Spectral characterization of some cubic graphs

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Abstract

It is proved that the Cartesian product of an odd cycle with the complete graph on 2 vertices, is determined by the spectrum of the adjacency matrix. We also present some computational results on the spectral characterization of cubic graphs on at most 20 vertices.

AMS Subject Classification: 05C50.
Keywords: Cartesian product of graphs, eigenvalues, spectral characterization.

1 Introduction

Let G be a simple graph on the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For a subset S of $V(G)$ by $\langle S \rangle$, we mean the subgraph of G induced on S. We denote the distance between u and v (degree of v) in G, by $d_G(u, v)$ ($d_G(v)$), for $u, v \in V(G)$. If there is no danger of confusion, we simply write $d(u, v)$ ($d(v)$). The *adjacency matrix* of G is an n by n matrix A_G whose (i, j)-th entry is 1 if vertices v_i and v_j are adjacent and 0, otherwise. The *spectrum* of G is the multiset of eigenvalues of A_G. Two non-isomorphic graphs G and H are called *cospectral* if they share the same spectrum. We say G is *determined by its spectrum* (*DS* for short) if it has no cospectral mate. The *Cartesian product* $G \times H$ of two graphs G and H, is a graph with the vertex set $V(G) \times V(H)$ such that two vertices (x, y) and (x', y') are adjacent if and only if either $x = x'$ and y is adjacent to y' in H or x is adjacent to x' in G and $y = y'$.

The question whether or not a given graph is DS, has been investigated by some authors. For a survey of results on this area we refer the reader to [4, 5]. Some families of regular graphs, specially distance regular graphs have attracted more attention, see for instance [1, 6, 8, 9, 11]. In [9], Heydemann proved that the lexicographic product of an odd cycle with a totally disconnected

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graph is DS. He also conjectured that the same assertion holds for even cycles. In this paper, we consider 3-regular (cubic) graphs. Let C_n denote the cycle of length n. We prove that for any positive integer t, the graph $G_t = C_{2t+1} \times K_2$ is determined by the spectrum. Moreover, we present some computational results on the spectral characterization of cubic graphs on at most 20 vertices.

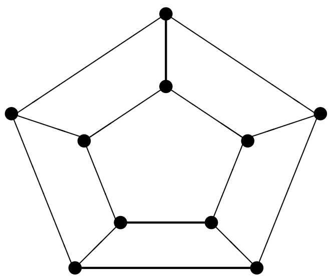

Figure 1: The graph $G_2 = C_5 \times K_2$

2 Preliminary results

There is no pair of cospectral regular graphs with less than 10 vertices and there are exactly four pairs of cospectral graphs on 10 vertices, none of them is cubic, see for instance [3]. Therefore, in our study we only need to consider graphs on at least 14 vertices and we can assume $t \geq 3$. In this section we investigate some structural properties of cospectral mates of G_t. Regularity and the number of smallest odd cycles in a graph is determined by the spectrum [5]. Using this, we conclude that any cospectral mate of G_t, say H_t, has two cycles of length $2t+1$. We denote these cycles of H_t by C and C'. Since C and C' are the only smallest odd cycles of H_t, it follows that they are induced subgraphs in H_t. Here, our objective is to prove that C and C' are disjoint. The claim is established through some lemmas.

Lemma 1 If there is a perfect matching between $V(C)$ and $V(H_t) \setminus V(C)$, then C and C' are disjoint.

Proof. Since H_t is cubic, by the assumption, $H' = \langle V(H_t) \setminus V(C) \rangle$ is a 2-regular graph and so a union of cycles. Since H' is of odd order and H_t has exactly two odd cycles of length $2t+1$, it easily follows that $H' = C'$ and the assertion holds. □

Lemma 2 Let u be a vertex in $V(H_t) \setminus V(C)$ and suppose that u has at least two neighbors in $V(C)$. Then $|V(C) \cap V(C')| = 2t$.

Proof. Assume v_1 and v_2 are two neighbors of u in $V(C)$. Let P be the path of even order (even number of vertices) from v_1 to v_2 on C. Such a path exists since C has odd length. The cycle C'' formed by vertices of P and u is an odd cycle whose length is smaller than or equal to $2t+1$. On the other hand $2t+1$ is the size of smallest odd cycles in H_t. Therefore, the length of C'' is $2t+1$ which implies that $C' = C''$ (because H_t has only two cycles of length $2t+1$) and P is of order $2t$. Since the vertices of P are on both C and C', the assertion follows. $\square$

Lemma 3 Let u and v be two adjacent vertices in $V(H_t) \setminus (V(C) \cup V(C'))$. Let $x, y \in V(C)$ be such that x is adjacent to u and y is adjacent to v. Then x is adjacent to y. The same assertion holds for C'.

Proof. Let P be the path of odd order (odd number of vertices) from x to y in C. The cycle C'' formed on $V(P) \cup \{u, v\}$ is an odd cycle of length $|V(P)| + 2$. Since $2t+1$ is the order of smallest odd cycles of H_t and C'' is distinct from C and C', we conclude that C'' has length at least $2t+3$ and that $|V(P) \geq 2t+1|$ involving that $|V(P) = 2t|$. Thus the assertion follows. $\square$

Lemma 4 Every vertex $u \in V(H_t) \setminus (V(C) \cup V(C'))$ has at most one neighbor in $V(C)$. The same assertion holds for C'.

Proof. Suppose that u is adjacent to two vertices v_1 and $v_2 \in C$. Let P be the path in C on an even number of vertices linking v_1 and v_2. Then $P \cup \{u\}$ is an odd cycle of length lower than or equal to $2t+1$ yielding a contradiction. So the assertion follows. $\square$

Lemma 5 $|V(C) \cap V(C')| \neq 2t$.

Proof. By contradiction, suppose that $K = \langle V(C) \cap V(C') \rangle$ has $2t$ vertices. Let $c_1 \in V(C) \setminus V(K)$ and $c_2 \in V(C') \setminus V(K)$. We claim that there is a perfect matching between two sets $F = \{c_1, c_2\} \cup \{x \in V(K) : d_K(x) = 2\}$ and $U = V(H_t) \setminus (V(C) \cup V(C'))$. The graph K is a path of length $2t-1$, the set U has $2t$ vertices and the graph $\langle V(C) \cup V(C') \rangle$ is of order $2t+2$ with degree sequence $3, 3, 2^{2t}$ (the power of 2 denotes the number of vertices of degree 2 in the graph). Note that, since H_t is a cubic graph, every vertex in F is adjacent to one vertex $u \in U$. If a perfect matching between U and F does not exist, then there is a vertex $u \in U$ with at least two neighbors in F. Using Lemma 4, we obtain that u is exactly adjacent to c_1 and c_2. Thus in this case, U has exactly one vertex with two neighbors in F. By Lemma 4, each vertex in $A = \{x \in V(K) : d_K(x) = 2\}$ is matched to some vertex in U. Therefore, $\langle U \rangle$ has degree sequence $3, 2^{2t-2}, 1$, and is a union of a lollipop graph with some cycles, see [2]. Using Lemma 3, u is not adjacent to those vertices of U which are matched to the vertices of A. Therefore, u is adjacent to the vertex of degree 3 in $\langle U \rangle$ which implies that the length of the cycle of the lollipop is $2t-1$, a contradiction. Hence, there is a perfect matching between U and F. So $\langle U \rangle$ is 2-regular. Now let c_1' be the vertex in $\langle U \rangle$-neighborhood of c_1. Since $\langle U \rangle$ is 2-regular, c_1' has a neighbor in $\langle U \rangle$ which is matched to a vertex of A. This is impossible by Lemma 3. Thus, the assertion follows. $\square$

Lemma 6 C and C' are disjoint.

Proof. Let $u \in V(H_t) \setminus V(C)$ have at least two neighbors in C. Lemma 2 implies that $|V(C) \cap V(C')| = 2t$ which is impossible by Lemma 5. Therefore, a perfect matching between $V(C)$ and $V(H_t) \setminus V(C)$ exists. Now the assertion follows by Lemma 1. $\square$

3 Main result

In this section, we prove that G_t is determined by the spectrum. We will take advantage of the following lemmas to prove our main result.

Lemma 7 [10] Let A be a square matrix. Then A equals zero if and only if $\mathrm{tr}(A^t A)$ is zero.

Lemma 8 Let G_1 and G_2 be two graphs with adjacency matrices,

$$A_{G_1} = \begin{pmatrix} A & I \\ I & A \end{pmatrix} \qquad \text{and} \qquad A_{G_2} = \begin{pmatrix} A & B \\ B^t & A \end{pmatrix},$$

where I is the identity matrix, A is a symmetric $(0,1)$-matrix and B is an automorphism of the graph corresponding to A. Then G_1 is isomorphic to G_2.

Proof. Let $Q = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}$. Then $Q^{-1} A_{G_2} Q = A_{G_1}$ which means that G_1 and G_2 are isomorphic. $\square$

Lemma 9 [3] Two simple graphs G and H are cospectral if and only if for any positive integer k, $\mathrm{tr}(A_G^k)$=$\mathrm{tr}(A_H^k)$.

Proposition 1 Given a $k-$regular graph G, let H be a cospectral mate of $\Gamma = G \times K_2$. If H has a disjoint union of two copies of G as a subgraph, then H is isomorphic to Γ.

Proof. Let the adjacency matrix of H be

$$\begin{pmatrix} A & B \\ B^t & A \end{pmatrix},$$

where A is the adjacency matrix of G. The graph G is k-regular and so H, Γ are $(k+1)$-regular which implies that the matrix B has one and only one non-zero entry per row and per column. Thus, B is a permutation matrix. By Lemma 8, it suffices to prove that B is an automorphism of G. Let $S = BA - AB$. We claim that $\mathrm{tr}(S^t S)$ equals zero. Since H is cospectral to Γ, by Lemma 9, we have $\mathrm{tr}(A_H^4) = \mathrm{tr}(A_\Gamma^4)$ which implies that,

$$6\mathrm{tr}(A^2) = \mathrm{tr}(BA^2B^t + B^tA^2B + ABAB^t + BAB^tA + B^tABA + AB^tAB).$$

It is known that similar matrices have the same trace. Thus the first two summands in the right hand side of the above relation are equal to $\mathrm{tr}(A^2)$. The other summands are equal to each other, since the trace operation is commuting with respect to matrix multiplication. It yields that

$$\mathrm{tr}(B^tABA) = \mathrm{tr}(AB^tAB) = \mathrm{tr}(A^2).$$

Therefore,

$$\mathrm{tr}(S^tS) = \mathrm{tr}(A^2 - AB^tAB - B^tABA + B^tA^2B) = 0.$$

We are done by Lemma 7. $\square$

Now we are ready to present our main result.

Theorem 1 For any positive integer t, the graph G_t is determined by the spectrum.

Proof. Let H_t be a graph cospectral to G_t. Then H_t has two cycles of length $2t+1$, say C and C'. By Lemma 6, C is disjoint from C' and so the assertion follows from Proposition 1. $\square$

We proved the Cartesian product of an odd cycle with K_2 is determined by the spectrum. For even n we do not know much about the spectral characterization of $C_n \times K_2$. From the following section, we only know that $C_n \times K_2$ is DS for $n = 6, 8, 10$. Note that as mentioned in the previous section, $C_4 \times K_2$ is also DS. We conjecture that, $C_n \times K_2$ is DS for all n.

4 Small cubic graphs

In this section we present some computational results on cubic graphs of order at most 20. We generated all such graphs up to isomorphism with an orderly algorithm. Using Maple and Matlab softwares, we computed the characteristic polynomial of these graphs and used them to find all families of cospectral pairs of small cubic graphs. A summary of results follows.

- There exist 112 cubic graphs up to order 12. All of them are DS.
- There are 509 cubic graphs of order 14. Exactly 3 pairs of these graphs are cospectral. The remaining 503 graphs are DS. Cospectral pairs are listed in Appendix A.
- Among 4060 cubic graphs with 16 vertices, there are exactly 42 pairs and one triple set of cospectral graphs. Moreover, the graph $C_8 \times K_2$ is determined by the spectrum. Graphs in the triple set are presented in Appendix A.
- There exist 41301 cubic graphs on 18 vertices. Among them, there are exactly 471 pairs, 15 triple sets and one quadruple set of cospectral graphs. Graphs in the quadruple set are listed in Appendix A.

- In order 20, there exist 510489 cubic graphs. We find 4799 pairs, 168 triple sets and 10 quadruple sets of cospectral graphs. Moreover, it is checked that the graph $C_{10} \times K_2$ is determined by the spectrum.

For order 20, the computation took about 3 days on a single computer. It would probably take much more time to carry out a similar computation for order 22. In the following table we present the ratio of DS cubic graphs to all cubic graphs for orders less than 20. Here, d_n and v_n denote the number of cubic and DS cubic graphs on n vertices, respectively.

n	v_n	d_n	$\frac{d_n}{v_n}$
≤ 12	112	112	1
14	509	503	0.9882
16	4060	3973	0.9786
18	41301	40310	0.9760
20	510489	500347	0.9801

Acknowledgement The authors would like to thank anonymous referees for their useful comments which considerably improved the final version of this paper.

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A List of cospectral graphs

For $i = 1, 2, 3$, the graphs L_i and L'_i, given below are cospectral. (For each vertex we give it's neighborhood). The notation $\sum_i$ stands for the spectrum of the graph L_i.

L_1	L'_1	L_2	L'_2	L_3	L'_3
1: 2, 3, 4	1: 2, 3, 4	1: 2, 3, 4	1: 2, 3, 4	1: 2, 3, 4	1: 2, 3, 4
2: 1, 3, 5	2: 1, 3, 5	2: 1, 3, 5	2: 1, 3, 5	2: 1, 3, 5	2: 1, 3, 5
3: 1, 2, 6	3: 1, 2, 6	3: 1, 2, 6	3: 1, 2, 6	3: 1, 2, 6	3: 1, 2, 6
4: 1, 7, 8	4: 1, 7, 8	4: 1, 7, 8	4: 1, 7, 8	4: 1, 7, 8	4: 1, 7, 8
5: 2, 7, 9	5: 2, 7, 9	5: 2, 7, 9	5: 2, 7, 9	5: 2, 7, 9	5: 2, 7, 9
6: 3, 7, a	6: 3, 8, 9	6: 3, a, b	6: 3, a, b	6: 3, a, b	6: 3, a, b
7: 4, 5, 6	7: 4, 5, a	7: 4, 5, a	7: 4, 5, c	7: 4, 5, c	7: 4, 5, c
8: 4, b, c	8: 4, 6, b	8: 4, b, c	8: 4, a, b	8: 4, 9, d	8: 4, a, d
9: 5, b, d	9: 5, 6, c	9: 5, c, d	9: 5, a, d	9: 5, 8, e	9: 5, c, e
a: 6, c, d	a: 7, d, e	a: 6, 7, e	a: 6, 8, 9	a: 6, c, d	a: 6, 8, e
b: 8, 9, e	b: 8, d, e	b: 6, 8, e	b: 6, 8, e	b: 6, c, e	b: 6, c, d
c: 8, a, e	c: 9, d, e	c: 8, 9, d	c: 7, d, e	c: 7, a, b	c: 7, 9, b
d: 9, a, e	d: a, b, c	d: 9, c, e	d: 9, c, e	d: 8, a, e	d: 8, b, e
e: b, c, d	e: a, b, c	e: a, b, d	e: b, c, d	e: 9, b, d	e: 9, a, d

$\sum_1 = [-2.8512, -1.8794, -1.8794, -1.8372, -1, -1, 0, 0.3473, 0.3473, 1.2555, 1.5321, 1.5321, 2.4329, 3]$
$\sum_2 = [-2.6970, -1.9319, -1.8521, -1.4142, -1.4142, -1, -0.5176, 0.2627, 0.5176, 1.4142, 1.4142, 1.9319, 2.2863, 3]$
$\sum_3 = [-2.4812, -2.4621, -2.0569, -1.6180, -0.8319, -0.6889, -0.4352, 0.1743, 0.6180, 1.1701, 1.4598, 1.9276, 2.2245, 3]$

A triple set of cospectral cubic graphs on 16 vertices:

L_1	L_2	L_3
1: 2, 3, 4	1: 2, 3, 4	1: 2, 3, 4
2: 1, 3, 5	2: 1, 3, 5	2: 1, 3, 5
3: 1, 2, 6	3: 1, 2, 6	3: 1, 2, 6
4: 1, 7, 8	4: 1, 7, 8	4: 1, 7, 8
5: 2, 7, 9	5: 2, 7, 9	5: 2, 7, 9
6: 3, a, b	6: 3, a, b	6: 3, a, b
7: 4, 5, c	7: 4, 5, c	7: 4, 5, a
8: 4, a, d	8: 4, a, d	8: 4, c, d
9: 5, a, e	9: 5, a, e	9: 5, c, e
a: 6, 8, 9	a: 6, 8, 9	a: 6, 7, f
b: 6, f, g	b: 6, d, f	b: 6, d, g
c: 7, d, f	c: 7, f, g	c: 8, 9, g
d: 8, c, g	d: 8, b, g	d: 8, b, f
e: 9, f, g	e: 9, f, g	e: 9, f, g
f: b, c, e	f: b, c, e	f: a, d, e
g: b, d, e	g: c, d, e	g: b, c, e

$$\sum_i = [-2.7361, -2.1408, -2, -1.7647, -1.5656, -1.1107, -0.8103, -0.2124, 0.3988, 0.6352, 0.9386, 1.5636, 1.5865, 1.8552, 2.3628, 3], \quad i = 1, 2, 3.$$

A quadruple set of cospectral cubic graphs on 18 vertices:

L_1	L_2	L_3	L_4
1: 2, 3, 4	1: 2, 3, 4	1: 2, 3, 4	1: 2, 3, 4
2: 1, 3, 5	2: 1, 3, 5	2: 1, 3, 5	2: 1, 3, 5
3: 1, 2, 6	3: 1, 2, 6	3: 1, 2, 6	3: 1, 2, 6
4: 1, 7, 8	4: 1, 7, 8	4: 1, 7, 8	4: 1, 7, 8
5: 2, 7, 9	5: 2, 7, 9	5: 2, 7, 9	5: 2, 7, 9
6: 3, a, b	6: 3, a, b	6: 3, a, b	6: 3, a, b
7: 4, 5, c	7: 4, 5, c	7: 4, 5, c	7: 4, 5, a
8: 4, a, d	8: 4, a, d	8: 4, a, d	8: 4, c, d
9: 5, b, d	9: 5, a, e	9: 5, a, e	9: 5, c, e
a: 6, 8, e	a: 6, 8, 9	a: 6, 8, 9	a: 6, 7, f
b: 6, 9, f	b: 6, f, g	b: 6, f, g	b: 6, g, h
c: 7, g, h	c: 7, h, u	c: 7, h, u	c: 8, 9, u
d: 8, 9, g	d: 8, f, h	d: 8, f, h	d: 8, f, g
e: a, h, u	e: 9, g, h	e: 9, f, u	e: 9, f, h
f: b, h, u	f: b, d, u	f: b, d, e	f: a, d, e
g: c, d, u	g: b, e, u	g: b, h, u	g: b, d, u
h: c, e, f	h: c, d, e	h: c, d, g	h: b, e, u
u: e, f, g	u: c, f, g	u: c, e, g	u: c, g, h

$$\sum_i = [-2.7743, -2.3824, -1.9190, -1.8506, -1.3648, -1.3227, -1.3097, -0.7806, -0.2846, 0.3515, 0.8103, 0.8308, 1.1805, 1.6825, 1.7480, 1.9306, 2.4545, 3], \quad i = 1, 2, 3, 4.$$