The spectrum of the McKay-Miller-Širáň graphs

A. Mohammadian and B. Tayfeh-Rezaie

Dedicated to Professor G.B. Khosrovshahi on the Occasion of His 70th Birthday.

Abstract. We determine the spectrum of the McKay-Miller-Širáň graphs. It turns out these graphs have at most five distinct eigenvalues and sometimes are integral.

1. Introduction

Throughout this note, we assume that q is a prime power congruent to 1 modulo 4. We denote the finite field of order q by $\mathbb{F}_q$. Let $\mathcal{S}_q$ and $\mathcal{S}'_q$ be the sets of non-zero squares and non-squares in $\mathbb{F}_q$, respectively. The McKay-Miller-Širáň graph $\mathcal{H}_q$ is defined as follows: The vertex set is $\{0,1\} \times \mathbb{F}_q \times \mathbb{F}_q$ and the edges are given by

$$(0, x, y) \text{ is adjacent to } (0, x, y') \text{ if and only if } y - y' \in \mathcal{S}_q;$$
$$(1, m, c) \text{ is adjacent to } (1, m, c') \text{ if and only if } c - c' \in \mathcal{S}'_q;$$
$$(0, x, y) \text{ is adjacent to } (1, m, c) \text{ if and only if } y = mx + c.$$

From the definition and as $|\mathcal{S}_q| = (q-1)/2$, the degree of every vertex of $\mathcal{H}_q$ is $(3q-1)/2$. These graphs $\mathcal{H}_q$, that were first introduced in [**3**], are the currently largest order known vertex-transitive graphs of diameter 2 and valency $(3q-1)/2$. The smallest of these graphs, $\mathcal{H}_5$, is the Hoffman-Singleton graph which is the largest order Moore graph known to exist. Note that none of these graphs $\mathcal{H}_q$ is a Cayley graph [**3**, Theorem 2]. Here, we compute the eigenvalues of the adjacency matrix of $\mathcal{H}_q$.

The original definition of the McKay-Miller-Širáň graphs relies on a suitable lift of the complete bipartite graph $K_{q,q}$. A simplified construction was presented in [**4**] based on compositions of regular coverings. It is also worth noting that the McKay-Miller-Širáň graphs are very rich in symmetries; their automorphism groups were determined in [**2**], using ideas related to combinatorial geometry.

1991 *Mathematics Subject Classification.* Primary 05C50; Secondary 15A18.

Key words and phrases. Adjacency matrix, eigenvalue, McKay-Miller-Širáň graph, integral graph.

The research of the first author was in part supported by a grant from IPM.

2. The spectrum of $\mathcal{H}_q$

We first recall that the Paley graph $\mathcal{P}_q$ has $\mathbb{F}_q$ as the vertex set with two vertices being adjacent if and only if their difference is in $\mathcal{S}_q$. It is well known that $\mathcal{P}_q$ is a strongly regular graph with parameters $(q,(q-1)/2,(q-5)/4,(q-1)/4)$ [**1**, p. 221]. For some element $a \in \mathcal{S}'_q$, by considering the map that sends a vertex x to ax, we find that $\mathcal{P}_q$ is isomorphic to its complement $\overline{\mathcal{P}}_q$. Therefore, if P denotes the adjacency matrix of $\mathcal{P}_q$, we may assume that the adjacency matrix of $\mathcal{H}_q$ has the form

$$H = I_{2q} \otimes P + \begin{bmatrix} 0 & E \\ E^{\mathsf{T}} & 0 \end{bmatrix},$$

where I_n, M^{T} and $A \otimes B$ denote the $n \times n$ identity matrix, the transpose of a matrix M and the Kronecker product of two matrices A and B, respectively, and E is a $q \times q$ block matrix whose entries are $q \times q$ permutation matrices.

The main step in computing the eigenvalues of H is to determine H^2. For this, we go to determine the number of common neighbors of each pair of vertices of $\mathcal{H}_q$. For any vertex v of $\mathcal{H}_q$, let $\mathcal{N}(u)$ be the set of neighbors of u. It is straightforward to check that

$$\mathcal{N}(0,x,y) = \{(0,x,y+\alpha) \mid \alpha \in \mathcal{S}_q\} \cup \{(1,a,y-ax) \mid a \in \mathbb{F}_q\} \tag{2.1}$$

and

$$\mathcal{N}(1,m,c) = \{(0,b,mb+c) \mid b \in \mathbb{F}_q\} \cup \{(1,m,c+\beta) \mid \beta \in \mathcal{S}'_q\}, \tag{2.2}$$

for all elements $x,y,m,c \in \mathbb{F}_q$. Assume that v and w are two vertices of $\mathcal{H}_q$. If the first components of v and w are the same, then by the definition of $\mathcal{H}_q$ and the parameters of the strongly regular graphs $\mathcal{P}_q$ and $\overline{\mathcal{P}}_q$, we find that $|\mathcal{N}(v) \cap \mathcal{N}(w)| = (q-5)/4$ or $(q-1)/4$, depending on v and w are adjacent or not. Otherwise, using (2.1) and (2.2), it is not hard to see that $|\mathcal{N}(v) \cap \mathcal{N}(w)| = 0$ or 1, depending on v and w are adjacent or not. From this argument, it follows that

$$\begin{aligned} H^2 = & \frac{3q-1}{2} I + \frac{q-5}{4} I_{2q} \otimes P + \frac{q-1}{4} I_{2q} \otimes (J - I - P) \\ & + \begin{bmatrix} (J_q - I_q) \otimes J_q & J - E \\ J - E^{\mathsf{T}} & (J_q - I_q) \otimes J_q \end{bmatrix}, \end{aligned}$$

where J_n denotes the $n \times n$ all one matrix. After simplifying, we obtain

$$H^2 + H = \frac{5q-1}{4} I + J + \frac{q-5}{4} I_{2q} \otimes J_q. \tag{2.3}$$

If $q = 5$, then from (2.3), it is observed that $\mathcal{H}_5$ is a strongly regular graph with parameters $(50,7,0,1)$ and so by [**1**, p. 219], the eigenvalues are $7,2,-3$ with multiplicities $1,28,21$, respectively. We therefore assume that $q > 5$. Clearly, the eigenvalues of the right hand side of (2.3) are $(9q^2-1)/4$, $(q^2-1)/4$, $(5q-1)/4$ with multiplicities 1, $2q-1$, $2q^2-2q$, respectively. Since $\mathcal{H}_q$ is a $(3q-1)/2$-regular graph, $\lambda_0 = (3q-1)/2$ is an eigenvalue of H with multiplicity $m_0 = 1$. Moreover, $2q-1$ eigenvalues of H satisfy $x^2 + x - (q^2-1)/4 = 0$ and $2q^2 - 2q$ eigenvalues of H satisfy $x^2 + x - (5q-1)/4 = 0$. Thus the other eigenvalues of H are $\lambda_1 = (-1+q)/2$, $\lambda_2 = (-1-q)/2$, $\lambda_3 = (-1+\sqrt{5q})/2$ and $\lambda_4 = (-1-\sqrt{5q})/2$. Let m_i be the multiplicity of λ_i in the spectrum of $\mathcal{H}_q$, for $i = 1,2,3,4$. We have $m_1 + m_2 = 2q-1$ and $m_3 + m_4 = 2q^2 - 2q$. It is well known that the trace of H^i is equal to the sum of the ith power of eigenvalues of H. Using this fact for $i = 1, 2$ and (2.3), it follows that

$$\begin{cases} m_0\lambda_0 + m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 + m_4\lambda_4 = 0 \\ m_0\lambda_0^2 + m_1\lambda_1^2 + m_2\lambda_2^2 + m_3\lambda_3^2 + m_4\lambda_4^2 = q^2(3q-1). \end{cases}$$

Now, by considering the equations $m_1 + m_2 = 2q - 1$, $m_3 + m_4 = 2q^2 - 2q$ and solving the obtained system, we have the following theorem.

THEOREM 2.1. *The eigenvalues of* $\mathcal{H}_q$ *are* $(3q-1)/2$, $(q-1)/2$, $(\sqrt{5q}-1)/2$, $-(\sqrt{5q}+1)/2$, $-(q+1)/2$ *with multiplicities* 1, $2q-2$, q^2-q, q^2-q, 1, *respectively.*

We remark that for any odd number r, $\mathcal{H}_{5^r}$ is an integral graph; that is, a graph whose spectrum of its adjacency matrix consists entirely of integers. Note that the integral graphs are very rare and difficult to be found.

Acknowledgments. The authors gratefully acknowledge valuable suggestions from the referee which helped to considerably shorten this paper. The authors would also like to express their gratitude to Professor Jozef Širáň for introducing the problem to them. The paper was written during the second author's visit to the Abdus Salam International Centre for Theoretical Physics (ICTP). It would be a pleasure to thank ICTP for the hospitality and facilities.

References

[1] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.
[2] P.R. Hafner, *Geometric realisation of the graphs of McKay-Miller-Širáň*, J. Combin. Theory Ser. B **90** (2004), 223–232.
[3] B.D. McKay, M. Miller and J. Širáň, *A note on large graphs of diameter two and given maximum degree*, J. Combin. Theory Ser. B **74** (1998), 110–118.
[4] J. Šiagiová, *A note on the McKay-Miller-Širáň graphs*, J. Combin. Theory Ser. B **81** (2001), 205–208.

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran
E-mail address: ali_m@ipm.ir

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran
E-mail address: tayfeh-r@ipm.ir