Some constructions of integral graphs

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Abstract

A graph is called integral if all eigenvalues of its adjacency matrix consist entirely of integers. Integral graphs are very rare and difficult to find. In this paper, we introduce some general methods for constructing such graphs. As a consequence, some infinite families of integral graphs are obtained.

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1 Introduction

Let G be a graph with the vertex set $\{v_1, \ldots, v_n\}$. The *adjacency matrix* of G is an $n \times n$ matrix $\mathcal{A}(G)$ whose (i, j)-entry is 1 if v_i is adjacent to v_j and is 0, otherwise. The zeros of the characteristic polynomial of $\mathcal{A}(G)$ are called the *eigenvalues* of G. The graph G is said to be *integral* if all the eigenvalues of G are integers. The notion of integral graphs was first introduced in [8]. Recently, integral graphs have found applications in quantum networks allowing perfect state transfer [12]. For a survey on integral graphs, we refer the reader to [3].

Integral graphs are very rare and difficult to find. For instance, out of 164,059,830,476 connected graphs on 12 vertices, there exist exactly 325 integral graphs [4]. In general, it seems impossible to give a complete characterization of integral graphs. This has led researchers to investigate integral graphs within restricted classes of graphs such as cubic graphs, 4-regular graphs, complete multipartite graphs and circulant graphs [3, 13, 15]. Meanwhile, some infinite

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families of integral graphs have been acquired through the generalizations of small integral graphs [14, 16, 17]. In this paper, we introduce some general methods based on the Kronecker product for constructing integral graphs. Using these methods, we present some infinite families of integral graphs.

Let us to recall some definitions and notation to be used throughout the paper. For a graph G, the *complement* of G, denoted by $\overline{G}$, is a graph on the vertex set of G such that two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. We denote the complete graph on n vertices and the complete bipartite graph with two parts of sizes m and n, by K_n and $K_{m,n}$, respectively. The $n \times n$ identity matrix and the $m \times n$ all one matrix will be denoted by I_n and $J_{m\times n}$, respectively, with the convention J_n instead of $J_{n\times n}$. We also drop the subscripts whenever there is no danger of confusion. If $A = [a_{ij}]$ is an $m \times n$ matrix and B is an $r \times s$ matrix, then the *Kronecker product* $A \otimes B$ is defined as the $mr \times ns$ matrix with the block form

$$\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

It is easy to see that there exists a permutation matrix P such that $A \otimes B = P(B \otimes A)P^{-1}$. Also, if $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_n$ are all eigenvalues of two square matrices A and B, respectively, then $\lambda_i \mu_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, are eigenvalues of $A \otimes B$.

2 A construction using commuting matrices

Some graph operations such as the Cartesian product and the strong product may be used to generate new integral graphs from given ones [8]. In this section, we give a new general method for constructing integral graphs using the Kronecker product and commuting sets of matrices with integral eigenvalues. One may use this construction to obtain some new infinite families of integral graphs. In the following, we give some examples.

Proposition 1. *Let $\mathcal{A} = \{A_1, \ldots, A_k\}$ be a commuting set of the adjacency matrices of k mutually edge-disjoint integral graphs on the same vertex set. Let $B_1, \ldots, B_k$ be symmetric $(0, 1)$-matrices with integral eigenvalues such that $\mathcal{B} = \{B_1, \ldots, B_k\}$ is a commuting set. Then the graph with the adjacency matrix $A_1 \otimes B_1 + \cdots + A_k \otimes B_k$ is integral.*

Proof. Obviously, $A_1 \otimes B_1 + \cdots + A_k \otimes B_k$ is the adjacency matrix of a graph which is integral, since $\{A_1 \otimes B_1, \ldots, A_k \otimes B_k\}$ is a commuting set of matrices whose eigenvalues are all integers. □

The following provides some examples of the sets $\mathcal{A}$ and $\mathcal{B}$ which satisfy the conditions of Proposition 1. Applying these examples to Proposition 1, one may obtain numerous infinite families of integral graphs. Notice that any instance of $\mathcal{A}$ may be used with any instance of $\mathcal{B}$ to find integral graphs.

Example 1. Given a regular integral graph G, one can take $\{\mathcal{A}(G), \mathcal{A}(\overline{G})\}$ as $\mathcal{A}$ in Proposition 1.

Example 2. Given a regular integral graph G, any subset of

$$\{I, J, J - I, \mathcal{A}(G), \mathcal{A}(\overline{G}), I + \mathcal{A}(G), I + \mathcal{A}(\overline{G})\}$$

may be used as $\mathcal{B}$ in Proposition 1.

Example 3. In [1, 2], it is shown that K_m ($K_{m,m}$) is decomposable into perfect matchings with mutually commuting adjacency matrices if and only if m is a power of 2. Using these results, we find some instances of $\mathcal{A}$ in Proposition 1. For an example, if $m = 2$, then we obtain two commuting matrices

$$A_1 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & J_2 - I_2 \\ J_2 - I_2 & 0 \end{bmatrix}$$

with integral eigenvalues. By Proposition 1, it is easily seen that for any positive integer n, $A_1 \otimes I_n + A_2 \otimes J_n$ represents an integral graph which is isomorphic to the Cartesian product of K_2 and $K_{n,n}$.

Example 4. Since any two circulant matrices of the same order commute, every set of the adjacency matrices of integral circulant graphs on the same vertex set is an example of $\mathcal{B}$ in Proposition 1. An explicit criterion for integrality of circulant graphs is given in [13].

3 The form $A \otimes I + B \otimes J$

In this section, we study integral graphs whose adjacency matrices have the form $A \otimes I + B \otimes J$, where A and B are the adjacency matrices of two edge-disjoint graphs. We first establish a theorem which has a crucial role in our proofs. For this, we need to recall the following theorem from linear algebra.

Theorem 1. [10, Theorem 1] *Let m and n be two positive integers and for any $i, j \in \{1, \ldots, n\}$, let A_{ij} be $m \times m$ matrices over a commutative ring that commute pairwise. Then*

$$\det \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} = \det \left(\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)} \right),$$

where S_n is the set of all permutations of $\{1, \ldots, n\}$.

Theorem 2. *Let A and B be two real matrices of the same order. Then $\det(A \otimes I_n + B \otimes J_n) = (\det A)^{n-1} \det(A + nB)$ for all positive integer n.*

Proof. Let $A = [a_{ij}]$ and $B = [b_{ij}]$. We have $\det(A \otimes I_n + B \otimes J_n) = \det[a_{ij} I_n + b_{ij} J_n]$. Since the block-entries of the matrix $[a_{ij} I_n + b_{ij} J_n]$ are commuting matrices, using Theorem 1 and the equality $J_n^k = n^{k-1} J_n$ for positive integer k, it is not very hard to see that

$$\det[a_{ij} I_n + b_{ij} J_n] = \det\left((\det A) I_n + \left(\frac{\det(A + nB) - \det A}{n}\right) J_n\right).$$

Since eigenvalues of J_n are 0 with multiplicity $n - 1$ and n with multiplicity 1, we conclude that

$$\det(A \otimes I_n + B \otimes J_n) = (\det A)^{n-1} \left(\det A + n\left(\frac{\det(A + nB) - \det A}{n}\right)\right) = (\det A)^{n-1} \det(A + nB),$$

as required. □

Corollary 1. *Let $n \geqslant 2$ and let A and B be two real matrices of the same order. Then all eigenvalues of $A \otimes I_n + B \otimes J_n$ are integers if and only if all eigenvalues of A and $A + nB$ are integers.*

Proof. By Theorem 2, we have

$$\det\big(xI - (A \otimes I_n + B \otimes J_n)\big) = \det\big((xI - A) \otimes I_n - B \otimes J_n\big) = \big(\det(xI - A)\big)^{n-1} \det\big(xI - (A + nB)\big),$$

which clearly implies the assertion. □

Example 5. The *Seidel matrix* of a graph Γ is defined as $\mathcal{A}(\overline{\Gamma}) - \mathcal{A}(\Gamma)$. Let G be a graph whose Seidel matrix has only integral eigenvalues. Then by Corollary 1, $\mathcal{A}(G) \otimes I_2 + \mathcal{A}(\overline{G}) \otimes (J_2 - I_2)$ represents a regular integral graph.

Lemma 1. *Let $r \neq 1$ be an integer and G be a k-regular graph on m vertices. Then all eigenvalues of the matrix*

$$M_r = \begin{bmatrix} \mathcal{A}(G) & I \\ I & \mathcal{A}(\overline{G}) \end{bmatrix} + r \begin{bmatrix} \mathcal{A}(\overline{G}) & 0 \\ 0 & \mathcal{A}(G) \end{bmatrix}$$

are integers if and only if $k = (m - 1)/2$ and $(r - 1)^2(2\lambda + 1)^2 + 4$ is a perfect square for every eigenvalue λ of G except k.

Proof. Applying Theorem 1, we have

$$\begin{aligned}\det(xI - M_r) =& \det\left[\begin{array}{c|c}(x+r)I - rJ + (r-1)\mathcal{A}(G) & -I \\ \hline -I & (x+1)I - J - (r-1)\mathcal{A}(G)\end{array}\right]\\
=& \det\Big(\big(x^2 + (r+1)x + r - 1\big)I - \big(x(r+1) - r(m-2) - k(r-1)^2\big)J\\
& -(r-1)^2\big(\mathcal{A}(G) + \mathcal{A}(G)^2\big)\Big)\\
=&\big(x^2 - (r+1)(m-1)x + r - 1 + mr(m-2) - k(k-m+1)(r-1)^2\big)\\
& \times \prod \big(x^2 + (r+1)x + r - 1 - (r-1)^2(\lambda + \lambda^2)\big),\end{aligned}$$

where the product runs over all the eigenvalues of G except k. The polynomial given in the first parenthesis in the last equality has integral roots if and only if $(r-1)^2(m-2k-1)^2+4$ is a perfect square. This implies that $k = (m-1)/2$. Also, the roots of $x^2+(r+1)x+r-1-(r-1)^2(\lambda+\lambda^2)$ are integers if and only if $(r-1)^2(2\lambda+1)^2+4$ is a perfect square, as desired. □

For a graph G and an integer $n \geqslant 2$, the graphs with the adjacency matrices

$$\begin{bmatrix}\mathcal{A}(G) & I\\ I & \mathcal{A}(\overline{G})\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}\mathcal{A}(G) & I\\ I & \mathcal{A}(\overline{G})\end{bmatrix} \otimes I_n + \begin{bmatrix}\mathcal{A}(\overline{G}) & 0\\ 0 & \mathcal{A}(G)\end{bmatrix} \otimes J_n$$

are denoted by $\Phi_0(G)$ and $\Phi_n(G)$, respectively. We mention that $\Phi_0(G)$ has been studied in [6] to characterize a family of graphs with specified eigenvalues. Combining Corollary 1 and Lemma 1, we obtain the following result.

Corollary 2. *Let $n \neq 1$ and G be a k-regular graph on m vertices. Then $\Phi_n(G)$ is integral if and only if $k = (m-1)/2$ and the numbers $(2\lambda+1)^2+4$ and $(n-1)^2(2\lambda+1)^2+4$ are perfect squares for every eigenvalue λ of G except k.*

A *strongly regular graph* with parameters (n, k, λ, μ) is a k-regular graph on n vertices such that any two adjacent vertices have λ common neighbors and any two nonadjacent vertices have μ common neighbors. A strongly regular graph with parameters $(n, (n-1)/2, (n-5)/4, (n-1)/4)$ is called a *conference graph*.

Theorem 3. *Let $n \neq 1$ and G be a connected strongly regular graph. Then $\Phi_n(G)$ is integral if and only if G is a conference graph on $m = 16s^2 + 24s + 5$ vertices for some integer $s \geqslant 0$ such that $m(n-1)^2 + 4$ is a perfect square.*

Proof. We know from [7, p. 219] that the distinct eigenvalues of a conference graph Γ on v vertices are $(v-1)/2$ and $(-1 \pm \sqrt{v})/2$. Hence for any eigenvalue λ of Γ except the largest one, we have $(2\lambda+1)^2 = v$. Therefore, if G is a conference graph on $m = 16s^2 + 24s + 5$ vertices for some $s \geqslant 0$ such that $m(n-1)^2 + 4$ is a perfect square, then all the necessary conditions in Corollary 2 hold and so $\Phi_n(G)$ is an integral graph. Now, we prove the converse. Let k be the degree of a strongly regular graph G and λ_1, λ_2 be the other distinct eigenvalues of G. By Corollary 2, $(2\lambda_i + 1)^2 + 4$ is a perfect square, thus λ_i is not integer, for $i = 1, 2$. Hence the multiplicities of λ_1 and λ_2 are equal and applying [7, Lemma 10.3.2], we deduce that G is a conference graph. Moreover, Corollary 2 implies that $m + 4$ and $m(n-1)^2 + 4$ are perfect squares, where m is the number of vertices of G. Since m is odd, $m + 4 = (2r + 1)^2$ for some positive integer r. To end the proof, it suffices to prove that r is odd. From [5], we know that m is the sum of two perfect squares. Furthermore, it is well known a positive integer a is a sum of two perfect squares if and only if all prime factors of a congruent to 3 modulo of 4 appear an even number of times in the factorization of a [9, p. 605]. Now, since m is the product of two relatively prime integers $2r - 1$ and $2r + 3$, one can easily conclude that r must be odd, as required. □

We recall that the *Fibonacci sequence* is recursively defined as $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$ with initial terms $F_0 = F_1 = 1$.

Example 6. It is well known that $5a^2 + 4$ for a positive integer a is a perfect square if and only if $a = F_{2k+1}$ for some integer $k \geqslant 0$ [9, p. 620]. Hence, if C_5 is the 5-cycle and $n \in \{0\} \cup \{F_{2k+1} + 1 \,|\, k \geqslant 0\}$, then by Theorem 3, $\Phi_n(C_5)$ is integral. Note that $\Phi_0(C_5)$ is the Petersen graph.

Example 7. Using the fact mentioned in Example 6, it is not hard to see that $45a^2 + 4$ for a positive integer a is a perfect square if and only if $a = F_{4k+3}/3$ for some integer $k \geqslant 0$. Thus, if G is one of the conference graphs on 45 vertices [11] and $n \in \{0\} \cup \{F_{4k+3}/3 + 1 \,|\, k \geqslant 0\}$, then by Theorem 3, $\Phi_n(G)$ is integral.

Remark 1. Note that if G is a conference graph on $16s^2 + 24s + 5$ vertices for some integer $s \geqslant 0$, then by Theorem 3, $\Phi_0(G)$ and $\Phi_2(G)$ are integral.

4 The form $\mathcal{A}(G) \otimes I + \mathcal{A}(\overline{G}) \otimes J$

For a graph G and an integer $n \geqslant 2$, we denote by $\Psi_n(G)$ the graph with the adjacency matrix $\mathcal{A}(G) \otimes I_n + \mathcal{A}(\overline{G}) \otimes J_n$. In this section, we investigate some graphs G for which $\Psi_n(G)$ are integral. Note that by Corollary 1, $\Psi_n(G)$ is integral if and only if all eigenvalues of $\mathcal{A}(G) + k\mathcal{A}(\overline{G})$ are integers for $k = 0, n$.

Example 8. Let G be any of the two integral graphs depicted in Figure 1. By an easy calculation, we find that for any integer n, the distinct eigenvalues of $\mathcal{A}(G_1) + n\mathcal{A}(\overline{G_1})$ and $\mathcal{A}(G_2) + n\mathcal{A}(\overline{G_2})$ are

$$-1, n-2, -3n+2, \frac{(2n+3) \pm \sqrt{5(2n+1)^2 + 4}}{2}$$

and

$$-1, n-2, -2n+1, \frac{(2n+3) \pm \sqrt{5(2n+1)^2+4}}{2},$$

respectively. It follows from Corollary 1 that $\Psi_n(G)$ is integral if and only if $5(2n+1)^2+4$ is a perfect square. Hence, by the fact mentioned in Example 6, $\Psi_n(G)$ is integral if and only if $2n+1 = F_{2\ell+1}$ for some integer $\ell \geqslant 0$ and it is not hard to check that this occurs if and only if $n \in \{(F_\ell - 1)/2 \,|\, \ell \equiv 1, 3 \pmod 6\}$.

Figure 1: Two integral graphs on 7 vertices.

Theorem 4. *Let $n \geqslant 2$, r, s be positive integers. Then the graph $\Psi_n(K_{r,s})$ is integral if and only if rs and $n^2(r-s)^2 + 4rs$ are perfect squares.*

Proof. We have

$$\det\Big(xI - \big(\mathcal{A}(K_{r,s}) + k\mathcal{A}(\overline{K_{r,s}})\big)\Big) = \det\left[\begin{array}{c|c} xI - k(J-I) & -J_{r\times s} \\ \hline -J_{s\times r} & xI - k(J-I) \end{array}\right]$$
$$= (x+k)^{r+s-2}\big(x^2 - k(r+s-2)x + k^2(r-1)(s-1) - rs\big).$$

We mention that in order to compute the above determinant, one can add a suitable multiple of the sum of the first r rows of the matrix to the last s rows to triangularize the matrix. By Corollary 1, $\Psi_n(K_{r,s})$ is integral if and only if all the roots of the above polynomial are integers for $k = 0, n$. Indeed, this occurs if and only if rs and $n^2(r-s)^2 + 4rs$ are perfect squares. $\square$

Remark 2. Notice that the numerical conditions of Theorem 4 are fulfilled for infinitely many integers r, s, n. For instance, if t is an arbitrary positive integer, then taking $r = t^2$, $s = (t+1)^2$ and

$$n = \begin{cases} \dfrac{(t+1)(2t-1)}{2}, & \text{if } t \text{ is odd;} \\[2ex] \dfrac{t(2t+3)}{2}, & \text{if } t \text{ is even;} \end{cases}$$

we find an infinite family of non-regular integral graphs.

Theorem 5. *Let $n \geqslant 2$, r, s be positive integers. If $H_{r,s}$ is the graph with the adjacency matrix*

$$\begin{bmatrix} J - I & J_{r\times s} \\ J_{s\times r} & 0 \end{bmatrix},$$

then $\Psi_n(H_{r,s})$ is integral if and only if $(r-1)^2+4rs$ and $(ns-n-r+1)^2+4rs$ are perfect squares.

Proof. By Corollary 1, it suffices to determine when all the roots of the polynomial

$$\det\Big(xI-\big(\mathcal{A}(H_{r,s})+k\mathcal{A}(\overline{H_{r,s}})\big)\Big)=\det\left[\begin{array}{c|c}(x+1)I-J & -J_{r\times s}\\ \hline -J_{s\times r} & (x+k)I-kJ\end{array}\right]$$
$$=(x+1)^{r-1}(x+k)^{s-1}$$
$$\times\big(x^2-(sk-k+r-1)x+k(r-1)(s-1)-rs\big)$$

are integers for $k=0,n$. This happens if and only if $(r-1)^2+4rs$ and $(ns-n-r+1)^2+4rs$ are perfect squares. □

Remark 3. We state that the numerical conditions of Theorem 5 are satisfied for infinitely many integers r,s,n. For example, if $(r,s,n)=(t,2t-1,t+1)$ where t is an arbitrary positive integer, then we find an infinite family $\{\mathcal{H}_t\}_{t\geqslant 2}$ of non-regular integral graphs. We recall that for a graph Γ on v vertices, the *Laplacian matrix* of Γ is defined as $\mathcal{L}(\Gamma)=\mathcal{D}(\Gamma)-\mathcal{A}(\Gamma)$, where $\mathcal{D}(\Gamma)$ denotes the diagonal matrix whose entries are the vertex degrees of Γ. The graph Γ is called *Laplacian integral* if all eigenvalues of its Laplacian matrix consist entirely of integers. With a similar calculation as we do in Theorem 5, one can verify that $\{\mathcal{H}_t\}_{t\geqslant 2}$ is also an infinite family of non-regular Laplacian integral graphs. The graph $\mathcal{H}_2$ is indicated in Figure 2.

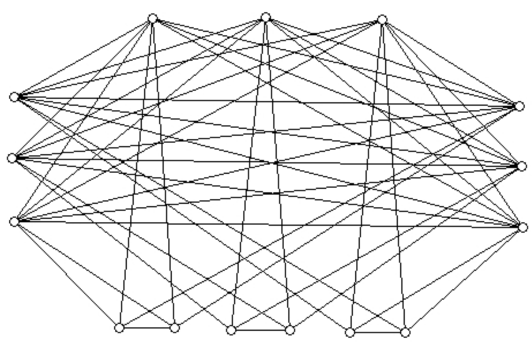

Figure 2: An integral graph on 15 vertices obtained from Theorem 5.

Remark 4. It is worth to mention that $\mathcal{A}(\overline{\Psi_n(G)})=(I+\mathcal{A}(G))\otimes(J_n-I_n)$, for every graph G and any integer $n\geqslant 2$. Hence G is integral if and only if $\overline{\Psi_n(G)}$ is integral. In particular, the complement of any integral graph obtained from Theorems 4 and 5 is also integral.

5 More block form constructions

In this section, we investigate the existence of more integral graphs whose adjacency matrices have the form $A \otimes I + B \otimes J$, where A and B have nice block structures. We start with the following theorem which makes use of the adjacency matrices of the complete graph and the complete bipartite graph as A and B, respectively.

Theorem 6. *Let $n \geqslant 2$, r, s be positive integers. Then the graph with adjacency matrix*

$$\begin{bmatrix} 0 & J_{r\times s} \\ J_{s\times r} & 0 \end{bmatrix} \otimes I_n + \begin{bmatrix} J_r - I_r & 0 \\ 0 & 0 \end{bmatrix} \otimes J_n$$

is integral if and only if rs and $n^2(r-1)^2 + 4rs$ are perfect squares.

Proof. Using Corollary 1, it is enough to determine when all the roots of the polynomial

$$\det\left(xI - \begin{bmatrix} k(J-I) & J_{r\times s} \\ J_{s\times r} & 0 \end{bmatrix}\right) = x^{s-1}(x+k)^{r-1}\big(x^2 - k(r-1)x - rs\big)$$

are integers for $k = 0, n$. Obviously, this holds if and only if rs and $n^2(r-1)^2 + 4rs$ are perfect squares. □

Theorem 7. *Let $n \geqslant 2$, r, s be positive integers. Then the graph with adjacency matrix*

$$\begin{bmatrix} 0 & J_{r\times s} \\ J_{s\times r} & 0 \end{bmatrix} \otimes I_n + \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \otimes (J_n - I_n)$$

is integral if and only if $1 + 4rs$ and $(n-1)^2 + 4rs$ are perfect squares.

Proof. Applying Corollary 1, it suffices to determine when all the roots of the polynomial

$$\det\left(xI - \begin{bmatrix} (k-1)I & J_{r\times s} \\ J_{s\times r} & 0 \end{bmatrix}\right) = x^{s-1}(x-k+1)^{r-1}\big(x^2 - (k-1)x - rs\big)$$

are integers for $k = 0, n$. For this, $1 + 4rs$ and $(n-1)^2 + 4rs$ must be perfect squares. □

Remark 5. It is easily checked that the graph constructed in Theorem 7 whenever $n = 2$ is integral if and only if rs is the product of two consecutive numbers. The infinite family of integral graphs obtained from Theorem 7 for $n = 2$ is first introduced in [17, Corollary 5(1)].

In the following, we present two instances of integral graphs with other variations of block forms of the adjacency matrices.

Theorem 8. *Let $n \geqslant 2$, r, s be positive integers. Then the graph with adjacency matrix*

$$\begin{bmatrix} 0 & J_{r\times s} & 0 \\ J_{s\times r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes I_n + \begin{bmatrix} 0_r & 0 & 0 \\ 0 & 0 & I_s \\ 0 & I_s & 0 \end{bmatrix} \otimes J_n$$

is integral if and only if rs and $n^2 + rs$ are perfect squares.

Proof. By Corollary 1, it suffices to determine when all the roots of the polynomial

$$\det \left(xI - \begin{bmatrix} 0 & J_{r\times s} & 0 \\ J_{s\times r} & 0 & kI \\ 0 & kI & 0 \end{bmatrix} \right) = x^r (x^2 - k^2)^{s-1} (x^2 - k^2 - rs)$$

are integers for $k = 0, n$. This occurs if and only if rs and $n^2 + rs$ are perfect squares. □

Theorem 9. *Let $n \geqslant 2$, r, s, t be positive integers. Then the graph with adjacency matrix*

$$\begin{bmatrix} 0 & J_{r\times s} & 0 \\ J_{s\times r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes I_n + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & J_{s\times t} \\ 0 & J_{t\times s} & 0 \end{bmatrix} \otimes J_n$$

is integral if and only if rs and $n^2 ts + rs$ are perfect squares.

Proof. Using Corollary 1, we need to determine when all the roots of the polynomial

$$\det \left(xI - \begin{bmatrix} 0 & J_{r\times s} & 0 \\ J_{s\times r} & 0 & kJ_{s\times t} \\ 0 & kJ_{t\times s} & 0 \end{bmatrix} \right) = x^{r+s+t-2} (x^2 - k^2 ts - rs)$$

are integers for $k = 0, n$. This happens if and only if rs and $n^2 ts + rs$ are perfect squares. □

Remark 6. Notice that in all theorems in this section, if we put $n = 1$, then we find some integral graphs if and only if the other parameters satisfy only the second numerical condition. For example, the stars K_{1,t^2} for any positive integer t and the graphs obtained in Theorem 8 whenever $n = r = 1$ consist the only integral trees with exactly one vertex of degree more than two [18].

Remark 7. It is well known that for two integers x and y, $x^2 + y^2$ is a perfect square if and only if one of x and y has form $2abc$ and the other has form $c(a^2 - b^2)$ for some integers a, b, c [9, p. 584]. Using this fact, we can find infinitely many integral graphs through the theorems of this section.

As a final result, we give another family of integral graphs whose adjacency matrices have a more complicated block form. It shows that one can construct new infinite families of integral graphs by considering similar block form matrices.

Theorem 10. *Let $n \geqslant 2$, r, s, t be positive integers. Then the graph with adjacency matrix*

$$\left[\begin{array}{c|c|c} 0 & J_{r\times sn} & 0 \\ \hline J_{sn\times r} & 0 & J_{s\times t}\otimes I_n \\ \hline 0 & J_{t\times s}\otimes I_n & I_t\otimes (J_n - I_n) \end{array}\right]$$

is integral if and only if all roots of the polynomial $x^3-(n-1)x^2-s(rn+t)x+rsn(n-1)$ are integers and $1+4st$ is a perfect square.

Proof. Let A be the matrix given in the statement of the theorem. From Theorem 2, we have

$$\begin{aligned}\det(xI-A) &= x^r \det\left(\begin{bmatrix} xI_s & -J \\ -J & (x+1)I_t \end{bmatrix}\otimes I_n + \begin{bmatrix} -\frac{r}{x}J_s & 0 \\ 0 & -I_t \end{bmatrix}\otimes J_n\right) \\ &= x^r\left(\det\begin{bmatrix} xI_s & -J \\ -J & (x+1)I_t \end{bmatrix}\right)^{n-1}\det\left(\begin{bmatrix} xI_s & -J \\ -J & (x+1)I_t \end{bmatrix}+n\begin{bmatrix} -\frac{r}{x}J_s & 0 \\ 0 & -I_t \end{bmatrix}\right).\end{aligned}$$

Hence, the graph with the adjacency matrix A is integral if and only if all roots of the polynomial

$$\det\left[\begin{array}{c|c} xI_s-\frac{kr}{x}J_s & -J \\ \hline -J & (x+1-k)I_t \end{array}\right] = x^{s-2}(x+1-k)^{t-1}\big(x^3-(k-1)x^2-s(rk+t)x+rsk(k-1)\big)$$

are integers for $k=0,n$. The assertion follows. □

Remark 8. Notice that if $(r,s,t,n)=(1,1,\ell^2+\ell,\ell^2+\ell+1)$ or $(r,s,t,n)=(\ell-1,2,1,2\ell)$ for some positive integer ℓ, then the conditions in Theorem 10 are satisfied. The infinite family of integral graphs corresponding to the first parameters is introduced in [16].

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