Star Saturation Number of Random Graphs

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Abstract

For a given graph F, the F-saturation number of a graph G is the minimum number of edges in an edge-maximal F-free subgraph of G. Recently, the F-saturation number of the Erdős–Rényi random graph $\mathbb{G}(n, p)$ has been determined asymptotically for any complete graph F. In this paper, we give an asymptotic formula for the F-saturation number of $\mathbb{G}(n, p)$ when F is a star graph.

Keywords: Random graph, Saturation, Star graph.

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1 Introduction

All graphs in this paper are assumed to be finite, undirected, and without loops or multiple edges. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For any subset S of $V(G)$, the induced subgraph of G on S is denoted by $G[S]$. For an integer $n \geqslant 1$ and a real number $p \in [0, 1]$, we denote by $\mathbb{G}(n, p)$ the probability space of all graphs on a fixed vertex set of size n where every two distinct vertices are adjacent independently with probability p.

In 1941, Turán posed one of the foundational problems in extremal graph theory [8]. His question was about the maximum number of edges in a graph on n vertices without a copy of a given graph F as a subgraph, a parameter which is now denoted by $ex(n, F)$. A dual idea called 'saturation number' was introduced by Zykov [10] and later independently by Erdős, Hajnal, and Moon [2]. It asks for the minimum number of edges in an edge-maximal F-free graph on n vertices. We below present this notion in a more general form.

Fix a positive integer n and a graph F. A graph G is called *F-saturated* if G contains no subgraph isomorphic to F but each graph obtained from G by joining a pair of non-adjacent vertices contains at least one copy of F as a subgraph. In other words, G is F-saturated if and only if it is an edge-maximal F-free graph. So, $ex(n, F)$ is equal to the maximum number of edges in an F-saturated graph on n vertices. The *saturation function* of F, denoted $sat(n, F)$, is the minimum number of edges in an F-saturated graph on n vertices. For instance, it was proved by Erdős, Hajnal, and Moon [2] that

$$sat(n, K_r) = (r-2)n - \binom{r-1}{2},$$

where $n \geqslant r \geqslant 2$ and K_r is the complete graph on r vertices.

For a given graph G, a spanning subgraph H of G is said to be an *F-saturated subgraph of G* if H contains no subgraph isomorphic to F but each graph obtained by adding an edge from $E(G) \setminus E(H)$ to H has at least one copy of F as a subgraph. The minimum number of edges in an F-saturated subgraph of G is denoted by $sat(G, F)$. Thus, $sat(n, F)$ is by definition equal to $sat(K_n, F)$. We refer the reader to [3] and the references therein for a survey on graph saturation.

In recent years, a new trend in extremal graph theory has been developed to extend the classical results, such as Ramsey's and Turán's theorems, to random analogues. The study reveals the behavior of extremal parameters for a typical graph. For instance, Korándi and Sudakov initiated the study of graph saturation for random graphs very recently [6]. They proved for every fixed $p \in (0, 1)$ and fixed integer $r \geqslant 3$ that

$$sat\big(\mathbb{G}(n,p), K_r\big) = \big(1 + o(1)\big) n \log_{\frac{1}{1-p}} n$$

with high probability. Let us recall that, for a sequence $X_1, X_2, \ldots$ of random variables, we write '$X_n = o(1)$ with high probability' if

$$\lim_{n\to\infty} \mathbb{P}\left(|X_n| \leqslant \epsilon\right) = 1,$$

for any $\epsilon > 0$.

Let $K_{1,r}$ be the star graph on $r+1$ vertices. In this paper, we investigate the $K_{1,r}$-saturation number of $\mathbb{G}(n, p)$. The classical version was resolved by Kászonyi and Tuza [5], where they proved that

$$sat(n, K_{1,r}) = \begin{cases} \binom{r}{2} + \binom{n-r}{2}, & \text{if } r+1 \leqslant n \leqslant \frac{3r}{2}; \\ \left\lceil \frac{(r-1)n}{2} - \frac{r^2}{8} \right\rceil, & \text{if } n \geqslant \frac{3r}{2}. \end{cases}$$

The first non-trivial case, namely $r = 2$, is especially interesting for the reason that $sat(G, K_{1,2})$ is by definition equal to the minimum cardinality of a maximal matching in G. It has been proven by Zito [9] that

$$\lim_{n\to\infty} \mathbb{P}\left(\tfrac{n}{2} - \log_{\frac{1}{1-p}}(np) < sat\big(\mathbb{G}(n,p), K_{1,2}\big) < \tfrac{n}{2} - \log_{\frac{1}{1-p}} \sqrt{n} \right) = 1. \quad (1)$$

Here we show that with high probability

$$sat\big(\mathbb{G}(n,p), K_{1,r}\big) = \frac{(r-1)n}{2} - \big(1 + o(1)\big)(r-1)\log_{\frac{1}{1-p}} n,$$

for every fixed $p \in (0,1)$ and fixed integer $r \geqslant 2$. Note that, for $r = 2$, our result gives an upper bound stronger than (1) whereas our lower bound is weaker. It is finally worth noting that for complete graphs the saturation number of random graphs is much larger than the classical version while the parameter for star graphs is slightly smaller than its classical value.

2 Results

Let G be a graph and k be a nonnegative integer. A subset S of $V(G)$ is called *k-independent* if the maximum degree of $G[S]$ is at most k. The *k-independence number* of G, denoted by $\alpha_k(G)$, is defined as the maximum cardinality of a k-independent set in G. In particular, $\alpha_0(G) = \alpha(G)$ is the usual independence number of G. The following theorem is well known and is proved as Theorem 7.3 in [4].

Theorem 2.1. (Matula [7]) *For any fixed number $p \in (0,1)$,*

$$\alpha(\mathbb{G}(n,p)) = \big(2 + o(1)\big)\log_{\frac{1}{1-p}} n$$

with high probability.

The following easy observation can be proved using a straightforward union bound argument. We apply it to obtain a generalized version of Theorem 2.1.

Lemma 2.2. *Let X be a binomial random variable with parameters n and $p \in (0,1)$. Then $\mathbb{P}(X \leqslant s) \leqslant \binom{n}{s}(1-p)^{n-s}$ for any $s \in \{0,1,\ldots,n\}$.*

Theorem 2.3. *For every fixed number $p \in (0,1)$ and fixed integer $k \geqslant 1$,*

$$\alpha_k(\mathbb{G}(n,p)) = \big(2+o(1)\big)\log_{\frac{1}{1-p}} n$$

with high probability.

Proof. Let $G \sim \mathbb{G}(n,p)$, $q = 1-p$, and $b = 1/q$. For any integer $s \geqslant 1$, let X_s be the number of induced subgraphs in G on s vertices with at most $sk/2$ edges. Clearly, $X_s = 0$ implies $\alpha_k(G) \leqslant s-1$. For any $S \subseteq V(G)$ with $|S| = s$, let Y_S count the number of edges in $G[S]$. By Lemma 2.2,

$$\begin{aligned}
\mathbb{E}(X_s) &= \sum_{\substack{S\subseteq V(G)\\ |S|=s}} \mathbb{P}\left(Y_S \leqslant \tfrac{ks}{2}\right)\\
&\leqslant \binom{n}{s}\binom{\binom{s}{2}}{\frac{ks}{2}} q^{\binom{s}{2}-\frac{ks}{2}}\\
&\leqslant \left(\frac{ne}{s}\right)^s \left(\frac{e\binom{s}{2}}{\frac{ks}{2}}\right)^{\frac{ks}{2}} q^{\binom{s}{2}-\frac{ks}{2}}\\
&\leqslant \left(\left(\frac{ne}{s}\right)^2 \left(\frac{se}{k}\right)^k q^{s-k-1}\right)^{\frac{s}{2}}\\
&\leqslant \left(Cn^2 s^k q^s\right)^{\frac{s}{2}},
\end{aligned}$$

for some fixed value C. Put $s = 2\log_b n + 2k\log_b\log_b n$. We have

$$\log_b\left(n^2 s^k q^s\right) = 2\log_b n + k\log_b s - s \longrightarrow -\infty$$

and so $n^2 s^k q^s \to 0$ as n tends to infinity. Therefore, $\mathbb{E}(X_s) \to 0$ and since $\mathbb{P}(X_s > 0) \leqslant \mathbb{E}(X_s)$ by the Markov inequality, it follows that $\mathbb{P}(X_s > 0) \to 0$ as n goes to infinity. This proves that $\alpha_k(G) \leqslant 2\log_b n + 2k\log_b\log_b n - 1$ with high probability. Now, the assertion follows from the fact $\alpha_k(G) \geqslant \alpha(G)$ and Theorem 2.1. □

The following lemma is later used to prove the lower bound on $sat(G, K_{1,r})$.

Lemma 2.4. *For every graph G on n vertices and integer $r \geqslant 2$,*

$$sat(G, K_{1,r}) \geqslant \frac{(r-1)\big(n-\alpha_{r-2}(G)\big)}{2}.$$

Proof. Let H be a $K_{1,r}$-saturated subgraph of G. Let A be the set of vertices of H with degree at most $r-2$ in H. Since H is a $K_{1,r}$-saturated subgraph of G, every vertex in $\overline{A} = V(G) \setminus A$ is of degree $r-1$ in H and $G[A] = H[A]$. This implies that $|A| \leqslant \alpha_{r-2}(G)$. We hence obtain that

$$|E(H)| \geqslant \frac{1}{2} \sum_{v \in \overline{A}} \deg_H(v) \geqslant \frac{(r-1)\big(n - \alpha_{r-2}(G)\big)}{2}. \qquad \square$$

We will make use of the next theorem in the proof of our main result.

Theorem 2.5. (Alon–Füredi [1]) *Let $G \sim \mathbb{G}(n,p)$ be a random graph and H be a fixed graph on n vertices with maximum degree Δ, where $(\Delta^2+1)^2 < n$. If*

$$p^{\Delta} > \frac{10 \log \left\lfloor \frac{n}{\Delta^2+1} \right\rfloor}{\left\lfloor \frac{n}{\Delta^2+1} \right\rfloor},$$

then the probability that G does not contain a copy of H is smaller than $1/n$.

Now we are in the position to prove our main result.

Theorem 2.6. *For every fixed number $p \in (0,1)$ and fixed integer $r \geqslant 2$,*

$$\operatorname{sat}\big(\mathbb{G}(n,p), K_{1,r}\big) = \frac{(r-1)n}{2} - \big(1+o(1)\big)(r-1)\log_{\frac{1}{1-p}} n$$

with high probability.

Proof. Let $G \sim \mathbb{G}(n,p)$, $q = 1-p$, and $b = 1/q$. Using Theorem 2.3 and Lemma 2.4, we find that

$$\lim_{n\to\infty} \mathbb{P}\left(\operatorname{sat}(G, K_{1,r}) \geqslant \tfrac{(r-1)n}{2} - (1+\epsilon)(r-1)\log_b n\right) = 1,$$

for any $\epsilon > 0$. So, it suffices to prove that

$$\lim_{n\to\infty} \mathbb{P}\left(\operatorname{sat}(G, K_{1,r}) \leqslant \tfrac{(r-1)n}{2} - (1-\epsilon)(r-1)\log_b n\right) = 1, \qquad (2)$$

for any $\epsilon > 0$. Fix ϵ and let ℓ be the least integer such that $\ell \geqslant (2-2\epsilon)\log_b n$ and $(n-\ell)(r-1)$ is even. Also, fix a regular graph L on $n-\ell$ vertices with degree $r-1$. For any $S \subseteq V(G)$ with $|S| = \ell$, let

$$X_S = \begin{cases} 1, & \text{if } S \text{ is an independent set in } G \text{ and} \\ & G[V(G)\setminus S] \text{ has a copy of } L \text{ as a subgraph;} \\ 0, & \text{otherwise.} \end{cases}$$

We assume n to be large enough whenever needed. It follows from Theorem 2.5 that $\mathbb{E}[X_S] \geqslant q^{\binom{\ell}{2}} \left(1 - \frac{1}{n-\ell}\right)$. Therefore, if we let

$$X = \sum_{\substack{S \subseteq V(G) \\ |S|=\ell}} X_S,$$

then $\mathbb{E}[X] \geqslant \binom{n}{\ell} q^{\binom{\ell}{2}} \left(1 - \frac{1}{n-\ell}\right)$. Moreover, for every subsets $S, T \subseteq V(G)$ of size ℓ with $|S \cap T| = i$, we easily see that $\mathbb{E}[X_S X_T] \leqslant q^{2\binom{\ell}{2} - \binom{i}{2}}$. By the Chebyshev inequality and noting that $n - \ell$ goes to infinity, we have

$$\begin{aligned}
\mathbb{P}(X = 0) &\leqslant \frac{\mathrm{var}(X)}{\mathbb{E}[X]^2} \\
&= \sum_{\substack{S,T \subseteq V(G) \\ |S|=|T|=\ell}} \frac{\mathbb{E}[X_S X_T] - \mathbb{E}[X_S]\mathbb{E}[X_T]}{\mathbb{E}[X]^2} \\
&= \sum_{i=0}^{\ell} \sum_{\substack{S,T \subseteq V(G) \\ |S|=|T|=\ell \\ |S \cap T|=i}} \frac{\mathbb{E}[X_S X_T] - \mathbb{E}[X_S]\mathbb{E}[X_T]}{\mathbb{E}[X]^2} \\
&\leqslant \binom{n}{\ell} \sum_{i=0}^{\ell} \frac{\binom{\ell}{i}\binom{n-\ell}{\ell-i} \left(q^{2\binom{\ell}{2} - \binom{i}{2}} - q^{2\binom{\ell}{2}} \left(1 - \frac{1}{n-\ell}\right)^2 \right)}{\binom{n}{\ell}^2 q^{2\binom{\ell}{2}} \left(1 - \frac{1}{n-\ell}\right)^2} \\
&= \sum_{i=0}^{\ell} \frac{\binom{\ell}{i}\binom{n-\ell}{\ell-i}}{\binom{n}{\ell} q^{\binom{i}{2}}} \frac{1 - q^{\binom{i}{2}} \left(1 - \frac{1}{n-\ell}\right)^2}{\left(1 - \frac{1}{n-\ell}\right)^2} \\
&\leqslant \frac{\binom{n-\ell}{\ell} + \ell\binom{n-\ell}{\ell-1}}{\binom{n}{\ell}} \frac{1 - \left(1 - \frac{1}{n-\ell}\right)^2}{\left(1 - \frac{1}{n-\ell}\right)^2} + 2\sum_{i=2}^{\ell} \frac{\binom{\ell}{i}\binom{n-\ell}{\ell-i}}{\binom{n}{\ell} q^{\binom{i}{2}}} \\
&\leqslant (\ell + 1)\frac{2n - 2\ell - 1}{(n - \ell - 1)^2} + 2\sum_{i=2}^{\ell} \frac{\binom{\ell}{i}\binom{n-\ell}{\ell-i}}{\binom{n}{\ell} q^{\binom{i}{2}}}.
\end{aligned}$$

Using the computations given in the proof of Theorem 7.3 of [4], the last summation above converges to 0 as $n \to \infty$ and hence $\mathbb{P}(X = 0) = o(1)$. This shows that with high probability there is $S \subseteq V(G)$ with $|S| = \ell$ such that S is an independent set in G and $G[V(G) \setminus S]$ has a copy L' of L as a subgraph. Denote the spanning subgraph of G with edge set $E(L')$ by H. It is easily seen that H is a $K_{1,r}$-saturated subgraph of G and

$$E(H) = \tfrac{(n-\ell)(r-1)}{2} \leqslant \tfrac{(r-1)n}{2} - (1-\epsilon)(r-1)\log_b n,$$

which concludes (2), as required. □

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