On the sum of Laplacian eigenvalues of graphs

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Abstract

Let k be a natural number and let G be a graph with at least k vertices. A.E. Brouwer conjectured that the sum of the k largest Laplacian eigenvalues of G is at most $e(G)+\binom{k+1}{2}$, where $e(G)$ is the number of edges of G. We prove this conjecture for $k = 2$. We also show that if G is a tree, then the sum of the k largest Laplacian eigenvalues of G is at most $e(G) + 2k - 1$.

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1 Introduction

Let G be a simple graph with the vertex set $V(G) = \{v_1, \ldots, v_n\}$. The *degree* of a vertex $v \in V(G)$, denoted by $d(v)$, is the number of neighbors of v. The *Laplacian matrix* of G is the $n \times n$ matrix $\mathcal{L}(G) = [\ell_{ij}]$ that records the vertex degrees $d(v_1), \ldots, d(v_n)$ on its diagonal and for any $i \neq j$, $1 \leqslant i, j \leqslant n$, $\ell_{ij} = -1$ if v_i and v_j are adjacent and $\ell_{ij} = 0$, otherwise. It is well-known that $\mathcal{L}(G)$ is positive semi-definite and so its eigenvalues are nonnegative real numbers. The eigenvalues of $\mathcal{L}(G)$ are called the *Laplacian eigenvalues* of G and are denoted by $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G)$. Note that each row sum of $\mathcal{L}(G)$ is 0 and therefore, $\mu_n(G) = 0$.

In this paper, we investigate the sum $\mathcal{S}_k(G) = \sum_{i=1}^{k} \mu_i(G)$ for $1 \leqslant k \leqslant n$. We denote the edge set of G by $E(G)$ and we let $e(G) = |E(G)|$. A.E. Brouwer [1] (see also [3]) has conjectured the following.

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Conjecture 1 *Let G be a graph with n vertices. Then $\mathcal{S}_k(G) \leqslant e(G) + \binom{k+1}{2}$ for $k = 1, \dots, n$.*

Using a computer, Brouwer [1] has checked Conjecture 1 for all graphs with at most ten vertices. For $k = 1$, the conjecture follows from the well-known inequality $\mu_1(G) \leqslant |V(G)|$ (see [7, p. 281]). Also the cases $k = n$ and $k = n-1$ are straightforward. Here, we prove Conjecture 1 for $k = 2$. We also show that $\mathcal{S}_k(T) \leqslant e(T) + 2k - 1$ for any tree T and any $1 \leqslant k \leqslant n$ from which the conjecture follows for trees.

Brouwer's conjecture is related to (and motivated by) the Grone-Merris conjecture [8]. Let $d_i^{\mathsf{T}} = |\{v \in V(G) \,|\, d(v) \geqslant i\}|$ for $i = 1, \dots, n$. The numbers $d_1^{\mathsf{T}} \geqslant d_2^{\mathsf{T}} \geqslant \cdots \geqslant d_n^{\mathsf{T}}$ are called the *conjugate degrees* of G. The Grone-Merris conjecture asserts that $\mathcal{S}_k(G) \leqslant \sum_{i=1}^{k} d_i^{\mathsf{T}}$ for $k = 1, \dots, n$. Since the Grone-Merris conjecture uses more detailed information from the graph than Brouwer's conjecture, one would expect that the Grone-Merris inequalities are better. For many graphs this is true, but not for all graphs. As an example, for the 4-cycle C_4, the Grone-Merris conjecture gives $\mathcal{S}_2(C_4) \leqslant 8$, whilst Brouwer's conjecture gives $\mathcal{S}_2(C_4) \leqslant 7$ (in fact, $\mathcal{S}_2(C_4) = 6$). The Grone-Merris conjecture is known to be true for (i) threshold graphs (see [8]), (ii) trees (see [10]), (iii) the cases $k \leqslant 2$ (see [4, Theorem 7.1]) and $k \geqslant n-1$ (trivial), and (iv) for all graphs with at most ten vertices (by computer; see [3]). Brouwer observed that his conjecture also holds for threshold graphs (see Section 3), and has verified the conjecture by computer for all graphs on at most ten vertices. Here we settle Brouwer's conjecture for trees and the case $k \leqslant 2$. Thus Brouwer's conjecture is true for all cases (i) to (iv), for which the Grone-Merris conjecture is known to be true.

For threshold graphs, the Grone-Merris conjecture holds with equality for every k. Examples that satisfy Brouwer's conjecture with equality are the complete graphs K_n with $k = n-1$, and the stars $K_{1,n-1}$ with $k = 1$.

Another related upper bound, worth mentioning, is (see [11]):

$$\mathcal{S}_k(G) \leqslant \frac{2mk + \sqrt{mk(n-k-1)(n^2-n-2m)}}{n-1},$$

where $1 \leqslant k < n$ and $m = e(G)$.

2 Notation and Preliminaries

We first present some notation and definitions. For a subset X of $V(G)$, $\mathcal{N}(X)$ denotes the set of vertices outside X, which have at least one neighbor in X. An *independent set* in G is a subset Y of $V(G)$ such that no two distinct vertices in Y are adjacent. Two distinct edges of G are called *independent* if they have no common endpoint. A set of pairwise independent edges in G is called a *matching*. The maximum size of a matching in G is known as the *matching number* of G, denoted by $m(G)$. For two graphs G_1 and G_2, the *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set is $E(G_1) \cup E(G_2)$. If $V(G_1) \cap V(G_2) = \varnothing$, then the union of G_1 and G_2 is denoted by $G_1 + G_2$. We denote the complete graph, star and path with n vertices by K_n, S_n and P_n, respectively. The complete bipartite graph with the part sizes m and n is denoted by $K_{m,n}$.

Brouwer [1] has checked Conjecture 1 for all graphs with at most ten vertices. For our purpose we only need the following statement.

Lemma 1 *For any graph G with at most eight vertices, $\mathcal{S}_2(G) \leqslant e(G) + 3$.*

We next state some lemmas and theorems which will be used in the subsequent sections.

Lemma 2 *Let n be a natural number.*

(i) *The Laplacian eigenvalues of K_n are n with multiplicity $n-1$, and 0.*

(ii) *The Laplacian eigenvalues of S_n are n, 1 with multiplicity $n-2$, and 0.*

The following lemma gives an affirmative answer to Conjecture 1 for $k = 1$.

Lemma 3 [7, p. 281] *If G is a graph with n vertices, then $\mu_1(G) \leqslant n$.*

Theorem 1 [7, p. 291] *Let G be a graph with n vertices and let G' be a graph obtained from G by inserting a new edge into G. Then the Laplacian eigenvalues of G and G' interlace, that is,*

$$\mu_1(G') \geqslant \mu_1(G) \geqslant \cdots \geqslant \mu_n(G') = \mu_n(G) = 0.$$

Theorem 2 [9] *Let G be a graph. Then $\mu_1(G) \leqslant \max\{d(v) + m(v) \,|\, v \in V(G)\}$, where $m(v)$ is the average of the degrees of the vertices of G adjacent to the vertex v.*

Theorem 3 [2] *Let G be a graph with n vertices and vertex degrees $d_1 \geqslant \cdots \geqslant d_n$. If G is not $K_s + (n-s)K_1$, then $\mu_s(G) \geqslant d_s - s + 2$ for $1 \leqslant s \leqslant n$.*

The following theorem from matrix theory plays a key role in our proofs. We denote the eigenvalues of a symmetric matrix M by $\lambda_1(M) \geqslant \cdots \geqslant \lambda_n(M)$.

Theorem 4 [5] (see also [6]) *Let A and B be two real symmetric matrices of size n. Then for any $1 \leqslant k \leqslant n$,*

$$\sum_{i=1}^{k} \lambda_i(A+B) \leqslant \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B).$$

An immediate consequence of Theorem 4 is the following corollary which will be used frequently.

Corollary 1 *Let $G_1, \ldots, G_r$ be some edge disjoint graphs. Then $\mathcal{S}_k(G_1 \cup \cdots \cup G_r) \leqslant \sum_{i=1}^{r} \mathcal{S}_k(G_i)$ for any k.*

The following Lemma asserts that to prove Conjecture 1 for $k = 2$, it suffices to consider connected graphs.

Lemma 4 *Let G be a graph. Then either $\mathcal{S}_2(G) = \mathcal{S}_2(H)$ for a connected component H of G or $\mathcal{S}_2(G) \leqslant e(G) + 2$.*

Proof. If the first statement does not hold, then G has two connected components H_1 and H_2 such that $\mu_1(G) = \mu_1(H_1)$ and $\mu_2(G) = \mu_1(H_2)$. By Lemma 3, we have $\mu_1(H_i) \leqslant |V(H_i)| \leqslant e(H_i) + 1$ for $i = 1, 2$. Therefore, $\mathcal{S}_2(G) \leqslant (e(H_1) + 1) + (e(H_2) + 1) \leqslant e(G) + 2$. $\square$

The next lemma is the key to our approach. Because of this result, it suffices to consider only a very restrictive class of graphs.

Lemma 5 *If Conjecture 1 is false for $k = 2$, then there exists a counterexample G for which $\mathcal{S}_2(H) > e(H)$ for every subgraph H of G.*

Proof. Let G be a counterexample for Conjecture 1 with $k = 2$ having a minimum number of edges. If G has a subgraph H that satisfies $\mathcal{S}_2(H) \leqslant e(H)$, then Corollary 1 gives $e(G) + 3 < \mathcal{S}_2(G) \leqslant \mathcal{S}_2(H) + \mathcal{S}_2(G - H)$. This implies that $\mathcal{S}_2(G - H) > e(G - H) + 3$, which contradicts the minimality of $e(G)$. $\square$

Lemma 6 *Let G be a graph with n vertices. Suppose that there exist two non-adjacent vertices $u, v \in V(G)$ such that $\mu_k(G) \geqslant d(u) + d(v) + 2$ for some integer k, $1 \leqslant k \leqslant n$. If G' is the graph obtained from G by inserting edge $e = \{u, v\}$ into G, then $\mathcal{S}_k(G') \leqslant \mathcal{S}_k(G) + 1$.*

Proof. For $i = 1, \ldots, n$, define $\epsilon_i = \mu_i(G') - \mu_i(G)$. By Theorem 1, $\epsilon_i \geqslant 0$ for any i. Let $d_1 \geqslant \cdots \geqslant d_n$ and $d'_1 \geqslant \cdots \geqslant d'_n$ be vertex degrees of G and G', respectively. Recall that for any graph $\mathit{\Gamma}$, considering the trace of the matrix $\mathcal{L}(\mathit{\Gamma})^2$, we have

$$\sum_{i=1}^{|V(\mathit{\Gamma})|} \mu_i(\mathit{\Gamma})^2 = \sum_{v \in V(\mathit{\Gamma})} d(v)^2 + 2e(\mathit{\Gamma}).$$

Applying this fact, we have

$$\begin{aligned}
\sum_{i=1}^{n} \mu_i(G')^2 &= \sum_{i=1}^{n} d_i'^2 + 2e(G') \\
&= \sum_{i=1}^{n} d_i^2 + 2e(G) + 2d(u) + 2d(v) + 4 \\
&= \sum_{i=1}^{n} \mu_i(G)^2 + 2\big(d(u) + d(v) + 2\big).
\end{aligned}$$

This yields that

$$2\mu_k(G)\sum_{i=1}^{k}\epsilon_i \leqslant \sum_{i=1}^{k}2\epsilon_i\mu_i(G) \leqslant \sum_{i=1}^{n}\mu_i(G')^2 - \sum_{i=1}^{n}\mu_i(G)^2 = 2\big(d(u)+d(v)+2\big).$$

Since $\mu_k(G) \geqslant d(u)+d(v)+2$, $\mathcal{S}_k(G') - \mathcal{S}_k(G) = \sum_{i=1}^{k}\epsilon_i \leqslant 1$ and the assertion follows. □

3 Trees and threshold graphs

In the following, we obtain an upper bound for the sum of the k largest Laplacian eigenvalues of a tree which implies Conjecture 1 for trees.

Theorem 5 *Let T be a tree with n vertices. Then $\mathcal{S}_k(T) \leqslant e(T) + 2k - 1$ for $1 \leqslant k \leqslant n$.*

Proof. We prove the assertion by induction on $|V(T)|$. If T is a star, then by Lemma 2(ii), $\mathcal{S}_k(T) = n+k-1$ for $1 \leqslant k < n$, and we are done. Thus assume that T is not a star. Then T has an edge whose removal leaves a forest F consisting of two trees T_1 and T_2, both having at least one edge. Suppose that k_i of the k largest eigenvalues of F come from the Laplacian spectrum of T_i for $i = 1, 2$, where $k_1 + k_2 = k$. If one of k_i, say k_2, is zero, then by $|V(T_2)| \geqslant 2$, Corollary 1, and the induction hypothesis, we conclude that $\mathcal{S}_k(T) = \mathcal{S}_k(F \cup K_2) \leqslant \mathcal{S}_{k_1}(T_1) + \mathcal{S}_k(K_2) \leqslant (e(T_1) + 2k_1 - 1) + 2 \leqslant n + 2k - 2 = e(T) + 2k - 1$. Otherwise, using Corollary 1 and the induction hypothesis, we have $\mathcal{S}_k(T) = \mathcal{S}_k(T_1 \cup T_2 \cup K_2) \leqslant \mathcal{S}_{k_1}(T_1) + \mathcal{S}_{k_2}(T_2) + \mathcal{S}_k(K_2) \leqslant (e(T_1) + 2k_1 - 1) + (e(T_2) + 2k_2 - 1) + 2 = e(T) + 2k - 1$. This completes the proof. □

A *threshold graph* is a graph obtained from K_1 by a sequence of operations of the form (i) adding an isolated vertex or (ii) taking the complement. It is clear that adding isolated vertices to a graph only increases the multiplicity of the Laplacian eigenvalue 0. This observation and the next theorem shows that Conjecture 1 is valid for threshold graphs.

Theorem 6 *Let G be a graph with n vertices and $1 \leqslant k \leqslant n-2$. If $S_k(G) \leqslant e(G) + \binom{k+1}{2}$, then $S_{n-k-1}(\overline{G}) \leqslant e(\overline{G}) + \binom{n-k}{2}$, where $\overline{G}$ is the complement of G.*

Proof. From [7, p. 280], we have $\mu_i(\overline{G}) = n - \mu_{n-i}(G)$ for $i = 1, \ldots, n-1$. Therefore,

$$\begin{aligned}
S_{n-k-1}(\overline{G}) &= n(n-k-1) - \big(\mu_{k+1}(G) + \cdots + \mu_{n-1}(G)\big)\\
&= n(n-k-1) - 2e(G) + \big(\mu_1(G) + \cdots + \mu_k(G)\big)\\
&= n(n-k-1) - \binom{n}{2} + e(\overline{G}) + \big(\mu_1(G) + \cdots + \mu_k(G)\big) - e(G)\\
&\leqslant e(\overline{G}) + n(n-k-1) - \binom{n}{2} + \binom{k+1}{2}\\
&= e(\overline{G}) + \binom{n-k}{2},
\end{aligned}$$

as desired. □

4 The case $k = 2$

In this section, we prove Conjecture 1 for $k = 2$. First we establish the conjecture for graphs with matching number at most three and then we conclude the assertion using Lemma 5.

Lemma 7 *Let G be a graph with $m(G) = 1$. Then $\mathcal{S}_2(G) \leqslant e(G) + 3$.*

Proof. Let $n = |V(G)|$. Since $m(G) = 1$, it is easily checked that either $G = S_m + (n - m)K_1$ for some m, $1 \leqslant m \leqslant n$ or $G = K_3 + (n - 3)K_1$. By Lemma 2, the assertion holds. □

We say that a connected graph has the form △ if it has a subgraph H isomorphic to K_3 such that every edge is incident with some vertex of H.

Lemma 8 *Let G be a graph of the form △. Then $\mathcal{S}_2(G) \leqslant e(G) + 3$.*

Proof. Let $n = |V(G)|$ and $d_1^\top \geqslant \cdots \geqslant d_n^\top$ be the conjugate degrees of G. If t is the number of vertices of degree 1 in G, then it is not hard to see that $2(n - t - 3) \leqslant e(G) - t - 3$. This implies that $d_2^\top = n - t \leqslant e(G) - n + 3$. Since $d_1^\top = n$, $d_1^\top + d_2^\top \leqslant e(G) + 3$. By [4, Theorem 7.1], the Grone-Merris conjecture is true for $k = 2$. Therefore, $\mathcal{S}_2(G) \leqslant d_1^\top + d_2^\top \leqslant e(G) + 3$. □

Lemma 9 *Let $n \geqslant 3$ and let G be a connected spanning subgraph of $K_{2,n-2}$. Then $\mathcal{S}_2(G) \leqslant e(G) + 3$.*

Proof. Assume that $\{\{v, w\}, B\}$ is the partition of $V(G)$. For simplicity, we write $\mu_i(G) = \mu_i$ for $1 \leqslant i \leqslant n$. Let $d_1 \geqslant \cdots \geqslant d_n$ be the vertex degrees of G and let r and s be the number of vertices of degree 1 and 2 in B, respectively. By Theorem 5, we can assume that G is not a tree. Hence $s \geqslant 2$ and the degrees $d_1, d_2 \geqslant 2$ are the degrees of v and w. It is easily seen that s rows of $2I - \mathcal{L}(G)$ are identical and therefore the multiplicity of 2 as an eigenvalue of $\mathcal{L}(G)$ is at least $s - 1$. Similarly, the multiplicity of 1 as eigenvalues of $\mathcal{L}(G)$ is at least $r - 2$. If $\mu_2 \leqslant 2$, then Lemma 3 implies that $\mu_1 + \mu_2 \leqslant n + 2 < e(G) + 3$. Hence we may assume that $\mu_2 > 2$ and so $\mu_1 \geqslant \mu_2 \geqslant \mu_a \geqslant \mu_b \geqslant \mu_n = 0$ are the five remaining eigenvalues. By $\mathrm{trace}(\mathcal{L}(G)) = \sum_{i=1}^n \mu_i = \sum_{i=1}^n d_i$, we have $\mu_1 + \mu_2 + \mu_a + \mu_b \leqslant d_1 + d_2 + 4$. Finally, by the interlacing theorem [7, p. 193] for the $(n - 2) \times (n - 2)$ submatrix $D = \mathrm{diag}(1, \ldots, 1, 2, \ldots, 2)$ of $\mathcal{L}(G)$, we find that $\mu_a \geqslant \mu_{n-2} \geqslant \lambda_{n-2}(D) \geqslant 1$. Hence $\mu_1 + \mu_2 \leqslant d_1 + d_2 + 4 - \mu_a - \mu_b \leqslant d_1 + d_2 + 3 = e(G) + 3$. □

Lemma 10 *Let G be a graph with $m(G) = 2$. Then $\mathcal{S}_2(G) \leqslant e(G) + 3$.*

Proof. By Lemmas 1 and 4, we may assume that G is a connected graph with at least 7 vertices. First suppose that G has a subgraph $H = K_3$ with $V(H) = \{u, v, w\}$. If every edge of G has at least one endpoint in $V(H)$, then by Lemma 8, we are done. Hence assume that there exists an edge $e = \{a, b\}$ whose endpoints are in $V(G) \setminus V(H)$. Let $M = V(G) \setminus \{a, b, u, v, w\}$.

Since $m(G) = 2$, there are no edges between $V(H)$ and M. Since $|M| \geqslant 2$, it is easily seen that all vertices in M are adjacent to one of the endpoints of e, say a. Hence there are no edges between b and $V(H)$. Now by ignoring the edges between a and $V(H)$, we find a subgraph K of G which is a disjoint union of K_3 and a star with the center a. Since the graph $L = G - E(K)$ is a star, Corollary 1 yields that $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(K) + \mathcal{S}_2(L) \leqslant (e(K) + 1) + (e(L) + 2) = e(G) + 3$, as required.

Next assume that G has no K_3 as a subgraph. Suppose that $e_1 = \{a_1, b_1\}$ and $e_2 = \{a_2, b_2\}$ are two independent edges in G. Since G contains no $3K_2$ and K_3 as subgraphs, $M = V(G) \setminus \{a_1, b_1, a_2, b_2\}$ is an independent set and at least one of the two endpoints of e_i has no neighborhood in M for $i = 1, 2$. Assume those endpoints to be b_1 and b_2. If b_1 and b_2 are adjacent, then $|M| \geqslant 2$ yields that all vertices in M are adjacent to only one of the two vertices a_1 and a_2, say a_1. This implies that G is a bipartite graph with the vertex set partition $\{\{a_1, b_2\}, V(G) \setminus \{a_1, b_2\}\}$ and so Lemma 9 yields the assertion. Now assume that b_1 and b_2 are not adjacent. If a_1 and a_2 are adjacent, then G is a tree and we are done by Theorem 5. Otherwise, G is a bipartite graph with the vertex set partition $\{\{a_1, a_2\}, V(G) \setminus \{a_1, a_2\}\}$ and using Lemma 9, the proof is complete. □

Lemma 11 *Let G be a graph with $m(G) = 3$. Then $\mathcal{S}_2(G) \leqslant e(G) + 3$.*

Proof. By Lemmas 1 and 4, we may assume that G is a connected graph with at least 9 vertices. Using Lemma 5, we may suppose that G has no subgraph H with $\mathcal{S}_2(H) \leqslant e(H)$. In particular, Lemma 2 implies that G has no subgraph $3S_3$. Suppose that G has a subgraph $K = K_3 + 2K_2$. Let $x \in V(G) \setminus V(K)$. Since $m(G) = 3$, the vertex x is not incident with the subgraph K_3 of K and so G has a subgraph $H = K_3 + S_3 + K_2$. Now by Lemma 2, we have $\mathcal{S}_2(H) = e(H)$ and therefore G has no subgraph $K_3 + 2K_2$.

Let $e_1 = \{a_1, b_1\}, e_2 = \{a_2, b_2\}$ and $e_3 = \{a_3, b_3\}$ be three independent edges in G. Since $m(G) = 3$, $M = V(G) \setminus V(\{e_1, e_2, e_3\})$ is an independent set. Since G has no $4K_2$ and $K_3 + 2K_2$ as subgraphs, either $\mathcal{N}(a_i) \cap M = \varnothing$ or $\mathcal{N}(b_i) \cap M = \varnothing$, for $i = 1, 2, 3$. With no loss of generality, we may assume that $\mathcal{N}(M) \subseteq \{a_1, a_2, a_3\}$. We consider the following three cases.

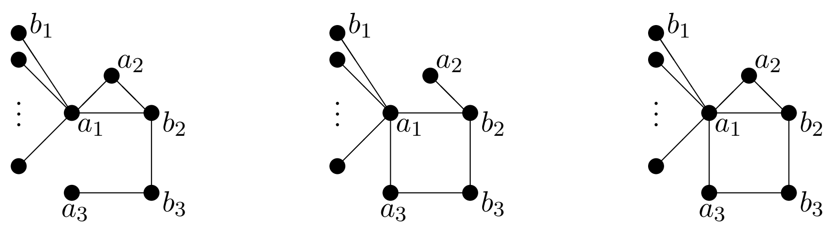
Case 1. $|\mathcal{N}(M)| = 3$. We have $\mathcal{N}(M) = \{a_1, a_2, a_3\}$. Since G has no $3S_3$, the bipartite subgraph $G - \{b_1, b_2, b_3\}$ has no perfect matching. By Hall's Theorem, there exists a subset of $\{a_1, a_2, a_3\}$ with 2 elements, say $\{a_2, a_3\}$, such that $|\mathcal{N}(\{a_2, a_3\}) \cap M| = 1$. This means that there exists exactly one vertex $y \in M$ which is adjacent to both a_2 and a_3. If $d(b_1) \geqslant 2$, then we clearly find a subgraph isomorphic to $3S_3$ in G, a contradiction. Therefore, $d(b_1) = 1$. Suppose that H is the star with center a_1 and $V(H) \subseteq \{a_1, a_2, a_3, b_2, b_3, y\}$. Then $G - E(H)$ is a disjoint union of a star S with center a_1 and a graph K containing P_5 with the vertex set $\{a_2, a_3, b_2, b_3, y\}$. Using Theorem 2, we have $\mu_1(P_5) \leqslant 4$ and by Lemma 2, we obtain that $\mu_1(K) \leqslant e(K)$. This yields that $\mathcal{S}_2(G - E(H)) \leqslant \mu_1(S) + \mu_1(K) \leqslant e(G - E(H)) + 1$. Thus $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(H) + \mathcal{S}_2(G - E(H)) \leqslant e(G) + 3$, as desired.

Case 2. $|\mathcal{N}(M)| = 2$. Without loss of generality, assume that $\mathcal{N}(M) = \{a_1, a_2\}$. Since $m(G) = 3$, b_1 is not adjacent to b_2. If b_1 is adjacent to a_3 or b_3, then changing the role of e_1, e_2, e_3 by three independent edges $\{a_1, z\}$, e_2, e_3 for some vertex $z \in M \cap \mathcal{N}(a_1)$, we have Case 1. Therefore, we may assume that b_1, and similarly b_2, is adjacent to none of the vertices a_3 and b_3. Let H be the induced subgraph on $\{a_1, a_2, a_3, b_3\}$.

First assume that H has a subgraph $L = K_3$. If $\{a_1, a_2\}$ is an edge of L, then clearly any edge of G is incident with L and by Lemma 8, there is nothing to prove. Now assume that exactly one of the two vertices a_1 and a_2, say a_1, is a vertex in L. Let K be the disjoint union of L and the induced subgraph of G on $\{a_2, b_2\} \cup (\mathcal{N}(a_2) \cap M)$ which is a star with at least three vertices. Note that $G - E(K)$ is a star or a disjoint union of two stars. Now, by Lemma 2 and Corollary 1, $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(K) + \mathcal{S}_2(G - E(K)) = (e(K) + 1) + (e(G - E(K)) + 2) = e(G) + 3$, as required.

Next suppose that H has no K_3 as a subgraph. Let $t = d(a_3) + d(b_3)$. We have $t \in \{3, 4\}$. It is not hard to see that $G - e_3$ contains two disjoint stars S_t with centers a_1 and a_2. Therefore, by Theorem 1, $\mu_2(G - e_3) \geqslant \mu_2(2S_t) = t$. Using Lemmas 6 and 10, we find that $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(G - e_3) + 1 \leqslant (e(G - e_3) + 3) + 1 = e(G) + 3$, as required.

Case 3. $|\mathcal{N}(M)| = 1$. Without loss of generality, assume that $\mathcal{N}(M) = \{a_1\}$. If $d(b_1) \geqslant 2$, then we clearly find three independent edges e'_1, e'_2, e'_3 in G such that the set $M' = V(G) \setminus V(\{e'_1, e'_2, e'_3\})$ is an independent set and $|\mathcal{N}(M')| \geqslant 2$ which is dealt with as the previous cases. Hence we assume that $d(b_1) = 1$. Suppose that H is the star with center a_1 and the vertex set $V(H) \subseteq \{a_1, a_2, a_3, b_2, b_3\}$. Then $G - E(H)$ is a disjoint union of a star S with center a_1 and a graph L containing $2K_2$ with $V(L) = \{a_2, a_3, b_2, b_3\}$. First assume that $L \neq P_4$. Using Lemma 2(i) and Lemma 3, we have $\mu_1(L) \leqslant e(L)$. This yields that $\mathcal{S}_2(G - E(H)) \leqslant \mu_1(S) + \mu_1(L) \leqslant e(G - E(H)) + 1$. Thus $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(H) + \mathcal{S}_2(G - E(H)) \leqslant e(G) + 3$, as desired. Next assume that $L = P_4$. With no loss of generality, suppose that L is the path $a_2 - b_2 - b_3 - a_3$. If $|\mathcal{N}(a_1) \cap L| = 1$, then G is a tree and the assertion follows from Theorem 5. If a_1 is adjacent to both b_2 and b_3, then by Lemma 8, there is nothing to prove. Suppose that a_1 is adjacent to none of b_2 and b_3. If we let K be the disjoint union of the star $G - V(L)$ and the edges $\{a_2, b_2\}$ and $\{a_3, b_3\}$, then the graph $G - E(K)$ is a disjoint union of a star with the center a_1 and the edge $\{b_2, b_3\}$. Now, by Lemma 2 and Corollary 1, we have $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(K) + \mathcal{S}_2(G - E(K)) \leqslant (e(K) + 1) + (e(G - E(K)) + 2) = e(G) + 3$. If none of the above cases occurs, then G is one of the following forms:



G_1 $\quad$ G_2 $\quad$ G_3

If $G = G_1$, then by Theorem 3, we have $\mu_2(G) \geqslant 3$. Since $d(a_3) + d(b_3) = 3$, applying Lemma 6 for the graph $G - e_3$ and using Lemma 10, we find that $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(G - e_3) + 1 \leqslant (e(G - e_3) + 3) + 1 =$

$e(G)+3$, as required. Hence assume that $G = G_2$ or $G = G_3$. First suppose that $\mu_2(G) \geqslant 4$. Since $d(a_3)+d(b_3) = 4$, applying Lemma 6 for the graph $G - e_3$ and using Lemma 10, the result follows. Now suppose that $\mu_2(G) < 4$. By Theorem 2, we have $\mu_1(G_2) \leqslant |V(G_2)|-1 = e(G_2)-1$ and by Lemma 3, $\mu_1(G_3) \leqslant |V(G_3)| = e(G_3)-1$. Therefore, $\mathcal{S}_2(G) < (e(G)-1)+4 = e(G)+3$. This completes the proof. □

We now present the main theorem of the paper.

Theorem 7 *Let G be a graph with at least two vertices. Then $\mathcal{S}_2(G) \leqslant e(G)+3$.*

Proof. Using Lemmas 7, 10 and 11, we may assume that G has a subgraph $H = 4K_2$, which satisfies $\mathcal{S}_2(H) = e(H)$. So the result follows by Lemma 5. □

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