The weak saturation number of $K_{2,t}$

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Abstract

For two graphs G and F, we say that G is weakly F-saturated if G has no copy of F as a subgraph and one can join all the nonadjacent pairs of vertices of G in some order so that a new copy of F is created at each step. The weak saturation number $\mathrm{wsat}(n, F)$ is the minimum number of edges of a weakly F-saturated graph on n vertices. In this paper, we examine $\mathrm{wsat}(n, K_{s,t})$, where $K_{s,t}$ is the complete bipartite graph with parts of sizes s and t. We determine $\mathrm{wsat}(n, K_{2,t})$ for all $n \geqslant t+2$ which particulary corrects a previous report in the literature. It is also shown that $\mathrm{wsat}(s+t, K_{s,t}) = \binom{s+t-1}{2}$ if $\gcd(s, t) = 1$ and $\mathrm{wsat}(s+t, K_{s,t}) = \binom{s+t-1}{2}+1$ otherwise.

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1. Introduction

All graphs throughout this paper are finite, undirected, and without loops or multiple edges. The edge set of a graph G is denoted by $E(G)$. For given two graphs G and F, a spanning subgraph H of G is said to be a *weakly F-saturated subgraph* of G if H has no copy of F as a subgraph and there is an ordering $e_1, e_2, \ldots$ of edges in $E(G) \setminus E(H)$ such that for $i = 1, 2, \ldots$ the addition of e_i

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to the spanning subgraph of G with the edge set $E(H) \cup \{e_1, \dots, e_{i-1}\}$ creates a copy of F that contains e_i. The minimum number of edges in a weakly F-saturated subgraph of G is called the *weak saturation number* of F in G and is denoted by $\mathrm{wsat}(G, F)$. For the purpose of simplification, a weakly F-saturated subgraph of K_n is said to be a *weakly F-saturated graph* and $\mathrm{wsat}(K_n, F)$ is written as $\mathrm{wsat}(n, F)$, where K_n is the complete graph on n vertices. For example, each path graph is weakly K_3-saturated and it is easily seen that $\mathrm{wsat}(n, K_3) = n-1$ due to the connectivity.

Determining the exact value of $\mathrm{wsat}(n, F)$ for a given graph F is often quite difficult. It is worth mentioning that the study of any extremal parameter is an important task in graph theory and often receives a great deal of attention. Weak saturation is closely related to the so-called 'graph bootstrap percolation' which was introduced for the first time in [2]. The notion of weak saturation was initially introduced by Bollobás [3] in 1968. Although the weak saturation number has been studied for a long time, related literature is still poor. Indeed, the main difficulty lies in proving lower bounds where usually combinatorial methods do not seem to work. Most arguments that have been used in this area are based on algebraic methods. However, our proofs in the current paper are all combinatorial. For results on weak saturation and related topics, we refer to the survey [6].

Lovász [13] proved that $\mathrm{wsat}(n, K_r) = (r-2)n - \binom{r-1}{2}$ when $n \geqslant r \geqslant 2$, settling a conjecture of Bollobás [3]. The result is also proved by Frankl [8], Kalai [10], Alon [1], and Yu [14]. Surprisingly, these proofs all are based on algebraic techniques and no combinatorial proof has been found so far.

After complete graphs, the next most natural problem to consider regarding weak saturation numbers is description of the behavior of $\mathrm{wsat}(n, K_{s,t})$, where $K_{s,t}$ is the complete bipartite graph with parts of sizes s and t. Borowiecki and Sidorowicz [4] proved that $\mathrm{wsat}(n, K_{1,t}) = \binom{t}{2}$ provided $n \geqslant t+1$. A short proof of this result is given in [7]. The equality $\mathrm{wsat}(n, K_{2,2}) = n$ follows from Theorem 16 of [4] for all $n \geqslant 4$. Faudree, Gould, and Jacobson [7] showed that $\mathrm{wsat}(n, K_{2,3}) = n+1$ for all $n \geqslant 5$. Using multilinear algebra, Kalai [9] established that $\mathrm{wsat}(n, K_{t,t}) = (t-1)n - \binom{t-1}{2}$ if $n \geqslant 4t-4$. This result is also proved by Kronenberg, Martins, and Morrison [12] for every $n \geqslant 3t-3$ by a linear algebraic argument. They also determined $\mathrm{wsat}(n, K_{t,t+1})$ for any $n \geqslant 3t-3$.

The authors of [5] have claimed that they determine $\mathrm{wsat}(n, K_{2,t})$ for $t \geqslant 4$ and $n \geqslant 2t-1$. We believe that their half page argument to prove the lower bound on $\mathrm{wsat}(n, K_{2,t})$ is not correct. In the current paper, we fill this gap in the literature by proving the following result.

Theorem 1.1. *For every two integers n, t with $t \geqslant 3$ and $n \geqslant t+2$, the following statements hold.*

(i) *If t is odd, then* $\mathrm{wsat}(n, K_{2,t}) = n - 2 + \binom{t}{2}$.

(ii) *If t is even and $n \leqslant 2t-2$, then* $\mathrm{wsat}(n, K_{2,t}) = n - 1 + \binom{t}{2}$.

(iii) *If t is even and $n \geqslant 2t-1$, then* $\mathrm{wsat}(n, K_{2,t}) = n - 2 + \binom{t}{2}$.

The proofs of the lower bounds of Theorem 1.1 which are presented in Section 3 form the most involved part of the paper. In Section 2, we establish the following theorem which particularly proves Theorem 1.1 for the initial case $n = t+2$. Generally, determination of $\mathrm{wsat}(n, F)$ for graphs F on n vertices seems to be an attractive problem.

Theorem 1.2. *For every two positive integers s and t,*

$$\operatorname{wsat}(s+t, K_{s,t}) = \begin{cases} \binom{s+t-1}{2} & \text{if } \gcd(s,t)=1, \\ \binom{s+t-1}{2}+1 & \text{otherwise.} \end{cases}$$

A relatively new trend in extremal graph theory is to extend the classical deterministic results to random analogues. Such study reveals the behavior of extremal parameters for a typical graph. For instance, the problem of determination of $\operatorname{wsat}(\mathbb{G}(n,p), K_{s,t})$ for given fixed integers s and t is still unsolved in general case, where $\mathbb{G}(n,p)$ denotes the Erdős–Rényi random graph model. Kalinichenko and Zhukovskii [11] presented some sufficient conditions for which $\operatorname{wsat}(\mathbb{G}(n,p), F) = \operatorname{wsat}(n, F)$ with high probability. Theorem 1.1 combined with Corollary 1 of [11] yields that with high probability $\operatorname{wsat}(\mathbb{G}(n,p), K_{2,t}) = n-2+\binom{t}{2}$ for each constant $p \in (0,1)$.

Below, we introduce more notation and terminology that we use in the rest of the paper. Let G be a graph. The vertex set of G is denoted by $V(G)$ and the *order* of G is defined as $|V(G)|$. We set $e(G) = |E(G)|$. For every two adjacent vertices u and v, we denote the edge joining u and v by uv. The *complement* of G, denoted by $\overline{G}$, is a graph with vertex set $V(G)$ in which $uv \in E(\overline{G})$ if $u \neq v$ and $uv \notin E(G)$. For a subset X of $V(G)$, we denote the induced subgraph of G on X by $G[X]$. For a subset Y of $E(G)$, we denote by $G - Y$ the graph obtained from G by removing the edges in Y. For a subset Z of $E(\overline{G})$, we adopt the notation $G+Z$ to denote the graph with vertex set $V(G)$ and edge set $E(G) \cup Z$. For simplicity, we write $G-e$ instead of $G - \{e\}$ and $G+e$ instead of $G+\{e\}$. For a vertex v of G, denote by $G - v$ the graph obtained from G by removing v and all edges incident to v. Also, define the set of neighbors of v as $N_G(v) = \{x \in V(G) \,|\, x \text{ is adjacent to } v\}$ and the *degree* of v as $\deg_G(v) = |N_G(v)|$. The minimum degree of vertices of G is denoted by $\delta(G)$. For the sake of convenience, we set $N_G[u] = \{u\} \cup N_G(u)$ and $N_G(u,v) = N_G(u) \cap N_G(v)$. For every two subsets A and B of $V(G)$, let $E_G(A,B)$ denote the set of edges of G having an endpoint in A and the other endpoint in B. We set $e_G(A,B) = |E_G(A,B)|$. For simplicity, we write $E_G(A)$ instead of $E_G(A,A)$ and $e_G(A)$ instead of $e_G(A,A)$. The *union* of two vertex disjoint graphs G_1 and G_2, denoted by $G_1 \sqcup G_2$, is the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. The *join* of two vertex disjoint graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained from $G_1 \sqcup G_2$ by joining every vertex in $V(G_1)$ to every vertex in $V(G_2)$.

2. Determination of $\operatorname{wsat}(s+t, K_{s,t})$

The problem of determining $\operatorname{wsat}(n, F)$ when the graph F is of order n is interesting to explore. In this section, we solve this problem when F is a complete bipartite graph. The following lemma helps us to get a lower bound.

Lemma 2.1. *Let s, t be positive integers and let G be a weakly $K_{s,t}$-saturated graph of order $s+t$. Then, $\overline{G}$ has no cycle. Moreover, if $\gcd(s,t) \neq 1$, then $\overline{G}$ is disconnected.*

Proof. Fix an order $e_1, e_2, \ldots$ of $E(\overline{G})$ that is obtained from a weakly $K_{s,t}$-saturation process on G. By contradiction, suppose that $\overline{G}$ has a cycle, say C. Let e_i be the first edge of C that appears in the order $e_1, e_2, \ldots$ In view of the definition of weakly $K_{s,t}$-saturation process, there is a partition $\{A, B\}$ of $V(G)$ with $|A| = s$ and $|B| = t$ such that e_i is the only missing edge between A and B in $G + \{e_1, \ldots, e_{i-1}\}$. So, both endpoints of each edge among $e_{i+1}, e_{i+2}, \ldots$ belong to one of A or B. This is impossible, since C has to pass through at least one edge e_j with $j > i$ having endpoints in both A and B. This shows that $\overline{G}$ has no cycle.

Now, assume that $\overline{G}$ is connected. As we saw above, $\overline{G}$ is a tree. Let $H_0 = \overline{G}$ and $H_i = \overline{G} - \{e_1, \ldots, e_i\}$ for any $i \geqslant 1$. We claim that for any $i \geqslant 0$, H_i is a forest whose connected components are of order divisible by d, where $d = \gcd(s, t)$. Since $\overline{G} - \{e_1, e_2, \ldots\} = \overline{K_{s+t}}$, we find that $d = 1$, as required.

We prove the claim by induction on i. The claim is clearly valid for $i = 0$. So, assume that $i \geqslant 1$. According to the definition of weakly $K_{s,t}$-saturation process, there is a partition $\{A, B\}$ of $V(G)$ with $|A| = s$ and $|B| = t$ such that e_i is the only missing edge between A and B in $G + \{e_1, \ldots, e_{i-1}\}$. Hence, e_i is the only edge in H_{i-1} between A and B. Let $C_1, \ldots, C_i$ be the connected components of H_{i-1}. Without loss of generality, assume that $e_i \in E(C_1)$. So, the connected components of H_i are $C_1', C_1'', C_2, \ldots, C_i$, where C_1' and C_1'' are respectively the induced subgraphs of C_1 on $A \cap V(C_1)$ and $B \cap V(C_1)$. As e_i is the only edge in H_{i-1} between A and B, we conclude that either $V(C_i) \subseteq A$ or $V(C_i) \subseteq B$ for any $i \geqslant 2$. It follows that A is a disjoint union of $V(C_1')$ and some sets among $V(C_2), \ldots, V(C_i)$. The induction hypothesis yields that $|V(C_2)|, \ldots, |V(C_i)|$ are multiples of d. This and the divisibility of $|A|$ by d imply that $|V(C_1')|$ is a multiple of d. A similar argument works for $|V(C_1'')|$. The claim is established. □

The following consequence immediately follows from Lemma 2.1.

Corollary 2.2. *For every integers s and t,* $\mathrm{wsat}(s+t, K_{s,t}) \geqslant \binom{s+t-1}{2}$. *Moreover, if* $\gcd(s,t) \neq 1$, *then* $\mathrm{wsat}(s+t, K_{s,t}) \geqslant \binom{s+t-1}{2} + 1$.

We present the following two lemmas to obtain a tight upper bound. We use the notation P_n for the path graph of order n.

Lemma 2.3. *Let s and t be positive integers. Then,* $\mathrm{wsat}(s+t, K_{s,t}) \leqslant \binom{s+t-1}{2} + 1$.

Proof. We prove that $G = \overline{P_{s+t-1} \sqcup K_1}$ is weakly $K_{s,t}$-saturated. Denote by $v_1, \ldots, v_{s+t-1}$ the vertices of P_{s+t-1} going in the natural order of the path and set $V(K_1) = \{v_{s+t}\}$. Let $e_1 = v_s v_{s+1}$, $e_i = v_{i-1}v_i$ for $i = 2, \ldots, s$, and $e_i = v_i v_{i+1}$ for $i = s+1, \ldots, s+t-2$. We claim that $e_1, \ldots, e_{s+t-2}$ is an order in which the weakly $K_{s,t}$-saturation process occurs. Let $H_0 = G$ and $H_i = G + \{e_1, \ldots, e_i\}$ for $i = 1, \ldots, s+t-2$. In order to prove the assertion, we find a partition $\{A_i, B_i\}$ of $V(G)$ such that $|A_i| = s$, $|B_i| = t$, and e_{i+1} is the only missing edge between A_i and B_i in H_i for $i = 0, 1, \ldots, s+t-3$. To do this, it is enough to introduce $A_0, A_1, \ldots A_{s+t-3}$. Let $A = \{v_1, \ldots, v_s\}$. Now, set $A_0 = A$, $A_i = (A \setminus \{v_i\}) \cup \{v_{s+t}\}$ for $i = 1, \ldots, s-1$, and $A_i = (A \setminus \{v_1\}) \cup \{v_{i+1}\}$ for $i = s, \ldots, s+t-3$. □

Lemma 2.4. *Let s and t be positive integers with* $\gcd(s,t) = 1$. *Then,* $\mathrm{wsat}(s+t, K_{s,t}) \leqslant \binom{s+t-1}{2}$.

Proof. We prove that $G = \overline{P_{s+t}}$ is weakly $K_{s,t}$-saturated. We proceed by induction on $s+t$. The assertion clearly holds for $s+t = 2$. Let $s+t \geqslant 3$ and denote by $v_1, \ldots, v_{s+t}$ the vertices of P_{s+t} going in the natural order of the path. Partition $V(G)$ into two subsets $A = \{v_1, \ldots, v_s\}$ and $B = \{v_{s+1}, \ldots, v_{s+t}\}$. Since the edge $e = v_s v_{s+1}$ is the only missing edge between A and B in G, we may consider e as the first element in an ordering of $E(\overline{G})$ in a weakly $K_{s,t}$-saturation process on G. For the sake of convenience, let $G' = G + e$ and without loss of generality, assume that $t \geqslant s$. Using the definition, in each step of a weakly $K_{s,t-s}$-saturation process on $G'[B]$, there is a partition $\{C, D\}$ of B such that $|C| = s$, $|D| = t - s$, and all edges between C and D are present except exactly one. Since there is no edge between A and B in $\overline{G'}$, every step of a weakly $K_{s,t-s}$-saturation process on $G'[B]$ corresponding to a vertex partition $\{C, D\}$ can be considered as a step of a weakly $K_{s,t}$-saturation process on G' corresponding to the vertex partition $\{C, A \cup D\}$. Hence, by the induction hypothesis, $G'[B]$ may be completed to reach to K_t through a weakly $K_{s,t}$-saturation process. Thus, it remains to show that $G'' = G' + \{v_i v_{i+1} \mid s+1 \leqslant i \leqslant s+t-1\}$ is weakly $K_{s,t}$-saturated. For $i = 1, \ldots, s-1$, the edge $e_i = v_i v_{i+1}$ is the only missing edge between $\{v_1, \ldots, v_i\} \cup \{v_{s+1}, \ldots, v_{2s-i}\}$ and $\{v_{i+1}, \ldots, v_s\} \cup \{v_{2s-i+1}, \ldots, v_{s+t}\}$ in G'' and therefore we may add e_i to G'' in the weakly $K_{s,t}$-saturation process. □

We end this section by pointing out that Theorem 1.2 is immediately concluded from Corollary 2.2, Lemma 2.3, and Lemma 2.4.

3. Determination of wsat$(n, K_{2,t})$

In this section, we establish Theorem 1.1 which is a direct consequence of Lemmas 3.2, 3.4, and 3.14. The following lemma is known, although it seems that it is not explicitly stated anywhere. We include a proof here for the sake of completeness.

Lemma 3.1. *Let F be a graph with $\delta(F) \geqslant 1$ and let G be a weakly F-saturated graph such that $|V(G)| \geqslant |V(F)| - 1$. Join a new vertex v to $\delta(F) - 1$ arbitrary vertices of G. Then, the resulting graph is also weakly F-saturated.*

Proof. Denote the resulting graph by G'. Since G is weakly F-saturated, we may add all edges in $\{uv \in E(\overline{G'}) \mid u, v \in V(G)\}$ to G' in some order to obtain a complete subgraph of G' on $V(G)$. Let $e \in E(F)$ be incident to a vertex of degree $\delta(F)$. For each vertex $x \in V(G) \setminus N_{G'}(v)$, there is a copy of $F - e$ in G' containing the vertices in $\{v, x\} \cup N_{G'}(v)$ and so, we may connect v to x in the weakly F-saturation process on G'. The assertion follows. □

The following lemma proves the upper bounds of Theorem 1.1.

Lemma 3.2. *For every two integers n, t with $t \geqslant 3$ and $n \geqslant t + 2$, the following statements hold.*

(i) *If t is odd, then* $\mathrm{wsat}(n, K_{2,t}) \leqslant n - 2 + \binom{t}{2}$.

(ii) *If t is even and $n \leqslant 2t - 2$, then* $\mathrm{wsat}(n, K_{2,t}) \leqslant n - 1 + \binom{t}{2}$.

(iii) *If t is even and $n \geqslant 2t - 1$, then* $\mathrm{wsat}(n, K_{2,t}) \leqslant n - 2 + \binom{t}{2}$.

Proof. Let H be a weakly $K_{2,t}$-saturated graph of order $t + 2$ with $\mathrm{wsat}(t + 2, K_{2,t})$ edges. Attach $n - t - 2$ pendent vertices to an arbitrary vertex of H to obtain a graph G of order n. By Lemma 3.1, G is a weakly $K_{2,t}$-saturated graph with $n - t - 2 + \mathrm{wsat}(t + 2, K_{2,t})$ edges. Parts (i) and (ii) follow from Theorem 1.2. The graph $\mathbb{G}_{n,t}$, depicted in Figure 1 and introduced in [5, 12], is weakly $K_{2,t}$-saturated for $n \geqslant 2t - 1$. This can be proved by using a proof similar to that of Proposition 14 in [12]. Since $\mathbb{G}_{n,t}$ has n vertices and $n - 2 + \binom{t}{2}$ edges, (iii) follows. □

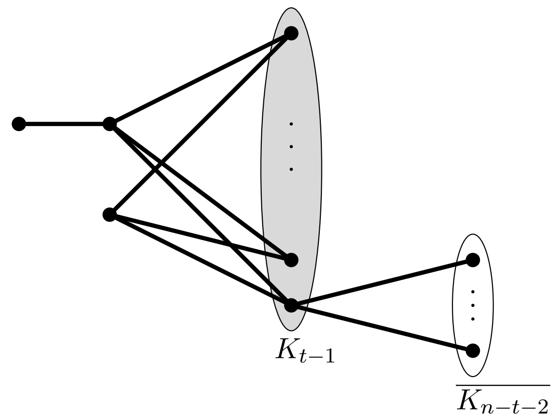

Figure 1. The graph $\mathbb{G}_{n,t}$. We have not drawn the edges between the vertices in the gray elliptical disk.

The following observation is trivially true. We state it for clarity.

Observation 3.3. *Fix a graph F and let G, H be two graphs with the same vertex set. Assume that the graph obtained from H by adding a sequence of edges in a weakly F-saturation process contains G as a subgraph. If G is weakly F-saturated, then so is H.*

The following lemma establishes a general lower bound on $\mathrm{wsat}(n, K_{2,t})$.

Lemma 3.4. *Let $t \geqslant 3$ and $n \geqslant t+2$. Then,* $\mathrm{wsat}(n, K_{2,t}) \geqslant n - 2 + \binom{t}{2}$.

Proof. Let G_0 be a weakly $K_{2,t}$-saturated graph. So, G_0 is connected. By Lemma 3.1, if we attach a new pendent vertex to a vertex of G_0, then the resulting graph is also weakly $K_{2,t}$-saturated whose number of vertices is one more than the number of vertices of G_0 and whose number of edges is one more than the number of edges of G_0. We attach t^3 new pendent vertices to each vertex of G_0 and we call the resulting graph by G. To prove the assertion, it suffices to show that $e(G) \geqslant n - 2 + \binom{t}{2}$ provided $n = |V(G)|$.

We define a process in which step, G is updated so that a special structure on G is preserved. In each step of the process, G looks as follows. The graph G contains G_0 as a subgraph. The edges of G are colored by two colors black and red. At the beginning of the process, all edges are black and the number of black edges does not change during the process. The spanning subgraph of G induced on black edges is connected. There exist two disjoint subsets A and B of $V(G)$ which are equipped with the following features. There is an ordering on A under which the vertices in A can be arranged as $x_1, \ldots, x_k$, where $k = |A|$. Every red edge has at least one endpoint in A. There exist the partition $\{A_1, \ldots, A_m\}$ of A and the partition $\{B_1, \ldots, B_m\}$ of B which are described below. Let $i \in \{1, \ldots, m\}$ and denote by x_{s_i} the first element among $x_1, \ldots, x_k$ which appears within A_i. The following properties will be held in each step of the process.

(P.1) For every two vertices $x, y \in A_i$, we have $N_G(x) \setminus \{y\} = N_G(y) \setminus \{x\}$.
(P.2) The set A_i is either a clique of size at least 2 or an independent set of size 2 or 3.
(P.3) Every edge between x_{s_i} and $A_i \setminus \{x_{s_i}\}$ is black whenever A_i is a clique.
(P.4) Every vertex in A_i is adjacent to every vertex in A_j whenever $i \neq j$.
(P.5) Each edge between x_{s_i} and $V(G) \setminus A$ is black.

(P.6) Every vertex $x_r \in A_i$ is adjacent to every vertex x_{s_j} by a black edge if $i \neq j$ and $s_j < r$.
(P.7) The size of B_i is $t - s_i$.
(P.8) For any vertex $x \in B_i$, we have $N_G(x) = A_i$.
(P.9) Any vertex $x_r \in A_i$ is adjacent to exactly α_r vertices in B_i by black edges, where

$$\alpha_r = \begin{cases} t - r & \text{if } r = s_i, \\ t - r + 2 & \text{if } A_i \text{ is an independent set of size} \\ & \text{3 and } x_r \text{ is the third element of } A_i, \\ t - r + 1 & \text{otherwise.} \end{cases}$$

The configuration described above is designed so that at each step of the process, the graph induced on black edges is weakly $K_{2,t}$-saturated, and red edges are the ones that are added through the weakly $K_{2,t}$-saturation process. This will be shown in Cases 3.5, 3.6, and 3.7 below.

At the beginning of the process, all edges are black and $A = B = \varnothing$. In Cases 3.5, 3.6, and 3.7, we explain how in each step of the process we update G and A, B, C to proceed to the next step, where $C = V(G) \setminus (A \cup B)$. More precisely, we repeat the process until one of the following occurs.

(T.1) $|A_i| \geqslant t - 1$ for some $i \in \{1, \ldots, m\}$.
(T.2) $k = t$, $|A_i| \leqslant t - 2$ for $i = 1, \ldots, m$, and there are two vertices $u, v \in C$ such that $|N_G(u, v)| \geqslant t - 1$ and $N_G(u) \setminus \{v\} \neq N_G(v) \setminus \{u\}$.
(T.3) $k = t + 1$, $|A_i| \leqslant t - 2$ for $i = 1, \ldots, m$, and there are two vertices $u, v \in C$ such that $|N_G(u, v)| \geqslant t - 1$ and $N_G(u) \setminus \{v\} \neq N_G(v) \setminus \{u\}$.
(T.4) $k = t + 1$, $|A_i| \leqslant t - 2$ for $i = 1, \ldots, m$, and there are two vertices $u \in A$ and $v \in C$ such that $|N_G(u, v)| \geqslant t - 1$ and $N_G(u) \setminus \{v\} \neq N_G(v) \setminus \{u\}$.

We now show what we do in each step of the process before termination. At the beginning of each step, G is weakly $K_{2,t}$-saturated and so there are two vertices a, b such that $|N_G(a, b)| \geqslant t-1$ and $N_G(a) \setminus \{b\} \neq N_G(b) \setminus \{a\}$. As (T.1) is not happened, (P.8) forces that $a, b \notin B$.

Case 3.5. $a, b \in C$.

Description. As (T.1)–(T.3) are not happened, $k \leqslant t-1$. Set $x_{k+1} = a$, $x_{k+2} = b$, $s_{m+1} = k+1$, and $A_{m+1} = \{x_{k+1}, x_{k+2}\}$. Suppose that x_{k+1} or x_{k+2} is not adjacent to x_{s_i} for some $i \in \{1, \ldots, m\}$. We find from (P.1) that $|N_G(x_{k+1}, x_{k+2}) \setminus A| \geqslant t-1-|N_G(x_{k+1}, x_{k+2}) \cap A| \geqslant t-1-(k-|A_i|) \geqslant |A_i| \geqslant 2$. So, we may remove two arbitrary edges between x_{k+2} and $N_G(x_{k+1}, x_{k+2}) \setminus A$ and join both x_{k+1} and x_{k+2} to x_{s_i} by black edges and to all vertices in $A_i \setminus \{x_{s_i}\}$ by red edges. By repeating this, we derive that $A \subseteq N_G(x_{k+1}, x_{k+2})$ and hence $|N_G(x_{k+1}, x_{k+2}) \setminus A| \geqslant t - 1 - k$. We remove $t - 1 - k$ arbitrary edges between x_{k+2} and $N_G(x_{k+1}, x_{k+2}) \setminus A$ and connect x_{k+2} to all vertices in a subset B_{m+1} consisting of $t - 1 - k$ arbitrary pendent vertices in $N_G(x_{k+1})$.

Now, update A to $A \cup A_{m+1}$ with the partition $\{A_1, \ldots, A_{m+1}\}$ and update B to $B \cup B_{m+1}$ with the partition $\{B_1, \ldots, B_{m+1}\}$. ◇

Case 3.6. $a \in A_i$ for some $i \in \{1, \ldots, m\}$ and $b \in C$.

Description. Since (T.1) and (T.4) are not happened, $k \leqslant t$. In view of (P.1) and without loss of generality, we may assume that $a = x_{s_i}$. Let $x_{k+1} = b$.

First, assume that $A_i \cup \{x_{k+1}\}$ is an independent set of size 3. Suppose that x_{k+1} is not adjacent to x_{s_j} for some $j \in \{1, \ldots, m\} \setminus \{i\}$. We obtain from (P.1) that $|N_G(x_{s_i}, x_{k+1}) \setminus A| \geqslant t - 1 - |N_G(x_{s_i}, x_{k+1}) \cap A| \geqslant t - 1 - (k - |A_j|) \geqslant |A_j| - 1 \geqslant 1$. So, we may remove an arbitrary edge between x_{k+1} and $N_G(x_{s_i}, x_{k+1}) \setminus A$ and join x_{k+1} to x_{s_j} by a black edge and to all vertices in $A_j \setminus \{x_{s_j}\}$ by red edges. By repeating this, we derive that $A \setminus A_i \subseteq N_G(x_{s_i}, x_{k+1})$ and thus $|N_G(x_{s_i}, x_{k+1}) \setminus A| \geqslant t - 1 - (k - |A_i|) \geqslant t - k + 1$. We now remove $t - k + 1$ arbitrary edges between x_{k+1} and $N_G(x_{s_i}, x_{k+1}) \setminus A$ and connect x_{k+1} to $t - k + 1$ arbitrary distinct vertices in B_i by black edges. This is possible, since $|B_i| \geqslant t - k + 1$ by (P.7) and using $s_i \leqslant k - 1$.

Next, assume that $A_i \cup \{x_{k+1}\}$ is not an independent set of size 3. Let x_{k+1} be not adjacent to x_{s_j} for some $j \in \{1, \ldots, m\}$. We find from (P.1) that $|N_G(x_{s_i}, x_{k+1}) \setminus A| \geqslant t - 1 - |N_G(x_{s_i}, x_{k+1}) \cap A| \geqslant t - 1 - (k - |A_j|) \geqslant |A_j| - 1 \geqslant 1$. So, we may remove an arbitrary edge between x_{k+1} and $N_G(x_{s_i}, x_{k+1}) \setminus A$ and join x_{k+1} to x_{s_j} by a black edge and to all vertices in $A_j \setminus \{x_{s_j}\}$ by red edges. By repeating this, we derive that $A \subseteq N_G(x_{k+1})$. If A_i is a clique, then $A \setminus \{x_{s_i}\} \subseteq N_G(x_{s_i}, x_{k+1})$. Suppose that A_i is an independent set. It follows from $A \subseteq N_G(x_{k+1})$ that $N_G(x_{s_i}, x_{k+1}) \cap A = A \setminus A_i$ and hence $|N_G(x_{s_i}, x_{k+1}) \setminus A| \geqslant t - 1 - (k - |A_i|) \geqslant |A_i| - 1$. We now remove $|A_i| - 1$ arbitrary edges between x_{k+1} and $N_G(x_{s_i}, x_{k+1}) \setminus A$ and join x_{s_i} to all vertices in $A_i \setminus \{x_{s_i}\}$ by black edges, resulting in $A \setminus \{x_{s_i}\} \subseteq N_G(x_{s_i}, x_{k+1})$. Therefore, regardless of whether A_i is a clique or an independent set, $|N_G(x_{s_i}, x_{k+1}) \setminus A| \geqslant t - 1 - (|A| - 1) = t - k$. Remove $t - k$ arbitrary edges between x_{k+1} and $N_G(x_{s_i}, x_{k+1}) \setminus A$ and connect x_{k+1} to $t - k$ arbitrary distinct vertices in B_i by black edges.

Now, update A_i to $A_i \cup \{x_{k+1}\}$ and A to $A \cup \{x_{k+1}\}$ with the partition $\{A_1, \ldots, A_m\}$. ◇

Case 3.7. $a \in A_i$ and $b \in A_j$ for some i, j with $1 \leqslant i < j \leqslant m$.

Description. In view of (P.1) and without loss of generality, we may assume that $a = x_{s_i}$ and $b = x_{s_j}$. Let $x_r \in A_j$. We know that $|B_i| > \alpha_r$ and there are exactly α_r black edges between x_r and B_j by (P.9). If A_j is an independent set of size 3 and x_r is the third element of A_j, then remove $\alpha_r - 1$ black edges between x_r and B_j and connect x_r to $\alpha_r - 1$ arbitrary vertices in B_i by black edges. Otherwise, remove α_r black edges between x_r and B_j and connect x_r to α_r arbitrary vertices in B_i by black edges.

Remove all black edges between x_{s_j} and x_r if $r > s_j$. The number of such edges is $k - s_j - q$, where $q = |A_j \setminus N_G[x_{s_j}]|$. Note that, $|N_G(x_{s_i}, x_{s_j}) \cap A| = k - p - q - 2$, where $p = |A_i \setminus N_G[x_{s_i}]|$. We have $|N_G(x_{s_i}, x_{s_j}) \setminus A| \geqslant t - 1 - (k - p - q - 2)$. Remove $t - 1 - (k - p - q - 2)$ black edges between x_{s_j} and $N_G(x_{s_i}, x_{s_j}) \setminus A$. Since $(k - s_j - q) + (t - 1 - (k - p - q - 2)) = p + (t - s_j) + 1$, we may connect x_{s_i} to all vertices in $(A_i \setminus N_G[x_{s_i}]) \cup B_j$ and x_{s_j} to an arbitrary vertex in B_i by black edges.

Now, update A_i to $A_i \cup A_j$ and consider the partition $\{A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_m\}$ for A and the partition $\{B_1, \ldots, B_{j-1}, B_{j+1}, \ldots, B_m\}$ for B. ◇

At the end of each of Cases 3.5, 3.6, and 3.7, we do the following. In order to establish (P.1) and (P.2), for each vertex w with $N_G(w) \cap A_i \neq \varnothing$, join w to all vertices in $A_i \setminus N_G(w)$ by red edges. In order to establish (P.5), for each vertex $w \in C$ with $N_G(w) \cap A_i \neq \varnothing$, since there exists at least a vertex $x_r \in A_i$ so that the color of the edge wx_r is black, we may switch the color of the edge wx_r with the color of the edge wx_{s_i}. Note that, after doing all these changes, the number of black edges does not change and the resulting graph have Properties (P.1)–(P.9). Moreover, by Observation 3.3, G is still weakly $K_{2,t}$-saturated.

Now, assume that the process is terminated. Denote by $G_{\rm b}$ the spanning subgraph of G induced on black edges. So, in order to establish the assertion, we should show that $e(G_{\rm b}) \geqslant n-2+\binom{t}{2}$. For each $i \geqslant 0$, let C_i be the set of vertices in C with the distance i from A in $G_{\rm b}$ and let $\{C_1, \ldots, C_d\}$ be a partition of C. Notice that $C_0 = A$. For any integer $i \in \{1, \ldots, d\}$ and any vertex $c \in C_i$, consider an arbitrary edge $e_c \in E_{G_{\rm b}}(\{c\}, C_{i-1})$ and set $E = \{e_c \,|\, c \in C\}$. Denote by $G_{\rm c}$ the spanning subgraph of $G_{\rm b}$ with $E(G_{\rm c}) = \{x_{s_i}x_r \in E(G_{\rm b}) \,|\, s_i < r\} \cup E_{G_{\rm b}}(A, B) \cup E$. Finally, set $F = E_{G_{\rm b}}(C, V(G)) \setminus E_{G_{\rm c}}(C, V(G))$. Note that, in $G_{\rm c}$, every C_i is an independent set and moreover, for $i = 1, \ldots, d$, every vertex in C_i has exactly one neighbor in C_{i-1}.

Let β and γ be respectively the number of independent sets among $A_1, \ldots, A_m$ of sizes 2 and 3. Also, let δ be 1 if $k = t-1$ and 0 otherwise. We have

$$\begin{aligned}
e(G_{\rm c}) &= e_{G_{\rm c}}(A) + e_{G_{\rm c}}(A, B) + e_{G_{\rm c}}\big(C, V(G)\big) \\
&= \left(\sum_{r=1}^{k} \big|\{i \,|\, s_i < r\}\big| - \beta - 2\gamma\right) + \left(\sum_{r=1}^{k} \alpha_r\right) + \left(n - |A| - \sum_{i=1}^{m} |B_i|\right) \\
&= \left(\sum_{i=1}^{m} \big|\{r \,|\, r > s_i\}\big| - \beta - 2\gamma\right) + \left(\sum_{r=1}^{t} (t-r+1) - m + \gamma - \delta\right) + \left(n - k - \sum_{i=1}^{m} (t - s_i)\right) \\
&= \left(\sum_{i=1}^{m} (k - s_i) - \beta - 2\gamma\right) + \left(\binom{t+1}{2} - m + \gamma - \delta\right) + \left(n - k - \sum_{i=1}^{m} (t - s_i)\right) \\
&= \left(n - 2 + \binom{t}{2}\right) + \Big((m-1)(k-t-1) - \beta - \gamma - \delta + 1\Big). \qquad (1)
\end{aligned}$$

We consider the termination states (T.1)–(T.4) in Cases 3.8, 3.9, 3.10, and 3.11. In each of these cases, we will use (1) to establish that $e(G_{\rm b}) \geqslant n - 2 + \binom{t}{2}$.

Case 3.8. (T.1) has happened.

If the second term in (1) is nonnegative, then there is nothing to prove. So, we may assume that $(m-1)(k-t-1) - \beta - \gamma - \delta + 1 < 0$. From $k \geqslant t - 1 + 2(m-1)$, $\beta + \gamma \leqslant m$, and the definition of δ, we deduce that one of situations

$$\begin{cases} k = t-1 \\ m = \beta + \gamma = 1 \\ \delta = 1 \end{cases} \qquad \text{or} \qquad \begin{cases} k = t+1 \\ m = \beta + \gamma = 2 \\ \delta = 0 \end{cases}$$

holds. Thus, the second term in (1) is equal to -1, yielding that $e(G_{\rm c}) \geqslant n - 3 + \binom{t}{2}$. If $F \neq \varnothing$, then $e(G_{\rm b}) \geqslant e(G_{\rm c}) + |F| \geqslant n-2+\binom{t}{2}$, we are done. Suppose by way of contradiction that $F = \varnothing$. Since there is an independent set of size $t-1$ among $A_1, \ldots, A_m$, one concludes that $t = 3$ or 4. We show that $G[A \cup B]$ is a bipartite graph. This is clearly seen for $m = 1$ and one may consider the vertex partition $\{A_1 \cup B_2, A_2 \cup B_1\}$ for $m = 2$. Using the connectivity of G and starting by $G[A \cup B]$, we may add vertices in C to $G[A \cup B]$ in some order such that in each step the resulting graph is bipartite. This means that G is bipartite which is impossible, since a bipartite graph is clearly not weakly $K_{2,t}$-saturated.

Case 3.9. (T.2) has happened.

As $k=t$, it follows from (1) that $e(G_{\rm c})=n-2+\binom{t}{2}-(m+\beta+\gamma-2)$. Since $e(G_{\rm b})\geqslant e(G_{\rm c})+|F|$, in order to prove the assertion, it suffices to show that $|F|\geqslant m+\beta+\gamma-2$.

We may assume that either $A\subseteq N_G(u,v)$ or $N_G(u,v)\cap A=A\setminus A_i$ for some $i\in\{1,\ldots,m\}$ with $|A_i|=2$. To see this, suppose that $|N_G(u,v)\cap A|\leqslant t-3$ and suppose that u or v is not adjacent to x_{s_j} for some $j\in\{1,\ldots,m\}$. We find that $|N_G(u,v)\setminus A|\geqslant t-1-|N_G(u,v)\cap A|\geqslant 2$. So, we may remove two arbitrary edges between v and $N_G(u,v)\setminus A$ and join both u and v to x_{s_j} by black edges and to all vertices in $A_j\setminus\{x_{s_j}\}$ by red edges. By repeating this, we get $|N_G(u,v)\cap A|\geqslant t-2$, as desired.

If $A\subseteq N_G(u,v)$, then there are $2m$ black edges between $\{u,v\}$ and $\{x_{s_1},\ldots,x_{s_m}\}$ by (P.5) which only two of them belong to E and thus $|F|\geqslant 2m-2\geqslant m+\beta+\gamma-2$, we are done.

So, assume that $N_G(u,v)\cap A=A\setminus A_i$ for some $i\in\{1,\ldots,m\}$ with $|A_i|=2$. We have $|N_G(u,v)\setminus A|=|N_G(u,v)|-|N_G(u,v)\cap A|\geqslant t-1-|A\setminus A_i|=1$. Fix $w\in N_G(u,v)\setminus A$. Obviously, $w\in C_1\cup C_2$. Then, there are $2m$ black edges between $\{u,v\}$ and $(\{x_{s_1},\ldots,x_{s_m}\}\setminus\{x_{s_i}\})\cup\{w\}$ which at most three of them belong to E. Hence, $|F|\geqslant 2m-3\geqslant m+\beta+\gamma-3$.

Towards a contradiction, suppose that the inequality $|F|\geqslant m+\beta+\gamma-2$ does not hold. We have $m+\beta+\gamma-3\geqslant|F|\geqslant 2m-3\geqslant m+\beta+\gamma-3$ which shows that $|F|=2m-3$ and $\beta+\gamma=m$. It follows from $\beta+\gamma=m$ that $A_1,\ldots,A_m$ are independent sets. Since there are $2m-2$ black edges between $\{u,v\}$ and $\{x_{s_1},\ldots,x_{s_m}\}\setminus\{x_{s_i}\}$ and only two of them belong to E, we deduce that $|F\cap E_{G_{\rm b}}(A,C)|\geqslant 2m-4$. Also, at least one of the black edges uw or vw belong to F, meaning that $|F\cap E_{G_{\rm b}}(C)|\geqslant 1$. Now, from $|F|=2m-3$, $|F\cap E_{G_{\rm b}}(A,C)|\geqslant 2m-4$, and $|F\cap E_{G_{\rm b}}(C)|\geqslant 1$, we derive that $uv\notin E(G)$ and $|F\cap E_{G_{\rm b}}(C)|=1$. Thus, $w\in C_2$ and $G_{\rm c}[C]=G_{\rm b}[C]-e$, where $e\in\{uw,vw\}$.

Denote by G' be the graph obtained from G by joining u to all vertices in $N_G(v)\setminus N_G(u)$ and joining v to all vertices in $N_G(u)\setminus N_G(v)$. Set $A'=A\cup\{u,v\}$, $C'=C\setminus\{u,v\}$, $A'_j=A_j$ for $j=1,\ldots,m$, and $A'_{m+1}=\{u,v\}$. We know that $A'_1,\ldots,A'_{m+1}$ are independent sets. For each $j\geqslant 0$, let C'_j be the set of vertices in C' with the distance j from A' in G' and let $\{C'_1,\ldots,C'_{d'}\}$ be a partition of C'. Since $G'[C']=G_{\rm c}[C']$, we observe in G' that $C'_1,\ldots,C'_{d'}$ are independent sets and moreover, for $j=2,\ldots,d'$, every vertex in C'_j has exactly one neighbor in C'_{j-1}. Further, for any vertex $c\in C'_1$, there is an index $j\in\{1,\ldots,m+1\}$ such that $N_{G'}(c)=A'_j$. Using these features, the following statements are easily obtained for two arbitrary distinct vertices $y,z\in V(G')$.

(i) Let $y\in C'$ and $z\in A'\cup C'$. Then, $N_{G'}(y,z)$ is one of $\varnothing$, $\{c\}$, or A'_j for some vertex $c\in C'$ and integer $j\in\{1,\ldots,m+1\}$. Hence, $|N_{G'}(y,z)|\leqslant t-2$.

(ii) Let $y,z\in A'$. If $y\in A'_j$ and $z\in A'_\ell$ for some indices $j\neq\ell$, then $N_{G'}(y,z)\subseteq A'\setminus(A'_j\cup A'_\ell)$. Thus, $|N_{G'}(y,z)|\leqslant t-2$.

As G' is not a complete graph and there is no pair $\{y,z\}$ of vertices of G' such that $|N_{G'}(y,z)|\geqslant t-1$ and $N_{G'}(y)\setminus\{z\}\neq N_{G'}(z)\setminus\{y\}$, one derives that G' and therefore G are not weakly $K_{2,t}$-saturated, a contradiction.

Case 3.10. (T.3) has happened.

As $k=t+1$, it follows from (1) that $e(G_{\rm c})=n-2+\binom{t}{2}-(\beta+\gamma-1)$. Since $e(G_{\rm b})\geqslant e(G_{\rm c})+|F|$, in order to prove the assertion, it is sufficient to show that $|F|\geqslant\beta+\gamma-1$.

We may assume that either $A\subseteq N_G(u,v)$ or $N_G(u,v)\cap A=A\setminus A_i$ for some $i\in\{1,\ldots,m\}$

with $|A_i| \in \{2,3\}$. To see this, suppose that $|N_G(u,v) \cap A| \leqslant t-3$ and u or v is not adjacent to x_{s_j} for some $j \in \{1,\ldots,m\}$. We find that $|N_G(u,v) \setminus A| \geqslant t-1-|N_G(u,v)\cap A| \geqslant 2$. So, we may remove two arbitrary edges between v and $N_G(u,v)\setminus A$ and join both u and v to x_{s_j} by black edges and to all vertices in $A_j \setminus \{x_{s_j}\}$ by red edges. By repeating this, we get $|N_G(u,v)\cap A| \geqslant t-2$, as desired.

If $A \subseteq N_G(u,v)$, then there are $2m$ black edges between $\{u,v\}$ and $\{x_{s_1},\ldots,x_{s_m}\}$ by (P.5) which only two of them belong to E and thus $|F| \geqslant 2m-2 \geqslant \beta+\gamma-1$, we are done.

So, assume that $N_G(u,v)\cap A = A \setminus A_i$ for some $i \in \{1,\ldots,m\}$ with $|A_i| \in \{2,3\}$. Thus, there are $2m-2$ black edges between $\{u,v\}$ and $\{x_{s_1},\ldots,x_{s_m}\} \setminus \{x_{s_i}\}$ which only two of them belong to E. Therefore, $|F| \geqslant 2m-4 \geqslant m-2 \geqslant \beta+\gamma-2$.

Working toward a contradiction, suppose that the inequality $|F| \geqslant \beta+\gamma-1$ is not valid. We have $\beta+\gamma-2 \geqslant |F| \geqslant 2m-4 \geqslant m-2 \geqslant \beta+\gamma-2$ which means that $m=\beta+\gamma=2$ and $F=\varnothing$. It follows from $m=\beta+\gamma=2$ that A_1, A_2 are independent sets and $A=A_1\cup A_2$. Since $t+1=|A_1|+|A_2| \leqslant 2\min\{3,t-2\}$, one concludes that $t=5$ and therefore $|A_1|=|A_2|=3$. Furthermore, it follows from $F=\varnothing$ that $|N_G(u,v)\cap C|=0$ and so $|N_G(u,v)|=|N_G(u,v)\cap A|=3$, contradicts with $|N_G(u,v)| \geqslant t-1$.

Case 3.11. (T.4) has happened.

As $k=t+1$, it follows from (1) that $e(G_{\rm c}) = n-2+\binom{t}{2}-(\beta+\gamma-1)$. Since $e(G_{\rm b}) \geqslant e(G_{\rm c})+|F|$, in order to prove the assertion, it is enough to show that $|F| \geqslant \beta+\gamma-1$. Before proceeding, we point out that $2m \leqslant |A_1|+\cdots+|A_m| \leqslant m(t-2)$ and so $2m \leqslant t+1 \leqslant m(t-2)$ which forces that $t \geqslant 5$.

From $u \in A$ and in view of (P.1), we may assume that $u = x_{s_i}$ for some $i \in \{1,\ldots,m\}$. Suppose that v is not adjacent to x_{s_j} for some $j \in \{1,\ldots,m\}\setminus\{i\}$. We have $|N_G(u,v)\setminus A| \geqslant t-1-|N_G(u,v)\cap A| \geqslant t-1-(k-|\{u\}\cup A_j|) = |A_j|-1 \geqslant 1$. So, we may remove an arbitrary edge between v and $N_G(u,v)\setminus A$ and join v to x_{s_j} by a black edge and to all vertices in $A_j\setminus\{x_{s_j}\}$ by red edges. By repeating this, we get $A\setminus A_i \subseteq N_G(u,v)$. Accordingly, there are $m-1$ black edges between v and $\{x_{s_1},\ldots,x_{s_m}\}\setminus\{x_{s_i}\}$ which only one of them belongs to E. Therefore, $|F| \geqslant m-2 \geqslant \beta+\gamma-2$.

Towards a contradiction, suppose that the inequality $|F| \geqslant \beta+\gamma-1$ does not hold. We have $\beta+\gamma-2 \geqslant |F| \geqslant m-2 \geqslant \beta+\gamma-2$ which means that $|F|=m-2$ and $\beta+\gamma=m$. It follows from $m=\beta+\gamma$ that $A_1,\ldots,A_m$ are independent sets. Also, it follows from $|F|=m-2$ that $uv \notin E(G)$ and $G_{\rm b}[C]=G_{\rm c}[C]$. The latter equality shows that $N_G(u,v)\cap C=\varnothing$. Since A_i is an independent set, we get $N_G(u,v)=A\setminus A_i$ which in turn yields that $|A_i|=2$.

Denote by G' the graph obtained from G by joining both vertices in A_i to all vertices in $N_G(v)\setminus N_G(u)$ and joining v to all vertices in $N_G(u)\setminus N_G(v)$. Set $A'=A\cup\{v\}$, $C'=C\setminus\{v\}$, $A'_i=A_i\cup\{v\}$, and $A'_j=A_j$ for any $j\in\{1,\ldots,m\}\setminus\{i\}$. We know that $A'_1,\ldots,A'_{m+1}$ are independent sets. For each $j \geqslant 0$, let C'_j be the set of vertices in C' with the distance j from A' in G' and let $\{C'_1,\ldots,C'_{d'}\}$ be a partition of C'. As $G'[C']=G_{\rm c}[C']$, we observe in G' that every C'_j is an independent set and moreover, for $j=2,\ldots,d'$, every vertex in C'_j has exactly one neighbor in C'_{j-1}. Further, for any vertex $c\in C'_1$, there is an index $j\in\{1,\ldots,m\}$ such that $N_{G'}(c)=A'_j$. Using these features and noting that $t\geqslant 5$, the following statements are straightforwardly obtained for two arbitrary distinct vertices $y,z\in V(G')$.

(i) Let $y\in C'$ and $z\in A'\cup C'$. Then, $N_{G'}(y,z)$ is one of $\varnothing$, $\{c\}$, or A'_j for some vertex $c\in C'$

and integer $j \in \{1, \dots, m\}$. Thus, $|N_{G'}(y,z)| \leqslant t-2$.

(ii) Let $y, z \in A'$. If $y \in A'_j$ and $z \in A'_\ell$ for some indices $j \neq \ell$, then $N_{G'}(y,z) \subseteq A' \setminus (A'_j \cup A'_\ell)$. Hence, $|N_{G'}(y,z)| \leqslant t-2$.

As G' is not a complete graph and there is no pair $\{y,z\}$ of vertices of G' such that $|N_{G'}(y,z)| \geqslant t-1$ and $N_{G'}(y) \setminus \{z\} \neq N_{G'}(z) \setminus \{y\}$, one deduces that G' and therefore G are not weakly $K_{2,t}$-saturated, a contradiction.

The proof is completed here. □

To complete the proof of Theorem 1.1, it remains to establish that $\mathrm{wsat}(n, K_{2,t}) \geqslant n-1+\binom{t}{2}$ for every integers n, t when t is even and $t+2 \leqslant n \leqslant 2t-2$. The following lemma is straightforwardly verified.

Lemma 3.12. *Let G be a graph with the partitioned vertex set $V(G) = X \cup Y \cup Z$ and the edge set $E(G) = \{ab \,|\, (a,b) \in (X,X) \cup (X,Y) \cup (Y,Z)\}$, where $|X| \geqslant 2$ and $|Y| \geqslant 1$. Then, G is weakly $K_{2,t}$-saturated if and only if either $|X| \geqslant t$ or $|Y| \geqslant t-1$ and $|X|+|Z| \geqslant t$.*

The following result has a crucial role in the proof of the last lemma.

Lemma 3.13. *Let $t \geqslant 3$ and let G be a weakly $K_{2,t}$-saturated graph of order n with $n \leqslant 2t-2$. Assume that v is a degree one vertex in G. Then, $G - v$ is also weakly $K_{2,t}$-saturated.*

Proof. Do a weakly $K_{2,t}$-saturation process on $V(G) \setminus \{v\}$ by joining nonadjacent vertices as far as possible. Suppose by way of contradiction that $G - v$ is not a complete graph. Define the relation $\approx$ on $V(G) \setminus \{v\}$ as $x \approx y$ if $|N_G(x,y)| \geqslant t-1$. Clearly, $\approx$ is an equivalence relation on $V(G) \setminus \{v\}$. Note that the equivalence classes are cliques or independent sets and the connections between two distinct equivalence classes are all present or all absent. Moreover, in view of $n \leqslant 2t-2$ and $\deg_G(v) = 1$, we observe that any independent equivalence class is of size at most $t-2$. Further, any clique equivalence class is of size at most $t-1$. Otherwise, we observe that K_{t+1} is a subgraph of $G - v$ and, by the connectivity of G and applying Lemma 3.1, we deduce that $G - v$ is a weakly $K_{2,t}$-saturated graph, a contradiction.

Assume that $\mathcal{A}$ is the equivalence class containing the neighbor of v. Note that $\mathcal{A} \neq V(G) \setminus \{v\}$. We continue the weakly $K_{2,t}$-saturation process on G by joining v to all vertices in $\mathcal{A}$. As G is not complete, there is a vertex w such that $|N_G(v,w)| \geqslant t-1$ and $N_G(v) \setminus \{w\} \neq N_G(w) \setminus \{v\}$. Since $N_G(v) = \mathcal{A}$, we deduce that $\mathcal{A}$ is a clique of size $t-1$ and moreover, $\mathcal{A} \subseteq N_G(w)$ implies that $w \notin \mathcal{A}$. Let $\mathcal{B}$ be the equivalence class containing w. We claim that $\mathcal{B}$ is an independent set. By contradiction, suppose that w has a neighbor $b \in \mathcal{B}$. Letting $a \in \mathcal{A}$, we have $(\mathcal{A} \setminus \{a\}) \cup \{w\} \subseteq N_G(a,b)$. This implies that $a \approx b$ which contradicts $\mathcal{A} \neq \mathcal{B}$, proving the claim. Now, add v to $\mathcal{B}$, join v to all vertices in $N_G(w) \setminus N_G(v)$, and join w to all vertices in $N_G(v) \setminus N_G(w)$. The resulting graph is not still complete, since $\mathcal{B}$ is an independent set. Thus, to proceed with the weakly $K_{2,t}$-saturation process, there should be two vertices $x \not\approx y$ such that $|N_G(x,y)| \geqslant t-1$ and $v \in N_G(x,y)$.

Since $x \not\approx y$, at most one of x, y belongs to $\mathcal{A}$. First, suppose that $x, y \notin \mathcal{A}$. From $x \not\approx y$ and $|\mathcal{A}| = t-1$, we conclude that $\mathcal{A} \nsubseteq N_G(x,y)$ which means that $\mathcal{A} \cap N_G(x,y) = \varnothing$. Then, it follows from $x, y \notin \mathcal{A}$ and $\{x,y\} \cup \mathcal{A} \cup N_G(x,y) \subseteq V(G)$ that $n \geqslant 2t$, a contradiction. Next, suppose without loss of generality that $x \in \mathcal{A}$ and $y \notin \mathcal{A}$. As $v \in N_G(x,y)$, we deduce that $w \in N_G(x,y)$ and so $(\mathcal{A} \setminus \{x\}) \cup \{w\} \subseteq N_G(x,y)$. This shows that $|N_G(x,y)| \geqslant t-1$ and therefore $x \approx y$, a contradiction. This contradiction completes the proof. □

The following lemma establishes a lower bound on $\mathrm{wsat}(n, K_{2,t})$ for even t.

Lemma 3.14. *Let $t \geqslant 4$ be even and let n be an integer with $t+2 \leqslant n \leqslant 2t-2$. Then,* $\mathrm{wsat}(n, K_{2,t}) \geqslant n - 1 + \binom{t}{2}$.

Proof. For $t = 4$, the assertion follows from Theorem 1.2. So, assume that $t \geqslant 6$ is fixed. Working toward a contradiction, consider a weakly $K_{2,t}$-saturated graph G_0 which is a counterexample with the minimum possible order. In view of Lemma 3.4, we find that $e(G_0) = n_0 - 2 + \binom{t}{2}$, where $n_0 = |V(G_0)|$. We have $t+3 \leqslant n_0 \leqslant 2t-2$ by Theorem 1.2 and $\delta(G_0) \geqslant 2$ by Lemma 3.13. Using Lemma 3.1, if we attach a new pendent vertex to a vertex of G_0, then the resulting graph is also weakly $K_{2,t}$-saturated whose number of vertices is one more than the number of vertices of G_0 and whose number of edges is one more than the number of edges of G_0. Attach t^3 new pendent vertices to each vertex of G_0 and call the resulting graph by G. By assuming $n = |V(G)|$, we have $e(G) = n - 2 + \binom{t}{2}$.

We apply the process defined in the proof of Lemma 3.4 to G and assume that the process has terminated. As the number of black edges does not change during the process, $e(G_{\mathrm{b}}) = n - 2 + \binom{t}{2}$. We need a careful exploration of termination states (T.1)–(T.4) to get a contradiction. We will do it below by distinguishing Cases 3.15, 3.16, 3.17, and 3.18.

Case 3.15. (T.1) has happened.

Since $e(G_{\mathrm{b}}) = n - 2 + \binom{t}{2}$, it follows from (1) that

$$(m-1)(k-t-1) - \beta - \gamma - \delta + 1 \leqslant 0. \tag{2}$$

Using $k \geqslant t - 1 + 2(m-1)$, $\beta + \gamma \leqslant m$, and $\delta \leqslant 1$, we derive from (2) that $m = 1$ or 2. First, assume that $m = 1$. Then, we deduce from $k \geqslant t - 1 \geqslant 5$ that $\beta = \gamma = 0$. Thus, it follows from (2) that $\delta = 1$ and so $k = t - 1$. Next, assume that $m = 2$. As $k \geqslant t - 1 + 2(m-1)$, we have $k \geqslant t + 1$ and so $\delta = 0$. Hence, it follows from (2) that $\beta + \gamma \geqslant 1$. If $\beta + \gamma = 2$, then we get $k \leqslant 6$ which contradicts $k \geqslant t + 1 \geqslant 7$. Therefore, $\beta + \gamma = 1$ and so we find from (2) that $k = t + 1$. This forces that $\beta = 1$ and $\gamma = 0$.

The above discussion indicates that the second term in (1) is equal to 0, implying that $G_{\mathrm{b}} = G_{\mathrm{c}}$. We know that there is no red edge in C, meaning that $G[C] = G_{\mathrm{c}}[C]$. From this and since G_0 is a subgraph of G with $\delta(G_0) \geqslant 2$, we deduce that $V(G_0) \cap C \subseteq C_1$. Further, as G_0 is a subgraph of G and $G_{\mathrm{b}} = G_{\mathrm{c}}$, we yield for any vertex $c \in V(G_0) \cap C$ that there is an index $i \in \{1, \ldots, m\}$ such that $N_{G_0}(c) \cap A \subseteq N_G(c) \cap A = A_i$.

We are now ready to describe the structure of G_0. If $m = 1$, then G_0 is obviously a subgraph of the graph $\mathbb{G} = K_s \vee \overline{K_{n_0 - s}}$ for some $s \leqslant t - 1$. But, this is a contradiction, since $\mathbb{G}$ and therefore G_0 are not weakly $K_{2,t}$-saturated. So, suppose that $m = 2$. As (T.1) has happened and $k = t + 1$, we may assume without loss of generality that $|A_1| = 2$ and $|A_2| = t - 1$. It follows from $\beta = 1$ that A_1 is an independent set. Now, it is easily seen that G_0 is a subgraph of the graph $\mathbb{H}$, depicted in Figure 2, with $X \subseteq A_2$, $Y_1 \subseteq A_1$, and $|V(\mathbb{H})| = n_0$. Since $|X| \leqslant t - 1$ and $n_0 \leqslant 2t - 2$, Lemma 3.12 implies that the described graph $\mathbb{H}$ and therefore G_0 are not weakly $K_{2,t}$-saturated, a contradiction.

Case 3.16. (T.2) has happened.

Since $k = t$, it follows from (1) that $n-2+\binom{t}{2} = e(G_{\rm b}) \geqslant e(G_{\rm c}) + |F| \geqslant n-2+\binom{t}{2} - (m+\beta+\gamma-2) + |F|$ and therefore $|F| \leqslant m+\beta+\gamma-2 \leqslant 2m-2$. As we proved in Case 3.9, either $A \subseteq N_G(u,v)$ or $N_G(u,v) \cap A = A \setminus A_i$ for some $i \in \{1,\ldots,m\}$ with $|A_i| = 2$.

Assume that $A \subseteq N_G(u,v)$. So, there are $2m$ black edges between $\{u,v\}$ and $\{x_{s_1},\ldots,x_{s_m}\}$ by (P.5) which only two of them belong to E. This yields that $|F| \geqslant 2m-2$ and thus $|F| = 2m-2$ and $\beta+\gamma = m$.

Assume that $N_G(u,v) \cap A = A \setminus A_i$ for some $i \in \{1,\ldots,m\}$ with $|A_i| = 2$. We have $|N_G(u,v) \setminus A| = |N_G(u,v)| - |N_G(u,v) \cap A| \geqslant t-1-|A \setminus A_i| = 1$. Consider an arbitrary vertex $w \in N_G(u,v) \setminus A$. As at most one of the edges uw and vw belongs to E, we may assume without loss of generality that $uw \in F$. Clearly, $w \in C_1 \cup C_2$. There are $2m$ black edges between $\{u,v\}$ and $(\{x_{s_1},\ldots,x_{s_m}\} \setminus \{x_{s_i}\}) \cup \{w\}$ which at most three of them belong to E. Therefore, $|F|$ is equal to either $2m-3$ or $2m-2$. Since $|F| \leqslant m+\beta+\gamma-2 \leqslant 2m-2$, we find that $m-1 \leqslant \beta+\gamma \leqslant m$.

Let $R = F \setminus E_G(\{u,v\},\{x_{s_1},\ldots,x_{s_m}\})$. According to what we saw above, $|R| \leqslant 2$. Set $A' = A \cup \{u,v\}$, $C' = C \setminus \{u,v\}$, $A'_j = A_j$ for $j = 1,\ldots,m$, and $A'_{m+1} = \{u,v\}$. We distinguish the following six cases.

(I.1) $A \subseteq N_G(u,v)$, $|F| = 2m-2$, and $R = \varnothing$. In this case, $A'_1,\ldots,A'_{m+1}$ are independent sets.

(I.2) $N_G(u,v) \cap A = A \setminus A_i$, A_i is an independent set, $|F| = 2m-3$, and $R = \{uw\}$. In this case, $A'_1,\ldots,A'_{m+1}$ are independent sets except possibly for A'_j, where $j \in \{1,\ldots,m\} \setminus \{i\}$.

(I.3) $N_G(u,v) \cap A = A \setminus A_i$, A_i is a clique, $|F| = 2m-3$, and $R = \{uw\}$. In this case, A'_i is a clique and $A'_1,\ldots,A'_{i-1},A'_{i+1},\ldots,A'_{m+1}$ are independent sets.

(I.4) $N_G(u,v) \cap A = A \setminus A_i$, $|F| = 2m-2$, and $R = \{uv,uw\}$. In this case, $A'_1,\ldots,A'_m$ are independent sets and A'_{m+1} is a clique.

(I.5) $N_G(u,v) \cap A = A \setminus A_i$, $|F| = 2m-2$, and $R = \{ab,uw\}$ for some $a \in A'$ and $b \in C'$. In this case, $A'_1,\ldots,A'_{m+1}$ are independent sets.

(I.6) $N_G(u,v) \cap A = A \setminus A_i$, $|F| = 2m-2$, and $R = \{b_1b_2,uw\}$ for some $b_1,b_2 \in C'$. In this case, $A'_1,\ldots,A'_{m+1}$ are independent sets.

We define a supergraph G' of G as follows. Denote by G' the graph obtained from G by joining u to all vertices in $N_G(v) \setminus N_G[u]$ and joining v to all vertices in $N_G(u) \setminus N_G[v]$. For any $j \geqslant 0$, let C'_j be the set of vertices in C' with the distance j from A' in G' and let $\{C'_1,\ldots,C'_{d'}\}$ be a partition of C'. In the cases (I.1)–(I.5), we have $E(G'[C']) = E(G_{\rm c}[C'])$ and therefore we observe in G' that $C'_1,\ldots,C'_{d'}$ are independent sets and moreover, every vertex in C'_j has exactly one neighbor in C'_{j-1} for $j = 2,\ldots,d'$. In the case (I.6), we have $E(G'[C']) = E(G_{\rm c}[C']) \cup \{b_1b_2\}$ and therefore we observe in G' that $C'_1,\ldots,C'_{d'}$, all except probably for one, are independent sets and moreover, every vertex in C'_j has exactly one neighbor in C'_{j-1} for all $j \in \{2,\ldots,d'\}$ except probably for one. Further, for every vertex $c \in C'_1 \setminus \{b\}$, there is an index $j \in \{1,\ldots,m+1\}$ such that $N_{G'}(c) \cap A' = A'_j$.

First, we consider the cases (I.1), (I.3), (I.4), and (I.6). In view of the structure of G' described above, the following statements are easily obtained for two arbitrary distinct vertices $y,z \in V(G')$.

(i) Let $y \in C'$ and $z \in A' \cup C'$. Then, $N_{G'}(y,z)$ is one of $\varnothing$, $\{c\}$, $\{c_1,c_2\}$, A'_j, or $A'_j \cup \{c\}$ for some vertices $c,c_1,c_2 \in C'$ and index $j \in \{1,\ldots,m+1\}$. Hence, $|N_{G'}(y,z)| \leqslant 4$.

(ii) Let $y,z \in A'$. Assume that $y \in A'_j$ and $z \in A'_\ell$ for some $j \neq \ell$. In the cases (I.1) and (I.6), we have $N_{G'}(y,z) \subseteq A' \setminus (A'_j \cup A'_\ell)$. In the case (I.3), we have $N_{G'}(y,z) \subseteq A' \setminus (A'_j \cup A'_{m+1})$ if $\ell = i$ and $N_{G'}(y,z) \subseteq A' \setminus (A'_j \cup A'_\ell)$ if $i \notin \{j,\ell\}$. In the case (I.4), we have $N_{G'}(y,z) \subseteq A' \setminus (A'_j \cup A'_i)$ if $\ell = m+1$ and $N_{G'}(y,z) \subseteq A' \setminus (A'_j \cup A'_\ell)$ if $m+1 \notin \{j,\ell\}$. Thus, in any case, $|N_{G'}(y,z)| \leqslant t-2$.

As G' is not a complete graph and there is no pair $\{y,z\}$ of vertices of G' such that $|N_{G'}(y,z)| \geqslant t-1$ and $N_{G'}(y)\setminus\{z\} \neq N_{G'}(z)\setminus\{y\}$, one finds that G' and therefore G are not weakly $K_{2,t}$-saturated, a contradiction.

Next, we consider the cases (I.2) and (I.5). Since G_0 is a subgraph of G' with $\delta(G_0) \geqslant 2$, we conclude that $V(G_0)\cap C' \subseteq C'_1$. In the case (I.2), set $\widetilde{A} = A'$ and in the case (I.5), set $\widetilde{A} = A'\cup\{b\}$. Now, it is easily seen that G_0 is a spanning subgraph of the graph $\mathbb{H}$, depicted in Figure 2, with $X \subseteq \widetilde{A}\setminus(A'_i \cup A'_{m+1})$ and $Y_1 \subseteq A'_i \cup A'_{m+1}$. As $|X| \leqslant t-1$ and $n_0 \leqslant 2t-2$, Lemma 3.12 implies that the described graph $\mathbb{H}$ and therefore G_0 are not weakly $K_{2,t}$-saturated, a contradiction.

Case 3.17. (T.3) has happened.

Since $k = t+1$, it follows from (1) that $n-2+\binom{t}{2} = e(G_{\mathrm{b}}) \geqslant e(G_{\mathrm{c}})+|F| \geqslant n-2+\binom{t}{2}-(\beta+\gamma-1)+|F|$. Therefore, $|F| \leqslant \beta+\gamma-1 \leqslant m-1$. As we proved in Case 3.10, either $A \subseteq N_G(u,v)$ or $N_G(u,v)\cap A = A\setminus A_i$ for some $i \in \{1,\dots,m\}$ with $|A_i| \in \{2,3\}$.

Suppose that $A \subseteq N_G(u,v)$. Then, there are $2m$ black edges between $\{u,v\}$ and $\{x_{s_1},\dots,x_{s_m}\}$ by (P.5) which only two of them belong to E. This yields that $|F| \geqslant 2m-2$ which along with $|F| \leqslant m-1$ gives $m \leqslant 1$, a contradiction.

So, we may assume that $N_G(u,v)\cap A = A\setminus A_i$ for some i with $|A_i| \in \{2,3\}$. Then, there are $2m-2$ black edges between $\{u,v\}$ and $\{x_{s_1},\dots,x_{s_m}\}\setminus\{x_{s_i}\}$ which only two of them belong to E. This gives $|F| \geqslant 2m-4$ which along with $|F| \leqslant \beta+\gamma-1 \leqslant m-1$ leads to $2m-3 \leqslant \beta+\gamma \leqslant m$. Hence, either $1 \leqslant \beta+\gamma \leqslant m = 2$ or $m = \beta+\gamma = 3$.

If $m = \beta+\gamma = 2$, then $t+1 = |A_1|+|A_2| \leqslant 6$ which contradicts $t \geqslant 6$. So, assume that either $m = 2$ and $\beta+\gamma = 1$ or $m = \beta+\gamma = 3$. In both cases, $|F| = 2m-4$ and so F is contained in the set of the black edges between $\{u,v\}$ and $\{x_{s_1},\dots,x_{s_m}\}\setminus\{x_{s_i}\}$. Therefore, $G[C] = G_{\mathrm{b}}[C] = G_{\mathrm{c}}[C]$. From this and since G_0 is a subgraph of G with $\delta(G_0) \geqslant 2$, we conclude that $V(G_0)\cap C \subseteq C_1$. As each edge in F is incident to either u or v, we observe in G_{b} that every vertex in $C\setminus\{u,v\}$ has exactly one neighbor in A. Therefore, for every vertex $c \in V(G_0)\cap C\setminus\{u,v\}$, we have $N_{G_0}(c)\cap A \subseteq N_G(c)\cap A = A_j$ for some $j \in \{1,\dots,m\}$. It results in that G_0 is a subgraph of the graph $\mathbb{H}$, depicted in Figure 2, with $X \subseteq A\setminus A_i$, $Y_1 \subseteq A_i\cup\{u,v\}$, and $|V(\mathbb{H})| = n_0$. Since $|X| \leqslant t-1$ and $n_0 \leqslant 2t-2$, Lemma 3.12 implies that the described graph $\mathbb{H}$ and therefore G_0 are not weakly $K_{2,t}$-saturated, a contradiction.

Case 3.18. (T.4) has happened.

Since $k = t+1$, it follows from (1) that $n-2+\binom{t}{2} = e(G_{\mathrm{b}}) \geqslant e(G_{\mathrm{c}})+|F| \geqslant n-2+\binom{t}{2}-(\beta+\gamma-1)+|F|$. Therefore, $|F| \leqslant \beta+\gamma-1 \leqslant m-1$. As we proved in Case 3.11, $A\setminus A_i \subseteq N_G(u,v)$ by assuming $u = x_{s_i}$. So, there are $m-1$ black edges between v and $\{x_{s_1},\dots,x_{s_m}\}\setminus\{x_{s_i}\}$ which only one of them belongs to E. Hence, $|F| \geqslant m-2$ which derives that $|F|$ equals either $m-2$ or $m-1$.

We show that $N_G(u,v)\cap A_i = \varnothing$. Suppose otherwise. This forces that A_i is a clique and $uv \in E(G)$. The latter implies that $|F| = m-1$ and so $\beta+\gamma = m$. In particular, A_i is an independent set, a contradiction.

If $N_G(u,v) = A\setminus A_i$, then $t-1 \leqslant |N_G(u,v)| = t+1-|A_i|$ and so $|A_i| = 2$.

If $N_G(u,v) \neq A\setminus A_i$, then there is a vertex $w \in N_G(u,v)\cap C_1$. Since $v,w \in C_1$, we have $vw \in F$. As F also contains $m-2$ edge between v and $\{x_{s_1},\dots,x_{s_m}\}\setminus\{x_{s_i}\}$, we conclude that $|F| = m-1$ and so $\beta+\gamma = m$. Note that $|F| = m-1$ forces that $N_G(u,v) = (A\setminus A_i)\cup\{w\}$.

Let $R = F \setminus E_G(\{u,v\},\{x_{s_1},\dots,x_{s_m}\})$. According to what we saw above, $|R| \leqslant 1$. Set $A' = A \cup \{v\}$, $C' = C \setminus \{v\}$, $A'_i = A_i \cup \{v\}$, and $A'_j = A_j$ for any $j \in \{1,\dots,m\} \setminus \{i\}$. We distinguish the following five cases.

(J.1) $N_G(u,v) = A \setminus A_i$, $|F| = m-2$, and $R = \varnothing$. In this case, $|A_i| = 2$ and all of $A'_1,\dots,A'_m$ except possibly for one are independent sets.

(J.2) $N_G(u,v) = A \setminus A_i$, $|F| = m-1$, and $R = \{uv\}$. In this case, A_i is an independent set of size 2 and $A'_1,\dots,A'_{i-1},A'_{i+1},\dots,A'_m$ are independent sets.

(J.3) $N_G(u,v) = A \setminus A_i$, $|F| = m-1$, and $R = \{ab\}$ for some $a \in A'$ and $b \in C'$. In this case, $|A_i| = 2$ and $A'_1,\dots,A'_m$ are independent sets. We divide this case to the following three subcases.

- (J.3.1) There is A'_ℓ such that $|A'_\ell| = 3$ and $N_G(b) \cap A'_\ell = \varnothing$.
- (J.3.2) Among $A'_1,\dots,A'_m$, there are exactly two sets A'_{j_1} and A'_{j_2} which meet $N_G(b)$. In addition, $t \geqslant 8$ and $|A'_{j_1}| = |A'_{j_2}| = 3$.
- (J.3.3) $t = 6$, $m = 3$, $|A'_1| = |A'_2| = 3$, and $|A'_3| = 2$. The vertex b has neighbors in both A'_1, A'_2 and no neighbor in A'_3.

(J.4) $N_G(u,v) = A \setminus A_i$, $|F| = m-1$, and $R = \{b_1b_2\}$ for some $b_1, b_2 \in C'$. In this case, $A'_1,\dots,A'_m$ are independent sets.

(J.5) $N_G(u,v) = (A \setminus A_i) \cup \{w\}$, $|F| = m-1$, and $R = \{vw\}$. In this case, $A'_1,\dots,A'_m$ are independent sets.

We define a supergraph G' of G as follows. Denote by G' the graph obtained from G by joining c to all vertices in $N_G(v) \setminus N_G[c]$ and joining v to all vertices in $N_G(c) \setminus N_G[v]$ for every vertex $c \in A_i$. Note that, if either $uv \in E(G)$ or A_i is a clique in G, then A'_i is a clique in G'. For any $j \geqslant 0$, let C'_j be the set of vertices in C' with the distance j from A' in G' and let $\{C'_1,\dots,C'_{d'}\}$ be a partition of C'. In the cases (J.1)–(J.3) and (J.5), we have $E(G'[C']) = E(G_{\mathrm{c}}[C'])$ and therefore we observe in G' that $C'_1,\dots,C'_{d'}$ are independent sets and moreover, every vertex in C'_j has exactly one neighbor in C'_{j-1} for $j = 2,\dots,d'$. In the case (J.4), we have $E(G'[C']) = E(G_{\mathrm{c}}[C']) \cup \{b_1b_2\}$ and therefore we observe in G' that $C'_1,\dots,C'_{d'}$, all except probably for one, are independent sets and moreover, every vertex in C'_j has exactly one neighbor in C'_{j-1} for all $j \in \{2,\dots,d'\}$ except probably for one. Further, for every vertex $c \in C'_1 \setminus \{b\}$, there is an index $j \in \{1,\dots,m\}$ such that $N_{G'}(c) \cap A' = A'_j$.

First, we consider the cases (J.1), (J.2), and (J.3.1). We claim that there exists an independent set A'_ℓ of size 3. There is nothing to prove in the case (J.3.1). In the cases (J.1) and (J.2), if A'_i is an independent set, then we let $\ell = i$. Otherwise, since $t-1 = |A'_1| + \cdots + |A'_{i-1}| + |A'_{i+1}| + \cdots + |A'_m|$ is odd and $A'_1,\dots,A'_{i-1},A'_{i+1},\dots,A'_m$ are independent sets of sizes 2 or 3, we find an index $\ell \in \{1,\dots,m\} \setminus \{i\}$ such that A'_ℓ is an independent set of size 3, as we claimed. Since G_0 is a subgraph of G' with $\delta(G_0) \geqslant 2$, we should have $V(G_0) \cap C' \subseteq C'_1$. Now, it is straightforwardly seen that G_0 is a spanning subgraph of the graph $\mathbb{H}$, depicted in Figure 2, with $X \subseteq A' \setminus A'_\ell$ and $Y_1 \subseteq A'_\ell$. As $|X| \leqslant t-1$ and $n_0 \leqslant 2t-2$, Lemma 3.12 implies that the described graph $\mathbb{H}$ and therefore G_0 are not weakly $K_{2,t}$-saturated, a contradiction.

Next, we consider the cases (J.3.2), (J.4), and (J.5). In view of the structure of G' described above, the following statements are easily obtained for two arbitrary distinct vertices $y, z \in V(G')$.

(i) Let $y \in C'$ and $z \in A' \cup C'$. Then, $N_{G'}(y,z)$ is one of $\varnothing$, $\{c\}$, $\{c_1,c_2\}$, A'_j, $A'_j \cup \{c\}$, $A'_{j_1} \cup A'_{j_2}$, where $c, c_1, c_2 \in C'$, $j \in \{1,\dots,m\}$, and j_1, j_2 are given in the case (J.3.2). This shows that $|N_{G'}(y,z)| \leqslant \max\{4, |A'_{j_1} \cup A'_{j_2}|\}$. As $t \geqslant 8$ in the case (J.3.2) and $t \geqslant 6$ in the cases (J.4) and (J.5), one deduces that $|N_{G'}(y,z)| \leqslant t-2$.

(ii) Let $y, z \in A'$. Assume that $y \in A'_{\ell_1}$ and $z \in A'_{\ell_2}$ for some $\ell_1 \neq \ell_2$. Then, $N_{G'}(y,z)$ is either $A' \setminus (A'_{\ell_1} \cup A'_{\ell_2})$ or $(A' \cup \{b\}) \setminus (A'_{j_1} \cup A'_{j_2})$. Note that the latter one occurs in the case (J.3.2) whenever $\ell_1 = j_1$ and $\ell_2 = j_2$. So, in any case, $|N_{G'}(y,z)| \leqslant t-2$.

As G' is not a complete graph and there is no pair $\{y,z\}$ of vertices of G' such that $|N_{G'}(y,z)| \geqslant t-1$ and $N_{G'}(y) \setminus \{z\} \neq N_{G'}(z) \setminus \{y\}$, we deduce that G' and therefore G are not weakly $K_{2,t}$-saturated, a contradiction.

Finally, we consider the case (J.3.3). Let $p \in A'_3$. It follows from $A'_1 \cup A'_2 \subseteq N_{G'}(b,p)$ that $|N_{G'}(b,p)| \geqslant t-1$. We define a supergraph G'' of G' as follows. Denote by G'' the graph obtained from G' by joining both vertices in A'_3 to all vertices in $N_G(b) \setminus N_G(p)$ and joining b to all vertices in $N_G(p) \setminus N_G(b)$. Set $A'' = A' \cup \{b\}$, $C'' = C' \setminus \{b\}$, $A''_1 = A'_1$, $A''_2 = A'_2$, and $A''_3 = A'_3 \cup \{b\}$. For any $j \geqslant 0$, let C''_j be the set of vertices in C'' with the distance j from A'' in G'' and let $\{C''_1, \ldots, C''_{d''}\}$ be a partition of C''. As $E(G''[C'']) = E(G_{\mathrm{c}}[C''])$, we observe in G'' that $C''_1, \ldots, C''_{d''}$ are independent sets and moreover, every vertex in C''_j has exactly one neighbor in C''_{j-1} for $j = 2, \ldots, d''$. Further, for every vertex $c \in C''_1$, there is an index $j \in \{1,2,3\}$ such that $N_{G''}(c) \cap A'' = A''_j$. Using these features, the following statements are easily obtained for two arbitrary distinct vertices $y, z \in V(G'')$.

(i) Let $y \in C''$ and $z \in A'' \cup C''$. Then, $N_{G''}(y,z)$ is one of $\varnothing$, $\{c\}$, or A''_j for some vertex $c \in C''$ and index $j \in \{1,2,3\}$. Hence, $|N_{G''}(y,z)| \leqslant 3$.

(ii) Let $y, z \in A''$. Assume without loss of generality that $y \in A''_1$ and $z \in A''_2$. So, $N_{G''}(y,z) = A''_3$ and thus $|N_{G''}(y,z)| = 3$.

As $t = 6$, there exists no pair $\{y,z\}$ of vertices of G'' such that $|N_{G''}(y,z)| \geqslant t-1$ and $N_{G''}(y) \setminus \{z\} \neq N_{G''}(z) \setminus \{y\}$. But, G'' is not a complete graph, so G'' and therefore G are not weakly $K_{2,t}$-saturated, a contradiction.

The proof is completed here. $\square$

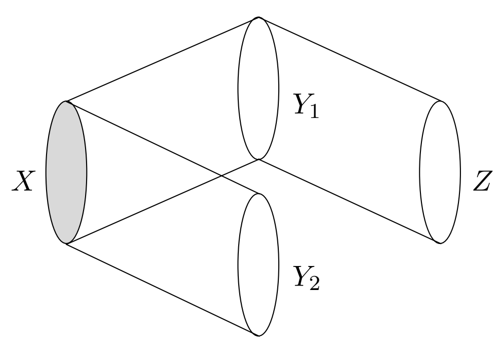

Figure 2. The graph $\mathbb{H}$. The set X is a clique and the sets Y_1, Y_2, Z are independent. Every vertex in X is adjacent to every vertex in $Y_1 \cup Y_2$ and every vertex in Z is adjacent to every vertex in Y_1.

We end the paper here by pointing out that Theorem 1.1 is concluded from Lemmas 3.2, 3.4, and 3.14.

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