Maximum order of trees and bipartite graphs with a given rank

E. Ghorbani[1,2] A. Mohammadian[2] B. Tayfeh-Rezaie[2]

[1]*Department of Mathematics, K.N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran*

[2]*School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran*

e_ghorbani@ipm.ir ali_m@ipm.ir tayfeh-r@ipm.ir

Abstract

The rank of a graph is that of its adjacency matrix. A graph is called reduced if it has no isolated vertices and no two vertices with the same set of neighbors. We determine the maximum order of reduced trees as well as bipartite graphs with a given rank and characterize those graphs achieving the maximum order.

Keywords: rank, tree, bipartite graph, adjacency matrix.

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1 Introduction

For a graph G, we denote by $V(G)$, the vertex set of G and the *order* of G is defined as $|V(G)|$. If $V(G) = \{v_1, \ldots, v_n\}$, then the *adjacency matrix* of G is an $n \times n$ real matrix $A(G)$ whose (i, j)-entry is 1 if v_i is adjacent to v_j and 0 otherwise. The *rank* of G, denoted by $\mathrm{rank}(G)$, is the rank of $A(G)$. The roots of the characteristic polynomial of $A(G)$ are called the *eigenvalues* of G.

We recall some definitions and notation used in the rest of paper. For a vertex v of a graph G, let $\mathcal{N}(v)$ denote the set of all vertices of G adjacent to v. The *degree* of v is defined by $d(v) = |\mathcal{N}(v)|$. We call a vertex v of G *pendant* if $d(v) = 1$. A vertex adjacent to a pendant vertex is said to be *pre-pendant*. A graph is called *reduced* if it has no isolated vertex and no two vertices v, w with $\mathcal{N}(v) = \mathcal{N}(w)$. For a subset S of $V(G)$, the notation $G - S$ represents the subgraph obtained by removing the vertices in S from G and also deleting all edges with at least one end vertex in S. For a vertex v of G, we use $G - v$ for $G - \{v\}$.

Let $r \geqslant 2$ be an integer. It is not hard to prove that every reduced graph of rank r has at most $2^r - 1$ vertices. Let $m(r)$ be the maximum possible order of a reduced graph of rank r. In [6], it was proved that there exists a constant c such that $m(r) \leqslant c \cdot 2^{r/2}$ and a construction was provided for the graphs of order

$$n(r) = \begin{cases} 2^{(r+2)/2} - 2 & \text{if } r \text{ is even,} \\ 5 \cdot 2^{(r-3)/2} - 2 & \text{if } r \text{ is odd.} \end{cases}$$

It is conjectured in [1] that, in fact, $m(r) = n(r)$. We know from [4] that if G is a reduced graph of rank r containing an induced matching of size $r/2$ or an induced subgraph consisting of the vertex disjoint union of a matching of size $(r-3)/2$ and a cycle of order 3, then the order of G is at most $m(r)$. Further, it is established in [7] that for every reduced graph G with no path of length 3 as an induced subgraph, $\text{rank}(G)$ is equal to the order of G. Finally, it is worth to mention that for any eigenvalue $\mu \notin \{-1, 0\}$ of a graph G with $n \geqslant 5$ vertices, there is an upper bound for n in terms of $r = \text{rank}(A(G) - \mu I)$, namely $n \leqslant r(r+1)/2$ by a result from [2].

In this paper, we show that every reduced tree of rank r has at most $\frac{3r}{2} - 1$ vertices and characterize all reduced trees of rank r and order $\frac{3r}{2} - 1$. We also prove that every reduced bipartite graph of rank r has at most $2^{r/2} + \frac{r}{2} - 1$ vertices and characterize all reduced bipartite graphs achieving this bound.

2 Reduced trees

For any integer r, let $t(r) = \frac{3r}{2} - 1$. In the following, we prove that any reduced tree of rank r has at most $t(r)$ vertices and characterize all reduced trees achieving this bound. Notice that a tree is reduced if and only if no two pendant vertices have the same neighbor. We first state the following well known fact.

Lemma 1. *Let G be a graph and u be a pendant vertex of G with the unique neighbor v. Then* $\text{rank}(G) = \text{rank}(G - \{u, v\}) + 2$.

We denote the path of order ℓ by P_ℓ. Let k be a positive integer and $n = 3k - 1$. We recursively define the family $\mathcal{R}_n$ of reduced trees of order n as follows: The set $\mathcal{R}_2$ contains just P_2, and $\mathcal{R}_n$ is constructed from $\mathcal{R}_{n-3}$ by attaching a pendant vertex of a P_3 to a pre-pendant vertex of a tree in $\mathcal{R}_{n-3}$, whenever $n \geqslant 5$. For example, the elements of $\mathcal{R}_{14}$ are depicted in Figure 1. By Lemma 1, any tree in $\mathcal{R}_n$ is of rank k.

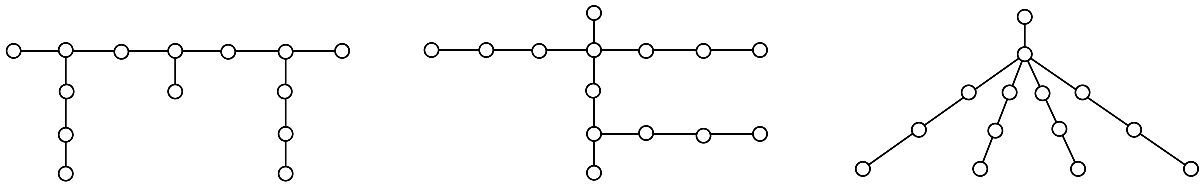

Figure 1: The family $\mathcal{R}_{14}$.

Theorem 2. *The order of any reduced tree of rank r is at most $t(r)$. Moreover, the set of all reduced trees of rank r and order $t(r)$ is $\mathcal{R}_{t(r)}$.*

Proof. Suppose that T is a reduced tree of order n and rank r. Since T is a bipartite graph, r is even. We proceed by induction on r. If $r = 2$, then by Lemma 1, T is a star and since T is reduced, we have $T = P_2$, as required. Let $r \geqslant 4$. Consider a path of maximum length in T and call its first two vertices from one end u and v, respectively. Clearly, u is a pendant vertex and $d(v) = 2$, since T is reduced and $T \neq P_2$. By Lemma 1, the tree $T' = T - \{u, v\}$ has rank $r - 2$. If T' is reduced, then $n - 2 \leqslant t(r - 2)$ and hence $n \leqslant t(r) - 1$. So assume that T' is not reduced. Then T has a vertex $w \in \mathcal{N}(v)$ of degree 2 which is a pendant vertex in T'. Since T is reduced, $T'' = T' - w$ is also a reduced tree. Thus $n - 3 \leqslant t(r - 2)$ and so $n \leqslant t(r)$. This proves the first statement of the theorem. Note that if $n = t(r)$, then T'' is a reduced tree of rank $r - 2$ and order $t(r - 2)$. By the induction hypothesis, $T'' \in \mathcal{R}_{t(r-2)}$ and T is obtained from T'' by attaching a pendant vertex of a P_3 to a pre-pendant vertex of T''. It follows that $T \in \mathcal{R}_{t(r)}$. $\square$

A reduced tree T is said to be *maximal* if any reduced tree containing T as a proper subtree has a higher rank. In what follows, we characterize all maximal trees of a given rank. The following lemma provides a simple description of maximal trees.

Lemma 3. *A reduced tree T is maximal if and only if for every vertex v which is not pre-pendant, $\operatorname{rank}(T) = \operatorname{rank}(T - v)$; or equivalently, there exists a vector $\boldsymbol{x}$ in the null space of $A(T)$ such that $\boldsymbol{x}(v) \neq 0$.*

Proof. By Lemma 1, attaching a new vertex to a vertex v of a tree T increases rank if and only if $\operatorname{rank}(T) = \operatorname{rank}(T - v)$. Therefore, a reduced tree T is maximal if and only if $\operatorname{rank}(T) = \operatorname{rank}(T - v)$, for any non-pre-pendant vertex v. On the other hand, it is known that for every vertex v of a graph G, $\operatorname{rank}(G) = \operatorname{rank}(G - v)$ if and only if there exists a vector $\boldsymbol{x}$ in the null space of $A(G)$ such that $\boldsymbol{x}(v) \neq 0$. This completes the proof. $\square$

Theorem 4. *Every maximal tree T of rank $r \geqslant 4$ is obtained from a maximal tree T' of rank $r-2$ in one of the two following ways:*

(i) *attaching a vertex of a P_2 to a vertex of T' which is neither pendant nor pre-pendant;*

(ii) *attaching a pendant vertex of a P_3 to a pre-pendant vertex of T'.*

Proof. By Lemma 1, it is easy to see that any tree resulting by (i) is maximal. Let T be a maximal tree of rank r which is not obtained by (i). We prove that T is obtained by (ii). Consider a path of maximum length in T and call its first four vertices from one end u, v, w, y, respectively. So u is a pendant vertex and $d(v) = 2$. We claim that w is not a pre-pendant vertex. Otherwise, for any vector $\boldsymbol{x}$ in the null space of $A(T)$, we have $\boldsymbol{x}(w) = 0$. Also, since the sum of the components of $\boldsymbol{x}$ corresponding to the neighbors of v is zero, $\boldsymbol{x}(u) = 0$ which contradicts Lemma 3. This proves the claim. Furthermore, if $d(w) \geqslant 3$, then $T - \{u, v\}$ would be a maximal tree of rank $r-2$ which contradicts our assumption on T. Thus $d(w) = 2$. We show that $T' = T - \{u, v, w\}$ is a reduced tree of rank $r-2$. Applying Lemmas 1 and 3, we find that $\mathrm{rank}(T') = \mathrm{rank}(T-u) - 2 = r-2$ and also $\mathrm{rank}(T-y) = \mathrm{rank}(T'-y) + 2 = \mathrm{rank}(T - \{w, y\}) = r-2$. This and Lemma 3 imply that y is a pre-pendant vertex and hence T' is reduced.

Let z be the pendant vertex adjacent to y and let $\{\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_{n-r-1}\}$ be a basis for the null space of $A(T')$. We define a basis $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-r}\}$ for the null space of $A(T)$ as follows. For $1 \leqslant i \leqslant n-r-1$, we let $\boldsymbol{x}_i(a) = \boldsymbol{x}'_i(a)$ for every $a \in V(T'-z)$ and we set $\boldsymbol{x}_i(u) = -\boldsymbol{x}_i(w) = \boldsymbol{x}_i(z)/2 = \boldsymbol{x}'_i(z)$ and $\boldsymbol{x}_i(v) = 0$. Moreover, $\boldsymbol{x}_{n-r}$ is defined as zero on $V(T'-z)$ and we put $\boldsymbol{x}_{n-r}(u) = -\boldsymbol{x}_{n-r}(w) = \boldsymbol{x}_{n-r}(z) = 1$ and $\boldsymbol{x}_{n-r}(v) = 0$. Now, in view of Lemma 3, T' is a maximal tree of rank $r-2$ and so T is obtained by (ii), as desired.

Note that, using Lemma 3, the argument appeared in the previous paragraph also shows that any tree resulting by (ii) is maximal. So the proof is complete. $\square$

3 Reduced bipartite graphs

For an even integer r, let $b(r) = 2^{r/2} + \frac{r}{2} - 1$. In this section, we show that every reduced bipartite graph of rank r has at most $b(r)$ vertices. We also prove that there exists a unique reduced bipartite graph of rank r and order $b(r)$. For a graph G, a subset S of $V(G)$ with more than one element is called a *duplication class* of G if $\mathcal{N}(u) = \mathcal{N}(v)$ for any $u, v \in S$. The proof of the following lemma can be found in [5, 6]. We remark that for every vertex v of a graph G with $d(v) \geqslant 1$, it is easily checked that $\mathrm{rank}(G - \mathcal{N}(v)) \leqslant \mathrm{rank}(G) - 2$.

Lemma 5. *Let G be a reduced graph and H be an induced subgraph of G with the maximum possible order subject to* $\mathrm{rank}(H) < \mathrm{rank}(G)$. *Then the following hold.*

(i) $|V(G) \setminus V(H)| \leqslant \min\big\{|\mathcal{N}(u)\triangle\mathcal{N}(v)| \,\big|\, u,v \in V(G)\big\}$, *where* $\triangle$ *denotes the symmetric difference operation on sets, and* $\operatorname{rank}(H) \geqslant \operatorname{rank}(G) - 2$.

(ii) *If* w *is an isolated vertex of* H*, then* $\mathcal{N}(w) = V(G) \setminus V(H)$.

(iii) *If* H *is not reduced, then* $\operatorname{rank}(H) = \operatorname{rank}(G) - 2$*, each duplication class of* H *has two elements and* H *has at most one isolated vertex.*

(iv) *If* H *is not reduced and* $\{v_1, v_1'\}, \ldots, \{v_s, v_s'\}$ *are all the duplication classes of* H*, then there exist two sets* S, S' *such that* $V(G) \setminus V(H) = S \cup S'$, $S \subseteq \mathcal{N}(v_i) \setminus \mathcal{N}(v_i')$ *and* $S' \subseteq \mathcal{N}(v_i') \setminus \mathcal{N}(v_i)$ *for any* $i \in \{1, \ldots, s\}$.

Lemma 6. *Let* G *be a graph of order* n *and let* S *be an independent set in* G *with* $|S| = \alpha \geqslant 2$*. Then*

$$\min\big\{|\mathcal{N}(u)\triangle\mathcal{N}(v)| \,\big|\, u,v \in S\big\} \leqslant \frac{\alpha(n-\alpha)}{2(\alpha-1)}.$$

Proof. Let $m = \min\big\{|\mathcal{N}(u)\triangle\mathcal{N}(v)| \,\big|\, u,v \in S\big\}$ and $s = \sum_{u,v\in S} |\mathcal{N}(u)\triangle\mathcal{N}(v)|$. We have $m\binom{\alpha}{2} \leqslant s$. On the other hand, a double counting argument shows that

$$s = \sum_{w\in V(G)\setminus S} d_S(w)\big(\alpha - d_S(w)\big),$$

where $d_S(w) = |\mathcal{N}(w) \cap S|$. Since $d_S(w)(\alpha - d_S(w)) \leqslant \alpha^2/4$ for any vertex $w \in V(G) \setminus S$, we find that $s \leqslant (n-\alpha)\alpha^2/4$. It follows that $m \leqslant \frac{\alpha(n-\alpha)}{2(\alpha-1)}$. □

We recall a family of bipartite graphs, see [3]. Let n be a positive integer. Suppose that B is a set with n elements and let $\mathscr{P}(B)$ denote the set of all non-empty subsets of B. We consider the bipartite incidence graph $\mathcal{B}_n$ with the vertex set $B \cup \mathscr{P}(B)$ and the edges connecting two vertices $x \in B$ and $X \in \mathscr{P}(B)$ if and only if $x \in X$. It is easy to see that $\mathcal{B}_n$ is a reduced bipartite graph of rank $2n$ and order $2^n + n - 1$.

Theorem 7. *The order of a reduced bipartite graph of rank* r *is at most* $b(r)$*. Moreover, every reduced bipartite graph of rank* r *and order* $b(r)$ *is isomorphic to* $\mathcal{B}_{r/2}$.

Proof. Let G be a reduced bipartite graph of rank r and order $n \geqslant b(r)$. Let $\{V_1, V_2\}$ be a partition of $V(G)$ into independent sets V_1 and V_2. By induction on r, we prove that G is isomorphic to $\mathcal{B}_{r/2}$. Since every graph of rank 2 is complete bipartite, there is nothing to prove when $r = 2$. Assume that $r \geqslant 4$. Let H be an induced subgraph of G with the maximum possible order such that $\operatorname{rank}(H) < \operatorname{rank}(G)$ and let $t = n - |V(H)|$. Suppose towards a contradiction that H has no duplication class. By the induction hypothesis and Lemma 5 (iii), $|V(H)| - 1 \leqslant b(r-2)$. Since the independence number of G is at least $n/2$, using Lemma 6, we have $t < (n+3)/4$. Hence $(3b(r) - 7)/4 < |V(H)| - 1 \leqslant b(r-2)$

which contradicts $r \geqslant 4$. Assume that $\{v_1, v_1'\}, \ldots, \{v_s, v_s'\}$ are the duplication classes of H and let K be the resulting graph after deleting the vertices $v_1', \ldots, v_s'$ and the possible isolated vertices from H. From Lemma 5 (iv), we may assume that $V(G) \setminus V(H) \subseteq V_1$ and $\{v_1, \ldots, v_s\} \subseteq V_2$. Let P be the subset of V_2 obtained by removing the vertices $v_1, v_1', \ldots, v_s, v_s'$ and the possible isolated vertices of H and let $Q = V_1 \cap V(K)$. Set $p = |P|$, $q = |Q|$ and $k = |V(K)|$. If we denote the number of isolated vertices of H by ε, then Lemma 5 (iii) implies that $\varepsilon \in \{0, 1\}$. Since K is a reduced bipartite graph with $\operatorname{rank}(K) \leqslant r - 2$, by the induction hypothesis, we have $k \leqslant b(r-2)$.

We assume that $k \geqslant (2n + r - 6)/4$. This implies that equality occurs in both $n \geqslant b(r)$ and $k \leqslant b(r-2)$. By the induction hypothesis, K is isomorphic to $\mathcal{B}_{(r-2)/2}$ and thus $q = \frac{r}{2} - 1$. Hence, by $n = 2k - p - q + t + \varepsilon$, we find that $t = p + 2 - \varepsilon$. If $t \geqslant 3$, then by Lemma 5 (iv), there are two vertices $x, y \in V(G) \setminus V(H)$ such that $\mathcal{N}(x) \triangle \mathcal{N}(y) \subseteq P$. By Lemma 5 (i), we deduce that $p \geqslant t$ which is impossible. Thus $t \leqslant 2$. If $t = 2$, then either $p = \varepsilon = 0$ or $p = \varepsilon = 1$. Hence the bipartite adjacency matrix of G, that is the submatrix of $A(G)$ whose rows and columns are respectively indexed by V_1 and V_2, has one of the forms

$$A_1 = \begin{bmatrix} B & B \\ \boldsymbol{j} & 0 \\ 0 & \boldsymbol{j} \end{bmatrix} \quad \text{or} \quad A_2 = \begin{bmatrix} B' & B' & 0 & \boldsymbol{b} \\ \boldsymbol{j} & 0 & 1 & \star \\ 0 & \boldsymbol{j} & 1 & \star \end{bmatrix},$$

where $B = [B' \ \boldsymbol{b}]$ is the bipartite adjacency matrix of $\mathcal{B}_{(r-2)/2}$ and $\boldsymbol{j}$ is the all one vector. Since $\boldsymbol{j}$ is not in the row space of B, $\operatorname{rank}(A_1) = \frac{r}{2} + 1$. Also, it is easy to see that $\operatorname{rank}(A_2) = \frac{r}{2} + 1$. Hence in both cases $\operatorname{rank}(G) = r + 2$, a contradiction. Therefore $t = 1$ and so $\varepsilon = 1$. Now, it is straightforward to check that G is isomorphic to $\mathcal{B}_{r/2}$.

It is clear that $\operatorname{rank}(G) \leqslant 2 \min\{|V_1|, |V_2|\}$, namely $\min\{|V_1|, |V_2|\} \geqslant r/2$. If $\min\{|V_1|, |V_2|\} = r/2$, then one can easily see that G is isomorphic to $\mathcal{B}_{r/2}$. Hence, to complete the proof, we assume, toward a contradiction, that $k < (2n+r-6)/4$ and $\min\{|V_1|, |V_2|\} \geqslant \frac{r}{2}+1$. Then the equalities $n = k+t+s+\varepsilon$ and $k = p + q + s$ yield that $p + q \leqslant \frac{r}{2} + t + \varepsilon - 4$. Therefore, $\frac{r}{2} + 1 \leqslant q + t \leqslant \frac{r}{2} + 2t + \varepsilon - p - 4$ which implies that $2 \leqslant (p - \varepsilon + 5)/2 \leqslant t$.

First suppose that $t \geqslant 3$. By Lemma 5 (iv), there are two vertices $x, y \in V(G) \setminus V(H)$ such that $\mathcal{N}(x) \triangle \mathcal{N}(y) \subseteq P$. By Lemma 5 (i), we deduce that $p \geqslant t$. We claim that $s \geqslant 2$. By contradiction, suppose that $s = 1$. From $t < (n+3)/4$ and $p + q \leqslant \frac{r}{2} + t + \varepsilon - 4$, we obtain that $n = p + q + t + 2s + \varepsilon < (n + r + 3)/2$. Since $n \geqslant b(r)$ and $r \geqslant 4$, we find that $r = 4$ and $n \leqslant 6$. However, $n \geqslant p + t + 2s \geqslant 2t + 2 \geqslant 8$, a contradiction. Hence $s \geqslant 2$. Since $\mathcal{N}(v_1) \triangle \mathcal{N}(v_2) \subseteq Q$, Lemma 5 (i) implies that $q \geqslant t$. Now, from $p + q \leqslant \frac{r}{2} + t + \varepsilon - 4$, we obtain that $t \leqslant q \leqslant \frac{r}{2} - 3$. Since $\mathcal{N}(v_1) \cap Q, \ldots, \mathcal{N}(v_s) \cap Q$ are distinct, we deduce that $s \leqslant 2^q - 1$ and so $s \leqslant 2^{(r-6)/2} - 1$. Moreover, we have $n = p+q+t+2s+\varepsilon \leqslant \frac{r}{2}+2t+2s-2 \leqslant \frac{3r}{2}+2s-6$ which in turn implies that $b(r) \leqslant 2^{(r-4)/2} + \frac{3r}{2} - 10$, a contradiction.

Next assume that $t = 2$. From $2 \leqslant (p - \varepsilon + 5)/2 \leqslant t$, we find that $p = 0$ and $\varepsilon = 1$. Furthermore, the inequalities $p + q \leqslant \frac{r}{2} + t + \varepsilon - 4$ and $q + t \geqslant \frac{r}{2} + 1$ show that $q = \frac{r}{2} - 1$. From $b(r) \leqslant n$ and

$s \leqslant 2^q - 1$, we obtain that $s = 2^{(r-2)/2} - 1$ and so the bipartite adjacency matrix of G has the form

$$A = \begin{bmatrix} B & B & 0 \\ \boldsymbol{j} & 0 & 1 \\ 0 & \boldsymbol{j} & 1 \end{bmatrix}.$$

Clearly, $\mathrm{rank}(A) = \frac{r}{2} + 1$ which means that $\mathrm{rank}(G) = r + 2$, a contradiction. □

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