

# On 2- $(v, 3)$ trades of minimum volume

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## Abstract

In this paper, Steiner and non-Steiner 2- $(v, 3)$  trades of minimum volume are considered. It is shown that these trades are composed of a union of some Pasch configurations and possibly some 2- $(v', 3)$  trades with  $6 \leq v' \leq 10$ . We determine the number of non-isomorphic Steiner 2- $(v, 3)$  trades of minimum volume. As for non-Steiner trades the same thing is done for all  $v$ 's, except for  $v \equiv 5 \pmod{6}$ .

## 1. Introduction

For given  $v, k$ , and  $t$ , let  $X = \{1, 2, \dots, v\}$  and let  $P_k(X)$  denote the set of all  $k$ -subsets of  $X$ . The elements of  $X$  and  $P_k(X)$  are called points and blocks, respectively.

A  $t$ -( $v, k$ ) trade  $T = \{T^+, T^-\}$ , consists of two disjoint collections of blocks  $T^+$  and  $T^-$  such that for every  $A \in P_t(X)$ , the number of blocks containing  $A$  is the same in both  $T^+$  and  $T^-$ .

The *foundation* of a trade is the set of elements covered by  $T^+$  and  $T^-$  and is denoted by  $\text{found}(T)$ . In a  $t$ -( $v, k$ ) trade, we take  $v$  to be the foundation size. The number of blocks in  $T^+(T^-)$  is called the volume of the trade  $T$  and is denoted by  $\text{vol}(T)$ .

A  $t$ -( $v, k$ ) trade  $T$  is called *Steiner*, if each element  $A \in P_t(X)$  occurs at most once in  $T^+(T^-)$ .  $T$  is called *simple*, if there are no repeated blocks in  $T^+(T^-)$ . Here, we are concerned only with simple 2- $(v, 3)$  trades.

A trade  $T$  is called *fundamental*, if it contains no proper trade.

Two trades  $T_1 = \{T_1^+, T_1^-\}$  and  $T_2 = \{T_2^+, T_2^-\}$  are called isomorphic, if there exists a bijection  $\sigma : \text{found}(T_1) \rightarrow \text{found}(T_2)$  such that  $\sigma(T_1) = \{\sigma(T_1^+), \sigma(T_1^-)\} = \{T_2^+, T_2^-\} = T_2$ .

Bryant [1] has determined the spectrum ( the set of allowable volumes) of Steiner 2- $(v, 3)$  trades in which  $v$  is the foundation size. In Table 1, the minimum volume of such trades is given. In this paper, we determine the number of non-isomorphic Steiner 2- $(v, 3)$  trades of minimum volume. When a 2- $(v, 3)$  trade is not Steiner, we determine the possible minimum volume for all  $v$ 's, except for  $v \equiv 5 \pmod{6}$ , and obtain the number of non-isomorphic 2- $(v, 3)$  non-Steiner trades of minimum volume.

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**Table 1.**  
**Minimum volume of Steiner 2-( $v, 3$ ) trades.**

| $v \pmod{6}$ | minimum volume   |
|--------------|------------------|
| 0            | $\frac{2v}{3}$   |
| 1            | $\frac{2v+4}{3}$ |
| 2            | $\frac{2v+2}{3}$ |
| 3            | $\frac{2v+3}{3}$ |
| 4            | $\frac{2v+4}{3}$ |
| 5            | $\frac{2v+2}{3}$ |

## 2. Preliminaries

We denote a trade  $T$  with foundation size  $f$  and volume  $s$  by  $T = T(s, f)$ . The number of occurrences of a point  $x$  in  $T^+(T^-)$  is denoted by  $r_x$ . If  $r_x = 2$ , we call  $x$  a *regular point*, otherwise  $x$  is said to be an *irregular point*. Symbols  $1, 2, \dots$ , and  $A, B, \dots$ , are used for regular and irregular points respectively, and  $x, y, \dots$ , to refer to either kind. The number of occurrences of the pair  $xy$  is denoted by  $\lambda_{xy}$ . If  $\lambda_{xy} \leq 1$ , we call the pair Steiner, otherwise non-Steiner. Clearly for a non-Steiner pair  $xy$ , we have  $r_x \geq 3$  and  $r_y \geq 3$ . The points  $x$  and  $y$  are said to be adjacent if  $\lambda_{xy} \neq 0$ , and the collection of all (not necessarily distinct) adjacencies of  $x$  is denoted by  $N(x)$ . The *elements occurrence sequence* of  $T$  (abbreviated to  $\text{EOS}(T)$ ) is the non-decreasing sequence of occurrences of points.

A trade  $T$  may consist of two trades such as  $T_1$  and  $T_2$ , then we use the notation  $T = T_1 + T_2$ . When foundations of  $T_1$  and  $T_2$  are disjoint  $T = T_1 \oplus T_2$  is used instead.

The unique trade  $T(4, 6)$  is commonly called a Pasch configuration or briefly a Pasch.

The special case of the following Lemma for  $r = 2$  and its corollary has been proved in [1].

**Lemma 2.1.** Let  $T$  be a fundamental trade. Let  $x \in \text{found}(T)$  with at most one irregular adjacency and let  $r_x = r$ . Then,  $T = T(2r, 2r + 2)$  and has a unique structure with  $\text{EOS}(T) = 2, \dots, 2, r, r$ .

**Proof.** Let  $N(x) = \{y_1, \dots, y_{2r}\}$ . With no loss of generality, we can assume that the blocks of

$T$  containing  $x$  are:

$$\begin{array}{cc}
\frac{T^+}{\quad} & \frac{T^-}{\quad} \\
xy_1y_2 & xy_1y_3 \\
xy_3y_4 & xy_2y_5 \\
xy_5y_6 & xy_4y_7 \\
\vdots & \vdots \\
xy_{2r-1}y_{2r} & xy_{2r-2}y_{2r}
\end{array}$$

$T$  must also contain the following blocks:

$$\begin{array}{cc}
y_1y_3^- & y_1y_2^- \\
y_2y_5^- & y_3y_4^- \\
y_4y_7^- & y_5y_6^- \\
\vdots & \vdots \\
y_{2r-2}y_{2r}^- & y_{2r-1}y_{2r}^-
\end{array}$$

Clearly, the only possible way to fill the blanks is to use a fixed new point.  $\square$

**Corollary 2.1.** If  $\text{EOS}(T) = 2, \dots, 2, r$ , then  $r$  is even and  $T$  is the union of disjoint Pasches, except possibly for some Pasches, which contain the irregular point.

### 3. Steiner trades of minimum volume

In this section, we are concerned with Steiner trades of minimum volume. Bryant [1] has determined the minimum volume of these trades (Table 1). Here, we investigate, up to isomorphism, the structure as well as the number of such trades. If  $f \equiv 0 \pmod{6}$ , then by Table 1, the minimum volume is  $2f/3$  and by Corollary 2.1, the trade has a unique structure. Thus we have:

**Lemma 3.1** [1]. If  $f \equiv 0 \pmod{6}$ , then Steiner trade of minimum volume is the union of disjoint Pasches.

For the sake of simplicity in the statements of lemmas, we make the following note, which is also used in Section 4.

**Note.** Let  $T$  be a trade. By  $T_c$ , we denote the part of  $T$  which is the union of disjoint Pasches consisting only of regular points. By  $P(A, B)$ , we mean a Pasch containing irregular points  $A, B$  in which  $\lambda_{AB} = 0$  and by  $P(AB)$ , we mean  $\lambda_{AB} = 1$ . The union of Pasches of  $T$  containing some irregular points is denoted by  $T_p$ . Clearly, if  $A \in \text{found}(T)$  and  $r_A = 3$ , then  $A \notin \text{found}(T_p)$ .

Apart from  $T_c$  and  $T_p$ , we have in  $T$  a Pasch-free trade  $T'$ . Let  $x \in \text{found}(T')$  be an irregular point with  $r_x = r$  such that  $N(x)$  contains at most one irregular point. By Lemma 2.1,  $x$  appears in a  $T(2r, 2r + 2)$ . We denote the union of such parts of  $T'$  by  $T_m$ . By  $T_m(A)$ , we

mean  $A \in \text{found}(T_m)$ . Therefore, we have  $T = T_c \oplus (T_p + T_m + T_r)$  where  $T_r = T(s_r, f_r)$  is a Pasch-free trade in which for each  $x \in \text{found}(T_r)$ ,  $N(x)$  contains at least two irregular points. Hence,  $f_r \leq R$  in which  $R$  is the sum of occurrences of irregular points of  $\text{found}(T_r)$ .

In the following lemmas, we use the above notations and determine  $T_p$ ,  $T_m$  and  $T_r$  to give a complete description of the structure of the trade  $T$ .  $T_c$  obviously has a unique structure and we omit it from our statements.  $A, B, \dots$  will denote irregular points in non-increasing occurrence order. The small trades which appear in the following lemmas are listed in Table 2.

**Lemma 3.2.** Let  $T = T\left(\frac{2f+2}{3}, f\right)$  be a Steiner trade. Then

- (i) If  $\text{EOS}(T) = 2, \dots, 2, 4$ , then  $f \equiv 5 \pmod{6}$ ,  $T_p = P(A) + P(A)$  and  $T_m = T_r = \emptyset$ .
- (ii) If  $\text{EOS}(T) = 2, \dots, 2, 3, 3$ , then  $f \equiv 2 \pmod{6}$ ,  $T_p = T_r = \emptyset$  and  $T_m = T(6, 8)$ .

**Proof.**

- (i) This is just Corollary 2.1.
- (ii) Clearly  $T_p = \emptyset$ . There are just two irregular points, hence  $T_r = \emptyset$ . By Lemma 2.1,  $T_m = T(6, 8)$ .  $\square$

**Lemma 3.3.** Let  $T = T\left(\frac{2f+3}{3}, f\right)$  be a Steiner trade. Then

- (i)  $\text{EOS}(T) \neq 2, \dots, 2, 5; 2, \dots, 2, 3, 4$ .
- (ii) If  $\text{EOS}(T) = 2, \dots, 2, 3, 3, 3$ , then  $f \equiv 3 \pmod{6}$ ,  $T_p = T_m = \emptyset$  and  $T_r = T(7, 9)$ .

**Proof.**

- (i) By Lemma 2.1 and Corollary 2.1, both cases are impossible.
- (ii) If  $T_m \neq \emptyset$ , then by Lemma 2.1,  $T_m = T(6, 8)$  and  $\text{EOS}(T_r) = 2, \dots, 2, 3$  which is by Corollary 2.1 impossible. So  $T_m = \emptyset$ . We have  $f_r \leq 9$ , hence  $T_r = T(7, 9)$  which is unique by Table 2.  $\square$

**Lemma 3.4.** Let  $T = T\left(\frac{2f+4}{3}, f\right)$  be a Steiner trade. Then

- (i) If  $\text{EOS}(T) = 2, \dots, 2, 6$ , then  $f \equiv 4 \pmod{6}$ ,  $T_p = P(A) + P(A) + P(A)$  and  $T_m = T_r = \emptyset$ .
- (ii) If  $\text{EOS}(T) = 2, \dots, 2, 3, 5$ , then  $f \equiv 1 \pmod{6}$ ,  $T_p = P(A)$ ,  $T_m(A) = T(6, 8)$  and  $T_r = \emptyset$ .

- (iii) If  $\text{EOS}(T) = 2, \dots, 2, 4, 4$ , then  $f \equiv 4 \pmod{6}$ ,  $T_r = \emptyset$  and for  $T_p$  and  $T_m$  one of the followings occurs:
  - (a)  $T_p = \emptyset$  and  $T_m = T_4(8, 10)$ ;
  - (b)  $T_p = (P(A) + P(A)) \oplus (P(B) + P(B))$  and  $T_m = \emptyset$ ;
  - (c)  $T_p = (P(A) \oplus P(B)) + P(AB)$  and  $T_m = \emptyset$ ;
  - (d)  $T_p = (P(A) \oplus P(B)) + P(A, B)$  and  $T_m = \emptyset$ ;
  - (e)  $T_p = P(A, B) + P(A, B)$  and  $T_m = \emptyset$ ;
  - (f)  $T_p = P(A, B) + P(AB)$  and  $T_m = \emptyset$ .
- (iv) If  $\text{EOS}(T) = 2, \dots, 2, 3, 3, 4$ , then  $f \equiv 1 \pmod{6}$ ,  $T_r = \emptyset$  and for  $T_p$  and  $T_m$  one of the following occurs:
  - (a)  $T_p = P(A) + P(A)$  and  $T_m = T(6, 8)$ ;
  - (b)  $T_p = P(A)$  and  $T_m(A) = T(6, 8)$ .
- (v) If  $\text{EOS}(T) = 2, \dots, 2, 3, 3, 3, 3$ , then  $T_p = \emptyset$  and for  $T_m$  and  $T_r$  one of the followings occurs:
  - (a)  $f \equiv 1 \pmod{6}$ ,  $T_m = \emptyset$  and  $T_r = T_1(6, 7)$ ;
  - (b)  $f \equiv 4 \pmod{6}$ ,  $T_m = \emptyset$  and  $T_r = T_1(8, 10)$ ;
  - (c)  $f \equiv 4 \pmod{6}$ ,  $T_m = T(6, 8) \oplus T(6, 8)$  and  $T_r = \emptyset$ .

**Proof.**

- (i) This is just Corollary 2.1.
- (ii) There are just two irregular points, so  $T_r = \emptyset$ . By Lemma 2.1,  $T_p = P(A)$  and  $T_m(A) = T(6, 8)$ .
- (iii) There are just two irregular points, so  $T_r = \emptyset$ . If  $T_p = \emptyset$ , then by Lemma 2.1,  $T_m = T_4(8, 10)$ . Otherwise  $T_m = \emptyset$  and  $T_p$  can be any of the forms (a)-(e).
- (iv) We have  $f_r \leq 10$ . But  $T_r = T(6, 7)$  and  $T_r = T(8, 10)$  with  $\text{EOS}(T_r) = 2, \dots, 2, 3, 3, 4$  do not exist (cf. Table 2). So  $T_r = \emptyset$ . If  $T_p = \emptyset$ , then  $\text{EOS}(T_m) = 2, \dots, 2, 3, 3, 4$  which forces a Pasch containing an irregular point, a contradiction.  
 If  $T_p = P(A)$ , then  $\text{EOS}(T_m) = 2, \dots, 2, 3, 3$  and by Lemma 2.1,  $T_m(A) = T(6, 8)$ . If  $T_p = P(A) + P(A)$ , then  $\text{EOS}(T_m) = 2, \dots, 2, 3, 3$  and by Lemma 2.1,  $T_m = T(6, 8)$ .
- (v) Clearly  $T_p = \emptyset$  and  $f_r \leq 12$ . If  $T_r = \emptyset$ , then by Lemma 2.1,  $T_m = T(6, 8) \oplus T(6, 8)$ , otherwise we have  $T_r = T_1(6, 7)$  or  $T_r = T_1(8, 10)$ , with  $\text{EOS}(T_r) = 2, \dots, 2, 3, 3, 3, 3$ . Hence,  $T_m = \emptyset$ .  $\square$

#### 4. Non-Steiner trades of minimum volume

In this section, we consider non-Steiner trades of minimum volume. In order to establish the main results, we need the following lemmas.

**Lemma 4.1.** Let  $T$  be a non-Steiner trade. If there is a point  $A \in \text{found}(T)$  such that  $r_A = 3$ , then  $T$  must have at least three irregular points.

**Proof.** Suppose  $T$  has only two irregular points  $A$  and  $B$ . Clearly the pair  $AB$  is non-Steiner. Thus  $T$  contains the following blocks:

| $\underline{T^+}$ | $\underline{T^-}$ |
|-------------------|-------------------|
| $AB1$             | $AB3$             |
| $AB2$             | $AB4$             |
| $A34$             | $A12$             |
| $B3-$             | $B1-$             |
| $B4-$             | $B2-$             |
| $12-$             | $34-$             |

The only possible way to complete the blocks containing 3 and 4 is to use a fixed point  $z$  which is irregular, a contradiction.  $\square$

**Lemma 4.2.** Let  $T = T(s, f)$  contain a non-Steiner pair  $AB$  with  $r_A = r_B = 3$ . Then  $s \geq \frac{2f+6}{3}$ . If equality holds, then  $T = T_c \oplus T(6, 6)$ .

**Proof.** With no loss of generality, we can assume that  $T$  contains the following blocks:

| $\underline{T^+}$ | $\underline{T^-}$ |
|-------------------|-------------------|
| $ABx$             | $ABz$             |
| $ABy$             | $ABt$             |
| $Azt$             | $Axy$             |
| $Bzt$             | $Bxy$             |

Now, the pairs  $xy$  and  $zt$  are non-Steiner, hence  $r_x, r_y, r_z, r_t \geq 3$ . Therefore  $s \geq \frac{2f+6}{3}$ . If  $s = \frac{2f+6}{3}$ , then  $xyz, xyt \in T^+$  and  $ztx, zty \in T^-$  and we have a  $T(6, 6)$ .  $\square$

For a non-Steiner trade  $T = T(s, f)$ , by Lemmas 4.1 and 4.2, we have  $\text{EOS}(T) \neq 2, \dots, 2, 3, 4; 2, \dots, 2, 3, 3, 3$ . Hence  $s \geq \frac{2f+4}{3}$ .

We make a modification to the note in section 3.

**Note** (continued). When  $T_r$  is non-Steiner, we improve the upper bound for  $f_r$  to  $R-2$  or  $R-3$ . We omit some details. If  $f_r = R$ , then each point has exactly two irregular adjacencies. To prove  $f_r \leq R-3$ , we show that there are some points whose total number of irregular adjacencies, except for the two of them for each point, is at least 6.

Let  $AB$  be a non-Steiner pair in  $T_r$ . If  $ABC \in T_r^+$ , then each of  $A$  and  $B$  has three irregular adjacencies. Now, let  $AB1 \in T_r^+$ . We can assume, with no loss of generality, that  $T_r$  contains the following blocks:

| $T_r^+$ | $T_r^-$ |
|---------|---------|
| $1AB$   | $1Ax$   |
| $1xy$   | $1By$   |
| $Ax-$   | $AB-$   |
| $By-$   | $xy-$   |

If  $x$  is an irregular point (so we can let  $x = D$ ), then  $1$  and  $A$  have at least three irregular adjacencies each and  $AD1 \in T_r^-$ . When both  $x$  and  $y$  are regular points (so let  $x = 2$  and  $y = 3$ ), then blanks in the blocks of  $T_r^+$  are necessarily filled with an irregular fixed point, thus  $A$  and  $B$  have at least three irregular adjacencies each and we have  $AE2, AE3 \in T_r^+$ . Therefore, we have four “groups” of blocks each of which can be of three types:  $ABC$  (type 1),  $AD1$  (type 2) and  $AE2, AE3$  (type 3).

Since at least two groups are in  $T_r^+$  or  $T_r^-$ , we have  $f_r \leq R-2$ . Furthermore, in the following cases  $f_r \leq R-3$ :

- At least three groups appear in  $T_r^+(T_r^-)$ .
- Two groups appear in each of  $T_r^+$  and  $T_r^-$  and one of these four groups is of type 2.
- There are three groups of type 1.

If two groups appear in each of  $T_r^+$  and  $T_r^-$  (all of types 1 or 3), we have:

- If all groups are of type 3, then  $T_r$  contains  $T(7, 9) + T(7, 9)$ .
- If  $T_r^+$  contains one group of type 1 and  $T_r^-$  contains one group of type 3, then  $T_r$  will contain a Pasch, a contradiction.

So we have the following lemma:

**Lemma 4.3.** If  $T_r$  is non-Steiner, then  $f_r \leq R-2$ . If equality holds,  $T_r$  contains  $T(7, 9) + T(7, 9)$ .

In what follows,  $A, B, \dots$  will denote irregular points in non-increasing occurrence order. Table 3 consists of the small trades which appear in the following lemmas.

**Lemma 4.4.** Let  $T = T\left(\frac{2f+4}{3}, f\right)$  be a non-Steiner trade. Then

- (i)  $\text{EOS}(T) \neq 2, \dots, 2, 3, 5; \quad 2, \dots, 2, 3, 3, 3, 3.$
- (ii) If  $\text{EOS}(T) = 2, \dots, 2, 4, 4$ , then  $f \equiv 4 \pmod{6}$ ,  $T_p = P(AB) + P(AB)$  and  $T_m = T_r = \emptyset$ .
- (iii) If  $\text{EOS}(T) = 2, \dots, 2, 3, 3, 4$ , then  $f \equiv 1 \pmod{6}$ ,  $T_p = T_m = \emptyset$  and  $T_r = T_2(6, 7)$ .

**Proof.**

- (i) This is clear by Lemmas 4.1 and 4.2.
- (ii) There are only two irregular points and  $\lambda_{AB} \geq 2$ , so  $T_m = \emptyset$ . We have  $f_r \leq 5$ , so  $T_r = \emptyset$  and consequently we have  $T_p = P(AB) + P(AB)$ .
- (iii) By Lemma 4.1,  $T_p = \emptyset$ . If  $T_m \neq \emptyset$ , then  $T_m = T(6, 8)$  and  $\text{EOS}(T_r) = 2, \dots, 2, 4$ , so  $T_m + T_r$  is a Steiner trade. Hence  $T_m = \emptyset$ . Since  $f_r \leq 7$ , we have  $T_r = T_2(6, 7)$  (Table 3).  
□

**Lemma 4.5.** Let  $T = T\left(\frac{2f+5}{3}, f\right)$  be a non-Steiner trade. Then

- (i)  $\text{EOS}(T) \neq 2, \dots, 2, 3, 6;$
- (ii)  $\text{EOS}(T) \neq 2, \dots, 2, 4, 5;$
- (iii)  $\text{EOS}(T) \neq 2, \dots, 2, 3, 3, 5;$
- (iv)  $\text{EOS}(T) \neq 2, \dots, 2, 3, 4, 4;$
- (v)  $\text{EOS}(T) \neq 2, \dots, 2, 3, 3, 3, 3, 3;$
- (vi) If  $\text{EOS}(T) = 2, \dots, 2, 3, 3, 3, 4$ , then  $f \equiv 2 \pmod{6}$ ,  $T_p = T_m = \emptyset$  and  $T_r = T(7, 8)$ .

**Proof.**

- (i) It follows from Lemma 4.1.
- (ii) By Corollary 2.1 and Lemma 4.1,  $T_p = \emptyset$ . As  $\lambda_{AB} \geq 2$  and there are only two irregular points, we have  $T_m = \emptyset$ . By  $f_r \leq 6$ , it follows that  $T_r = \emptyset$ .
- (iii) By similar arguments as in (ii), we have  $T_p = \emptyset$ . If  $T_m \neq \emptyset$ , then  $T_m = T(6, 8)$  and  $\text{EOS}(T_r) = 2, \dots, 2, 5$  which is impossible. Since  $f_r \leq 8$ , we have  $T_r = T(7, 8)$  with  $\text{EOS}(T_r) = 2, \dots, 2, 3, 3, 5$  and such a trade does not exist(cf. Table 3) .



- (iv) By similar arguments as in (iii),  $T_p = T_m = \emptyset$  and  $T_r = T(7, 8)$  with  $\text{EOS}(T_r) = 2, \dots, 2, 3, 4, 4$ , which does not exist (cf. Table 3).
- (v) By Lemma 4.2, this is obvious.
- (vi) By Lemma 4.2,  $T_p = \emptyset$ . If  $T_m \neq \emptyset$ , then  $T_m = T(6, 8)$  and  $\text{EOS}(T_r) = 2, \dots, 2, 3, 4$ , which is by Lemma 4.1 impossible. So  $T_m = \emptyset$  and since  $f_r \leq 10$ , we have  $T_r = T(7, 8)$ .  $\square$

**Lemma 4.6.** Let  $T = T\left(\frac{2f+6}{3}, f\right)$  be a non-Steiner trade. Then

- (i)  $\text{EOS}(T) \neq 2, \dots, 2, 3, 7; 2, \dots, 2, 5, 5$ .
- (ii) If  $\text{EOS}(T) = 2, \dots, 2, 4, 6$ , then  $f \equiv 3 \pmod{6}$ ,  $T_p = P(AB) + P(AB) + P(A)$  and  $T_m = T_r = \emptyset$ .
- (iii) If  $\text{EOS}(T) = 2, \dots, 2, 3, 3, 6$ , then  $f \equiv 0 \pmod{6}$ ,  $T_p = P(A)$ ,  $T_m = \emptyset$  and  $T_r(A) = T_2(6, 7)$ .
- (iv) If  $\text{EOS}(T) = 2, \dots, 2, 3, 4, 5$ , then  $f \equiv 0 \pmod{6}$ ,  $T_p = P(A)$ ,  $T_m = \emptyset$  and  $T_r(A) = T_2(6, 7)$ .
- (v) If  $\text{EOS}(T) = 2, \dots, 2, 4, 4, 4$ , then  $f \equiv 3 \pmod{6}$ ,  $T_m = \emptyset$ . We have either  $T_p = \emptyset$  and  $T_r = T_2(8, 9)$ , or  $T_r = \emptyset$  and for  $T_p$ , we have one of the followings:
  - (a)  $(P(A) + P(A)) \oplus (P(BC) + P(BC))$ ;
  - (b)  $P(A) + P(BC) + P(ABC)$ ;
  - (c)  $P(A) + P(BC) + P(AB, BC)$ ;
  - (d)  $P(ABC) + P(AB, AC)$ ;
  - (e)  $P(AB, AC) + P(AB, AC)$ ;
  - (f)  $P(AB, AC) + P(AB, BC)$ .
- (vi) If  $\text{EOS}(T) = 2, \dots, 2, 3, 3, 3, 5$ , then  $f \equiv 3 \pmod{6}$ ,  $T_p = T_m = \emptyset$  and  $T_r = T_6(8, 9)$ .
- (vii) If  $\text{EOS}(T) = 2, \dots, 2, 3, 3, 4, 4$ , then  $f \equiv 0 \pmod{6}$  and for  $T_p, T_m$  and  $T_r$  one of the followings occurs:
  - (a)  $T_p = P(A) + P(A), T_m = \emptyset$  and  $T_r = T_2(6, 7)$ ;
  - (b)  $T_p = P(A), T_m = \emptyset$  and  $T_r(A) = T_2(6, 7)$ ;
  - (c)  $T_p = P(AB) + P(AB), T_m = T(6, 8)$  and  $T_r = \emptyset$ ;
  - (d)  $T_p = P(AB), T_m(AB) = T(6, 8)$  and  $T_r = \emptyset$ .

(viii) If  $\text{EOS}(T) = 2, \dots, 2, 3, 3, 3, 3, 4$ , then  $f \equiv 3 \pmod{6}$ ,  $T_p = \emptyset$  and for  $T_m$  and  $T_r$  one of the followings occurs:

(a)  $T_m = T(6, 8)$  and  $T_r = T_2(6, 7)$ ;

(b)  $T_m = \emptyset$  and  $T_r = T_3(8, 9)$ .

(ix) If  $\text{EOS}(T) = 2, \dots, 2, 3, 3, 3, 3, 3$ , then  $f \equiv 0 \pmod{6}$ ,  $T_p = T_m = \emptyset$  and  $T_r = T(6, 6)$ .

**Proof.**

(i) By Lemma 4.1,  $\text{EOS}(T) \neq 2, \dots, 2, 3, 7$ .

Let  $\text{EOS}(T) = 2, \dots, 2, 5, 5$ . By Lemmas 4.1 and 4.2,  $T_p = T_m = \emptyset$ .

Since  $f_r \leq 7$ , we must have  $T_r = T(6, 6)$ . Since this trade has a different EOS(cf. Table 3),  $T_r = \emptyset$ .

(ii) Clearly  $T_m = \emptyset$ . Since  $f_r \leq 7$ , we must have  $T_r = T(6, 6)$ , which has a different EOS(cf. Table 3), hence  $T_r = \emptyset$ . As  $\lambda_{AB} \geq 2$ , we must have  $T_p = P(AB) + P(AB) + P(A)$ .

(iii) Clearly  $T_m = \emptyset$ . If  $T_p = P(A)$ , then  $\text{EOS}(T - T_p) = 2, \dots, 2, 3, 3, 4$  and by Lemma 4.4(iii)  $T_m = \emptyset$  and  $T_r = T_2(6, 7)$ . So let  $T_p = \emptyset$ . Since  $f_r \leq 9$ , we have  $T_r = T(6, 6)$  or  $T_r = T(8, 9)$ . Neither of these two trades with the specified EOS exists(cf. Table 3).

(iv) By similar arguments as in (iii), the result follows.

(v) Clearly  $T_m = \emptyset$ . Since  $f_r \leq 9$ , we have  $T_r = T(6, 6)$ , which has a different EOS (cf. Table 3),  $T_r = T_3(8, 9)$ (cf. Table 3) or  $T_r = \emptyset$ . If  $T_r = \emptyset$ , then for  $T_p$ , one of the cases (a)-(f) occurs.

(vi) By Lemmas 4.1 and 4.2  $T_m = T_p = \emptyset$ . As  $f_r \leq 11$ ,  $T_r = T(6, 6)$ , which has a different EOS, or  $T_r = T_7(8, 9)$  (cf. Table 3).

(vii) Suppose  $T_p \neq \emptyset$ . If  $T_p = P(A)$  or  $T_p = P(A) + P(A)$ , then  $\text{EOS}(T - T_p) = 2, \dots, 2, 3, 3, 4$ . By Lemma 4.4(iii),  $T_m = \emptyset$  and for  $T_r$  we have  $T_r(A) = T_2(6, 7)$  or  $T_r = T_2(6, 7)$ . If  $T_p = P(AB)$  or  $T_p = P(AB) + P(AB)$ , then  $\text{EOS}(T - T_p) = 2, \dots, 2, 3, 3$ . By Lemma 2.1, we have  $T_r = \emptyset$  and  $T_m(AB) = T(6, 8)$  or  $T_m = T(6, 8)$ . Now let  $T_p = \emptyset$ . If  $T_m \neq \emptyset$ , it must necessarily be  $T(6, 8)$  and hence  $\text{EOS}(T_r) = 2, \dots, 2, 4, 4$ . But then, Lemma 4.4(ii) forces  $T_p \neq \emptyset$ . So  $T_m = \emptyset$ . Since  $f_r \leq 11$ , we have  $T_r = T(6, 6)$ , or  $T_r = T(8, 9)$ . None of these has the specified EOS (cf. Table 3).

(viii) By Lemma 4.2,  $T_p = \emptyset$ . If  $T_m \neq \emptyset$ , then  $T_m = T(6, 8)$  and  $\text{EOS}(T_r) = 2, \dots, 2, 3, 3, 4$ . Therefore, by Lemma 4.4(iii),  $T_r = T_2(6, 7)$  (Table 3). We now assume that  $T_m = \emptyset$ .

Since  $f_r \leq 13$ ,  $T_r = T_3(8, 9)$  (Table 3) or  $T_r = T(10, 12)$ . We show such  $T(10, 12)$  does not exist. Suppose  $r_A = 4$  and  $r_B = 3$ . With no loss of generality, we consider the following blocks:

| $T_r^+$ | $T_r^-$ |
|---------|---------|
| $AB1$   | $AB_y$  |
| $ABx$   | $ABz$   |
| $Byz$   | $B1x$   |
| $Ayr$   | $A1r$   |
| $Azs$   | $Axs$   |
| $1xr$   |         |
| $xs-$   |         |

The three remaining blocks in  $T_r^+$  must contain four new regular points and are of the form  $234, 25-, 35-$ , but then 2 has at most one irregular adjacency which is a contradiction.

(ix) By Lemma 4.2, the result follows.  $\square$

## 5. Conclusions

In the following theorems, we summarize the results of sections 3 and 4 on the number of non-isomorphic trades of minimum volume.

**Theorem 5.1.** Up to isomorphism, the number of Steiner trades of minimum volume is as follows:

| $f \pmod{6}$ | minimum volume   | # trades | exceptions                             |
|--------------|------------------|----------|--|
| 0            | $\frac{2f}{3}$   | 1        |  |
| 1            | $\frac{2f+4}{3}$ | 4        | $\#T(6, 7) = 1,$<br>$\#T(10, 13) = 3$  |
| 2            | $\frac{2f+2}{3}$ | 1        |  |
| 3            | $\frac{2f+3}{3}$ | 1        |  |
| 4            | $\frac{2f+4}{3}$ | 9        | $\#T(8, 10) = 4,$<br>$\#T(12, 16) = 8$ |
| 5            | $\frac{2f+2}{3}$ | 1        |  |

**Theorem 5.2.** Up to isomorphism, the number of non-Steiner trades of minimum volume is as follows:

| $f \pmod{6}$ | minimum volume   | # trades | exceptions                           |
|--------------|------------------|----------|--------------------------------------|
| 0            | $\frac{2f+6}{3}$ | 7        | $\#T(6,6) = 1,$<br>$\#T(10,12) = 5$  |
| 1            | $\frac{2f+4}{3}$ | 1        |                                      |
| 2            | $\frac{2f+5}{3}$ | 1        |                                      |
| 3            | $\frac{2f+6}{3}$ | 10       | $\#T(8,9) = 6,$<br>$\#T(12,15) = 10$ |
| 4            | $\frac{2f+4}{3}$ | 1        |                                      |

## 6. Appendix

Table 2 contains, up to isomorphism, all Steiner trades of minimum volume for  $7 \leq f \leq 10$  [2, Table 2]. In Table 3, all non-isomorphic non-Steiner trades of minimum volume are given for  $6 \leq f \leq 9$  [3,4].

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**Table 2.**  
**Steiner trades of minimum volume for  $7 \leq f \leq 10$ .**

| $T_1(6, 7)$ | $T(6, 8)$ | $T(7, 9)$ | $T_1(8, 10)$ | $T_2(8, 10)$ | $T_3(8, 10)$ | $T_4(8, 10)$ |
|-------------|-----------|-----------|--------------|--------------|--------------|--------------|
| 123         | 123       | 123       | 127          | 123          | 123          | 123          |
| 167         | 145       | 145       | 138          | 145          | 145          | 145          |
| 247         | 167       | 167       | 28A          | 167          | 167          | 167          |
| 256         | 248       | 248       | 379          | 189          | 189          | 189          |
| 346         | 368       | 358       | 459          | 24A          | 24A          | 24A          |
| 357         | 578       | 369       | 46A          | 268          | 35A          | 36A          |
|             |           | 579       | 57A          | 279          | 68A          | 58A          |
|             |           |           | 689          | 35A          | 79A          | 79A          |
| 127         | 124       | 124       | 128          | 124          | 124          | 124          |
| 136         | 136       | 136       | 137          | 135          | 135          | 136          |
| 235         | 157       | 157       | 27A          | 168          | 168          | 158          |
| 246         | 238       | 238       | 389          | 179          | 179          | 179          |
| 347         | 458       | 359       | 45A          | 23A          | 23A          | 23A          |
| 567         | 678       | 458       | 469          | 267          | 45A          | 45A          |
|             |           | 679       | 579          | 289          | 67A          | 67A          |
|             |           |           | 68A          | 45A          | 89A          | 89A          |

**Table 3.**  
**Non-Steiner trades of minimum volume for  $6 \leq f \leq 9$ .**

| $T(6, 6)$ | $T_2(6, 7)$ | $T(7, 8)$ | $T_1(8, 9)$ | $T_2(8, 9)$ | $T_3(8, 9)$ | $T_4(8, 9)$ | $T_5(8, 9)$ | $T_6(8, 9)$ |
|-----------|-------------|-----------|-------------|-------------|-------------|-------------|-------------|-------------|
| 123       | 123         | 123       | 123         | 123         | 123         | 123         | 123         | 123         |
| 124       | 124         | 147       | 145         | 145         | 145         | 145         | 145         | 145         |
| 156       | 156         | 148       | 247         | 248         | 248         | 246         | 239         | 248         |
| 256       | 157         | 156       | 268         | 249         | 249         | 248         | 247         | 249         |
| 345       | 267         | 245       | 356         | 267         | 347         | 239         | 268         | 347         |
| 346       | 345         | 358       | 357         | 345         | 356         | 347         | 346         | 356         |
|           |             | 678       | 239         | 568         | 589         | 356         | 357         | 467         |
|           |             |           | 589         | 579         | 679         | 789         | 489         | 789         |
| 125       | 126         | 124       | 124         | 124         | 124         | 124         | 124         | 124         |
| 126       | 127         | 138       | 135         | 135         | 135         | 135         | 135         | 135         |
| 134       | 135         | 145       | 236         | 234         | 234         | 234         | 236         | 234         |
| 234       | 145         | 167       | 237         | 268         | 289         | 236         | 237         | 289         |
| 356       | 234         | 235       | 289         | 279         | 367         | 289         | 289         | 367         |
| 456       | 567         | 478       | 359         | 458         | 458         | 379         | 349         | 456         |
|           |             | 568       | 457         | 459         | 479         | 456         | 457         | 478         |
|           |             |           | 568         | 567         | 569         | 478         | 468         | 479         |

## References

1. D. E. Bryant, *On the volume of trades in the triple systems*, Australas. J. Combin. **15** (1997), 161-176.
2. G. B. Khosrovshahi and H. R. Maimani, *On Steiner  $2-(v, 3)$  trades of small volumes*, to appear.
3. G. B. Khosrovshahi, H. R. Maimani and R. Torabi, *Classification of  $2-(6, 3)$  and  $2-(7, 3)$  trades*, submitted.
4. Z. Eslami, G. B. Khosrovshahi and B. Tayfeh-Rezaie, *On Classification of  $2-(8, 3)$  and  $2-(9, 3)$  trades*, in preparation.