b)-n. \tag{4}$$

Roughly specking, the following theorem states that there is no large gap between the types of quadruples of rows of a Hadamard matrix whose order is not a power of 2.

**Theorem 8.** *Let $H$ be a Hadamard matrix of order $n$ and let $r<n/16$. Suppose that for every three distinct rows $i,j,k$ of $H$, there exists a row $\ell$ with $T_{ijk\ell}\leqslant r$ and no row $x$ with $r<T_{ijkx}\leqslant 2r$. Then $n$ must be a power of* 2.

*Proof.* By Lemma 5, for every three distinct rows $i,j,k$ of $H$, there exists a unique row $\ell$ with $T_{ijk\ell}\leqslant r$. We call a set $\mathcal{S}$ of rows of $H$ to be 'full' if for every distinct rows $i,j,k\in\mathcal{S}$, the unique row $\ell$ with $T_{ijk\ell}\leqslant r$ is contained in $\mathcal{S}$. Trivially, $H$ has a full set of size 4. We claim that any full set of size $s<n$ can be extended to a full set of size $2s$. Clearly, the claim implies the assertion of the theorem.

Suppose that $\mathcal{S}=\{a_1,\ldots,a_s\}$ is a full set in $H$. Choose an arbitrary row $b_1$ in $H$ outside of $\mathcal{S}$ and, for $i=2,\ldots,s$, let $b_i$ be the unique row in $H$ such that $T_{a_1a_ib_1b_i}\leqslant r$. For $i=1,\ldots,s$, we may write $b_i=a_1\circ a_i\circ b_1\circ\beta_i$ for a suitable $(-1,1)$-vector $\beta_i$. Note that let $\beta_1=\mathbf{1}_n$, and for any $i\geqslant 2$,

$$\sigma(\beta_i)=\sigma(a_1\circ a_i\circ b_1\circ b_i)=n-8T_{a_1a_ib_1b_i}\geqslant n-8r. \tag{5}$$

Since $\mathcal{S}$ is a full set and $b_1$ is not in $\mathcal{S}$, $b_2,\ldots,b_s$ are not in $\mathcal{S}$. If $i\neq j$ and $b_i=b_j$, then $T_{a_1b_1b_i\ell}\leqslant r$ for $\ell=a_i$ and $\ell=a_j$ which is a contradiction. So, $\mathcal{S}'=\mathcal{S}\cup\{b_1,\ldots,b_s\}$ is of size $2s$. Now, we prove that $\mathcal{S}'$ is full. It clearly suffices to establish that

(i) $T_{a_ia_jb_ib_j}\leqslant r$, for every $1\leqslant i<j\leqslant s$;

(ii) $T_{a_ia_jb_kb_\ell}\leqslant r$, for any quadruple $\{a_i,a_j,a_k,a_\ell\}$ of type at most $r$;

(iii) $T_{b_ib_jb_kb_\ell}\leqslant r$, for any quadruple $\{a_i,a_j,a_k,a_\ell\}$ of type at most $r$.

In order to prove (i), in view of (4) and (5), we may write $n-8T_{a_ia_jb_ib_j}=\sigma(a_i\circ a_j\circ b_i\circ b_j)=\sigma(\beta_i\circ\beta_j)\geqslant\sigma(\beta_i)+\sigma(\beta_j)-n\geqslant n-16r$ which implies that

$T_{a_ia_jb_ib_j} \leqslant 2r$. Using the assumption of the theorem, we have $T_{a_ia_jb_ib_j} \leqslant r$, proving (i). Now, from (i), we find that

$$\sigma(\beta_i \circ \beta_j) = \sigma(a_i \circ a_j \circ b_i \circ b_j) = n - 8T_{a_ia_jb_ib_j} \geqslant n - 8r, \tag{6}$$

for every $1 \leqslant i < j \leqslant s$. Next, fix a quadruple $\{a_i, a_j, a_k, a_\ell\}$ of type at most $r$. In order to prove (ii), using (4) and (5), we may write

$$\begin{aligned} n - 8T_{a_ia_jb_kb_\ell} &= \sigma(a_i \circ a_j \circ b_k \circ b_\ell) \\ &= \sigma(a_i \circ a_j \circ a_k \circ a_\ell \circ \beta_i \circ \beta_j) \\ &\geqslant \sigma(a_i \circ a_j \circ a_k \circ a_\ell) + \sigma(\beta_i \circ \beta_j) - n \\ &\geqslant n - 16r, \end{aligned}$$

yielding $T_{a_ia_jb_kb_\ell} \leqslant 2r$. The assumption of the theorem results in $T_{a_ia_jb_kb_\ell} \leqslant r$, proving (ii). Note that $\sigma(a_i \circ a_j \circ a_k \circ a_\ell \circ \beta_i \circ \beta_j) = \sigma(a_i \circ a_j \circ b_k \circ b_\ell) = n - 8T_{a_ia_jb_kb_\ell} \geqslant n - 8r$. The latter inequality along with (6) give

$$\begin{aligned} n - 8T_{b_ib_jb_kb_\ell} &= \sigma(b_i \circ b_j \circ b_k \circ b_\ell) \\ &= \sigma(a_i \circ a_j \circ a_k \circ a_\ell \circ \beta_i \circ \beta_j \circ \beta_k \circ \beta_\ell) \\ &\geqslant \sigma(a_i \circ a_j \circ a_k \circ a_\ell \circ \beta_i \circ \beta_j) + \sigma(\beta_k \circ \beta_\ell) - n \\ &\geqslant n - 16r, \end{aligned}$$

implying $T_{b_ib_jb_kb_\ell} \leqslant 2r$. It follows from the assumption of the theorem that $T_{b_ib_jb_kb_\ell} \leqslant r$. This proves (iii) and completes the proof. □

The following consequence immediately follows from Theorem 8.

**Corollary 9.** *Let $H$ be a Hadamard matrix of order $n$ such that for every three distinct rows $i, j, k$ of $H$, there exists a row $\ell$ with $T_{ijk\ell} < n/24$. Then $n$ is a power of 2.*

Consider a Hadamard matrix $H$ of order $n > 12$. Assume that $n = 4m$ is not a power of 2 and $H$ has exactly two distinct types $\alpha$ and $\beta$ for the quadruples of rows with $\alpha < \beta$. By Lemma 1 and Corollary 3, we have

$$m^2 = \kappa_\alpha(m-2\alpha)^2 + \kappa_\beta(m-2\beta)^2 < (\kappa_\alpha + \kappa_\beta)(m-2\alpha)^2 = (4m-3)(m-2\alpha)^2,$$

which implies that $\alpha < \frac{n}{8}\big(1 - \frac{1}{\sqrt{n-3}}\big)$. Similarly, $\beta > \frac{n}{8}\big(1 - \frac{1}{\sqrt{n-3}}\big)$. In conclusion, by Corollary 9, we obtain

$$\frac{n}{24} \leqslant \alpha < \frac{n}{8}\left(1 - \frac{1}{\sqrt{n-3}}\right) < \beta \leqslant \frac{n}{8}.$$

We recall that the *Sylvester Hadamard matrices* are recursively defined as follows:

$$\mathsf{H}_1 = [1] \quad \text{and} \quad \mathsf{H}_{2^r} = \begin{bmatrix} \mathsf{H}_{2^{r-1}} & \mathsf{H}_{2^{r-1}} \\ \mathsf{H}_{2^{r-1}} & -\mathsf{H}_{2^{r-1}} \end{bmatrix} \quad \text{for } r = 1, 2, \ldots$$

It follows from Theorem 4 of [1] that every quadruple of rows of $\mathsf{H}_{2^r}$ is of type 0 or $2^{r-3}$ for all $r \geqslant 3$. We below show that the converse is also true.

**Theorem 10.** *Let $H$ be a Hadamard matrix of order $8t$ whose quadruples of rows are all of type $0$ or $t$. Then $H$ is equivalent to the Sylvester Hadamard matrix.*

*Proof.* Fix three rows of $H$ and let $\kappa_0$ and $\kappa_t$ be the number of other rows which respectively are of type 0 and $t$ with the these fixed rows. By applying Lemma 1, we find that $\kappa_0 = 1$ and $\kappa_t = n - 4$, where $n = 8t$. It is easy to see that, for every triple $\{i, j, k\}$ of rows of $H$, the vector $i \circ j \circ k$ is equal to the unique row $\ell$ in $H$ with $T_{ijk\ell} = 0$ up to negation. This means that if we write the first three rows of $H$ as the form

$$\begin{matrix} 2t & 2t & 2t & 2t \\ + & + & + & + \\ + & + & - & - \\ + & - & + & - \end{matrix}$$

then we may consider

$$\begin{matrix} 2t & 2t & 2t & 2t \\ + & - & - & + \end{matrix}$$

as the forth row of $H$. By a sequence of column permutations, we may consider the $4 \times n$ top submatrix of $H$ as

$$\left[\ \mathsf{H}_4 \,\middle|\, \cdots \,\middle|\, \mathsf{H}_4\ \right].$$

In order to proceed, assume that $n$ is divisible by $2^r$, for some $r \geqslant 2$, and the $2^r \times n$ top submatrix of $H$ is written as

$$\left[\ \mathsf{H}_{2^r} \,\middle|\, \cdots \,\middle|\, \mathsf{H}_{2^r}\ \right].$$

Again, by a sequence of column permutations, we may consider the $2^r \times n$ top submatrix of $H$ as

$$H' = \Big[\ \underbrace{K_1 \cdots K_1}_{\frac{n}{2^r}} \,\Big|\, \cdots \,\Big|\, \underbrace{K_{2^r} \cdots K_{2^r}}_{\frac{n}{2^r}}\ \Big], \tag{7}$$

where $K = \mathsf{H}_{2^r}$ and $K_i$ is the $i$th column of $K$ for $i = 1, \ldots, 2^r$. Let

$$x \; : \; x_1 \quad \cdots \quad x_{2^r}$$

be any of the remaining rows of $H$. In view of (7), by a column permutation, we may assume that

$$\begin{array}{ccc} & \alpha_i & \beta_i \\ x_i \; : & + & - \end{array}$$

for any $i$. Since $H'x^\top = 0$, it is not hard to see that

$$K \left[ \begin{array}{c} \alpha_1 - \beta_1 \\ \hline \vdots \\ \hline \alpha_{2^r} - \beta_{2^r} \end{array} \right] = \mathbf{0}.$$

As $K$ is an invertible matrix, we conclude that $\alpha_i = \beta_i$ for any $i$. Thus, we may rewrite the first $2^r + 1$ rows of $H$ in the form

$$\begin{array}{cccccc} K & \cdots & K & K & \cdots & K \\ + & \cdots & + & - & \cdots & -. \end{array}$$

For any $i \in \{2, 3, \ldots, 2^r\}$, $H$ has a unique row $\rho_i = 1 \circ i \circ \rho_1$ corresponding to the rows 1, $i$, and $\rho_1 = 2^r + 1$ with $T_{1i\rho_1\rho_i} = 0$. So, one can easily deduce that the first $2^{r+1}$ rows of $H$ have the form

$$\begin{array}{cccccc} K & \cdots & K & K & \cdots & K \\ K & \cdots & K & -K & \cdots & -K. \end{array}$$

This shows in particular that $n$ is divisible by $2^{r+1}$. Also, by a sequence of column permutations, we may consider the $2^{r+1} \times n$ top submatrix of $H$ as

$$\left[ \; \mathsf{H}_{2^{r+1}} \; \middle| \; \cdots \; \middle| \; \mathsf{H}_{2^{r+1}} \; \right].$$

Now, the assertion clearly follows by repeating the above process. □

The following result is an analogue of Theorem 10 and is easily derived from Corollary 9.

**Corollary 11.** *Let $H$ be a Hadamard matrix of order $n = 8t + 4$ whose quadruples of rows are all of type 1 or $t$. Then $n \in \{4, 12, 20\}$.*

## 4 Concluding remarks

Let us summarize our results. We classified Hadamard matrices of order $n$ having two types $\alpha$ and $\beta$ with $(\alpha, \beta) \in \{(0, \frac{n}{8}), (1, \frac{n-4}{8}), (\frac{n}{16}, \frac{n}{8})\}$. In the case $(\alpha, \beta) = (0, \frac{n}{8})$, only Sylvester Hadamard matrices exist. If $(\alpha, \beta) = (1, \frac{n-4}{8})$, then $n = 12, 20$, and there exists no example if $(\alpha, \beta) = (\frac{n}{16}, \frac{n}{8})$. These results along with our Lemma 1 and Tables 1–2 in [5] are enough to obtain the number of equivalence classes of Hadamard matrices of order up to 32 with exactly two types. The findings are shown in Table 2. In attempt to

| $n$ | 8 | 16 | 20 | 24 | 28 | 32 |
|---|---|---|---|---|---|---|
| # | 1 | 1 | 3 | 1 | 1 | 2 |

**Table 2.** The number of Hadamard matrices of order $n \leqslant 32$ with exactly two types.

find more examples of Hadamard matrices with exactly two types, we looked at the Paley Hadamard matrices of order $p + 1$ for primes $p$ less than 200. Examples were only in the orders $8, 20, 24, 32, 44, 60$. The search within the Paley Hadamard matrices of order $2(p + 1)$ for primes $p$ less than 100 gave no examples. The definition of Paley Hadamard matrices can be found in [4].

The classification of Hadamard matrices with exactly two distinct values for type of quadruples of rows seems to be a hard problem. Even, in order 36 the problem is already challenging. We carried out a non-exhaustive computer search for Hadamard matrices of order 36 having types 3 and 4. We obtained only five such matrices which had previously been found in [9]. It is an interesting question if there exists an infinite family of Hadamard matrices with exactly two distinct types besides the Sylvester Hadamard matrices.

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