Graphs cospectral with starlike trees

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Abstract

A tree which has exactly one vertex of degree greater than two is said to be starlike. In spite of seemingly simple structure of these trees, not much is known about their spectral properties. In this paper, we introduce a generalization of the notion of cospectrality called m-cospectrality which turns out to be useful in constructing cospectral graphs. Based on this, we construct cospectral mates for some starlike trees. We also present a set of necessary and sufficient conditions for divisibility of the characteristic polynomial of a starlike tree by the characteristic polynomial of a path.

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1 Introduction

In this paper, we are concerned only with undirected simple graphs (loops and multiple edges are not allowed). Let G be a graph of order n with the adjacency matrix A. We denote $\det(\lambda I - A)$, the characteristic polynomial of G, by $\chi(G) = \chi(G, \lambda)$. The multiset of eigenvalues of A is called the *adjacency spectrum*, or simply the *spectrum* of G. Since A is a symmetric matrix, the eigenvalues of A (or G) are real. Two nonisomorphic graphs with the same spectrum are called *cospectral*. We say that a graph *is determined by the spectrum* (*DS* for short) if there is no other nonisomorphic graph with the same spectrum.

A tree which has exactly one vertex of degree greater than two is said to be *starlike*. The vertex of maximum degree is called the *central* vertex. We denote by $\mathrm{S}(n_1, n_2, \ldots, n_k)$ a starlike tree in which removing the central vertex leaves disjoint paths $P_{n_1}, P_{n_2}, \ldots, P_{n_k}$. We say that $\mathrm{S}(n_1, n_2, \ldots, n_k)$ has branches of length $n_1, n_2, \ldots, n_k$. Note that it has $n_1 + n_2 + \cdots + n_k + 1$ vertices. In spite of seemingly simple structure of these trees, not much is known about their spectral properties. A summary of the main known results is as follows. In [9, 12], bounds on the maximum eigenvalue are given and also integral and hyperbolic starlike trees are characterized. In [8], it is shown that no two nonisomorphic starlike trees are cospectral. It has also been proved that starlike trees are determined by their Laplacian eigenvalues [10]. All cospectral mates of starlike trees with three branches have been found in [14]. For more results, we refer the reader to [5, 7, 15].

In recent years, there has been a growing interest to find new families of DS graphs. For a survey of the subject, the reader can consult [2, 3]. The problem of determination of DS starlike trees has been investigated by some researchers. For example, in [14], DS starlike trees with three branches have been recognized. The problem seems to be hard even in the case of trees with four branches. For this reason we have considered a related problem which hopefully will be useful in tackling the main problem. In this paper, we present a general method for constructing cospectral graphs and make use of it to find many infinite families of graphs cospectral with starlike trees. The method is developed from an example given in [7]: If we take the cospectral graphs $K_{1,4}$ and $C_4 + K_1$ and attach the path P_n to each vertex in these graphs, then the resulting graphs, one of them $\mathrm{S}(n, n, n, n, n-1)$, are still cospectral.

All known graphs cospectral with starlike trees have a component which is a path. This has motivated us to establish a set of necessary and sufficient conditions for divisibility of the characteristic polynomial of a starlike tree by the characteristic polynomial of a path. We show how the characterization could be useful in determining cospectral mates of starlike trees.

2 m-Cospectrality

In this section we introduce the notion of m-cospectrality. It is used to find cospectral mates for many infinite families of starlike trees.

2.1 The generalized characteristic polynomial

Let $Q = \{Q_1, Q_2, \ldots, Q_m\}$ be an ordered partition of vertices of a graph G with the vertex set $\{1, 2, \ldots, n\}$ and the adjacency matrix A. The *generalized characteristic polynomial of G with respect to Q*, denoted by $\chi_Q(G) = \chi_Q(G; \lambda_1, \lambda_2, \ldots, \lambda_m)$, is defined as $\det(\sum_{i=1}^{m} \lambda_i I_{(i)} - A)$, where $I_{(k)}$ is a $(0,1)$-matrix of order n in which $I_{(k)}(i,j) = 1$ if and only if $i = j \in Q_k$.

Let σ be a permutation on V. Then σG is a graph on V such that $\{i, j\}$ is an edge of G if and only if $\{\sigma(i), \sigma(j)\}$ is an edge of σG. Two graphs G and G' on the same vertex set V are called *m-cospectral* if there exist an ordered partition Q of size m of V and a permutation σ on V such that $\chi_Q(G) = \chi_Q(\sigma G')$.

It is obvious that an $(m+1)$-cospectral pair is at the same time an m-cospectral pair. Also clearly, 1-cospectrality is the same as cospectrality. On the other hand, at the other extreme case, we have the following.

Proposition 1 *Let G and G' be two graphs of order n. Then G and G' are isomorphic if and only if they are n-cospectral.*

Proof. Assume that G and G' are n-cospectral. Since the only possible partition with n parts is $Q = \{\{1\}, \{2\}, ..., \{n\}\}$, there is a labeling of the vertices of G and G' such that

$$\det(\operatorname{diag}(\lambda_1, ..., \lambda_n) - A) = \det(\operatorname{diag}(\lambda_1, ..., \lambda_n) - A'),$$

where A and A' are the adjacency matrices of G and G', respectively. Let i and j be two distinct vertices of G (and also of G'). The coefficient of $\prod_{k\neq i,j}\lambda_k$ in $\chi_Q(G)$ ($\chi_Q(G')$) is equal to the determinant of two by two submatrix of A (A') corresponding to rows and columns i,j. Therefore, ij is an edge of G if and only if it is an edge of G'. This implies that the graphs are isomorphic. The converse is obvious. $\square$

It is an easy task to construct nonisomorphic pairs of m-cospectral graphs of order n for m close to n. Let H be an arbitrary graph on $n-5$ vertices. Then it is not hard to show that $H+K_{1,4}$ and $H+C_4+K_1$ are $(n-3)$-cospectral (see the example below).

Example 1 Figure 1 depicts the smallest pair of cospectral graphs. We show that they are 2-cospectral but not 3-cospectral. Let A and A' be the adjacency matrices of G and

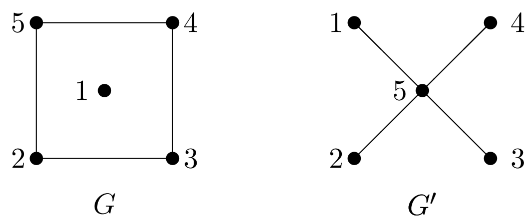

Figure 1: 2-cospectral graphs

G', respectively. Using the labeling in Figure 1, the generalized characteristic polynomials of G and G' with respect to the partition $Q=\{\{1\},\{2\},\dots,\{5\}\}$ are as follows:

$$\chi_Q(G)=\det(\mathrm{diag}(x_1,\dots,x_5)-A)=x_1(x_2x_3x_4x_5-x_2x_3-x_2x_5-x_4x_5-x_3x_4),$$
$$\chi_Q(G')=\det(\mathrm{diag}(y_1,\dots,y_5)-A')=y_1y_2y_3y_4y_5-y_1y_3y_4-y_1y_2y_4-y_1y_2y_3-y_2y_3y_4.$$

Now suppose that Q is a partition of $\{1,2,\dots,5\}$ for which $\chi_Q(G)=\chi_Q(G')$. We claim that Q cannot have more than two parts. Since the multisets $\{x_1,\dots,x_5\}$ and $\{y_1,\dots,y_5\}$ are the same and x_1 is a factor of $\chi_Q(G)$, with no loss of generality we may assume that

$x_1 = y_1 = y_2$. Dividing both sides by x_1, we find out that that equality holds if and only if $x_1 = x_2 = y_1 = y_2$ and $x_3 = x_4 = x_5 = y_3 = y_4 = y_5$. This yields $Q = \{\{1, 2\}, \{3, 4, 5\}\}$ or $Q = \{\{1, 2, \ldots, 5\}\}$ and therefore G and G' are 2-cospectral but not 3-cospectral.

Example 2 The graphs in Figure 2 are 4-cospectral but not 5-cospectral. Let A and

Figure 2: 4-cospectral graphs

A' be the adjacency matrices of G and G', respectively. Using the labeling in Figure 2, the generalized characteristic polynomials of G and G' with respect to the partition $Q = \{\{1\}, \{2\}, \ldots, \{7\}\}$ are as follows:

$$\begin{aligned}
\chi_Q(G) =& \det(\mathrm{diag}(q_1, \ldots, q_7) - A) \\
=& q_1q_2q_3q_4q_5q_6q_7 - q_1q_2q_3q_4q_5 - q_1q_2q_3q_6q_7 - q_1q_4q_5q_6q_7 - q_2q_3q_4q_5q_7 \\
& - q_2q_3q_5q_6q_7 - q_3q_4q_5q_6q_7 + q_1q_2q_3 + q_1q_4q_5 + q_1q_6q_7 + q_2q_3q_5 + q_2q_3q_7 \\
& + q_3q_4q_5 + q_3q_6q_7 + q_4q_5q_7 + q_5q_6q_7 - q_1 - q_3 - q_5 - q_7, \\
\chi_Q(G') =& \det(\mathrm{diag}(y_1, \ldots, y_7) - A') \\
=& y_1(y_2y_3y_4y_5y_6y_7 - y_2y_3y_4y_5 - y_2y_3y_4y_7 - y_2y_3y_6y_7 - y_2y_5y_6y_7 \\
& - y_3y_4y_5y_6 - y_4y_5y_6y_7 + y_2y_3 + y_2y_5 + y_2y_7 + y_3y_4 + y_3y_6 \\
& + y_4y_5 + y_4y_7 + y_5y_6 + y_6y_7 - 4).
\end{aligned}$$

Now suppose that Q is a partition of $\{1, 2, \ldots, 7\}$ for which $\chi_Q(G) = \chi_Q(G')$. We claim that Q cannot have more than four parts. Comparing the sentences of degree one, we find that $y_1 = q_1 = q_3 = q_5 = q_7$. After dividing both sides by y_1, since the sum of sentences of degree two should be equal, we obtain $3y_1(q_2 + q_4 + q_6) = (y_3 + y_5 + y_7)(y_2 + y_4 + y_6)$. It is not hard to see that with no loss of generality, equality holds if and only if $y_1 = y_3 = y_5 = y_7 = q_1 = q_3 = q_5 = q_7$, $y_2 = q_2, y_4 = q_4$ and $y_6 = q_6$. Therefore, G is 4-cospectral with G' with respect to $Q = \{\{2\}, \{4\}, \{6\}, \{1, 3, 5, 7\}\}$. The argument also proves that G is not 5-cospectral with G'.

Two rooted graphs G and G' are called *cospectrally rooted* if they are cospectral and also remain cospectral by removing their roots [11]. It is easily seen that if G and G' are cospectrally rooted, then they are 2-cospectral.

2.2 Constructing cospectral graphs

The notion of m-cospectrality can be used to construct new cospectral graphs from given m-cospectral pairs. Let $\mathcal{H}$ be a sequence of rooted graphs $H_1, H_2, \ldots, H_m$ with the corresponding roots $r_1, r_2, \ldots, r_m$, respectively. Let $Q = \{Q_1, Q_2, \ldots, Q_m\}$ be an ordered partition of the vertex set of graph G. The *rooted product of G by $\mathcal{H}$ with respect to Q*, denoted by $G_Q[\mathcal{H}]$, is obtained from G by identifying each vertex $v \in Q_i$ by the root of H_i. (This definition is a generalization of the rooted product given by Godsil and Mckay in [4].) The characteristic polynomial of $G_Q[\mathcal{H}]$ is given by Godsil and McKay. The following is their result in a slightly different form.

Theorem 1 [4]

$$\chi(G_Q[\mathcal{H}], \lambda) = \prod_{i=1}^{m} \chi(H_i - r_i, \lambda)^{|Q_i|} \chi_Q \left(G; \frac{\chi(H_1, \lambda)}{\chi(H_1 - r_1, \lambda)}, \frac{\chi(H_2, \lambda)}{\chi(H_2 - r_2, \lambda)}, \ldots, \frac{\chi(H_m, \lambda)}{\chi(H_m - r_m, \lambda)} \right).$$

We make use of the rooted product to construct new cospectral pairs from given m-cospectral graphs. The method is based on the following theorem which is a direct consequence of Theorem 1.

Theorem 2 *Let G and G' be m-cospectral graphs with respect to an ordered partition Q of vertices and let $\mathcal{H}$ be a sequence of m rooted graphs. Then $G_Q[\mathcal{H}]$ and $G'_Q[\mathcal{H}]$ are m-cospectral and so cospectral.*

Example 3 The graphs in Figure 3 are 2-cospectral and they are constructed from graphs in Example 1 by letting $\mathcal{H} = P_n, P_m$ in Theorem 2 (the branches with solid vertices are P_m and the rest are P_n).

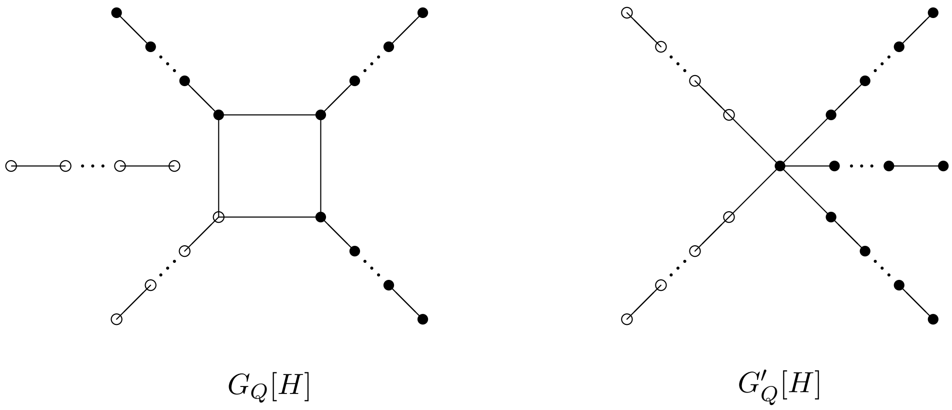

Figure 3: Cospectral pairs obtained from graphs in Example 1

2.3 Graphs cospectral with starlike trees

We use Theorem 2 to construct cospectral mates for some starlike trees. First, we give an infinite family of m-cospectral pairs for any positive integer m.

Theorem 3 *For any integers $m \geq n \geq 1$, the graphs $K_{1,mn}$ and $K_{m,n} + (m-1)(n-1)K_1$ are m-cospectral with respect to the partition of vertices given in Figure* 4.

Proof. Let $Q = \{Q_1, Q_2, \ldots, Q_m\}$ be the ordered partition of vertices in such a way that Q_i $(1 \leq i \leq m)$ consists of the vertices labeled i (see Figure 4). By expansion of the determinants one can prove the following inductively:

$$\chi_Q(G) = \chi_Q(G') = y_1 \prod_{j=1}^{m} y_j^n - n \prod_{j=1}^{m} y_j^{n-1} (\sum_{i=1}^{m} \prod_{j=1, j\neq i}^{m} y_j).$$

□

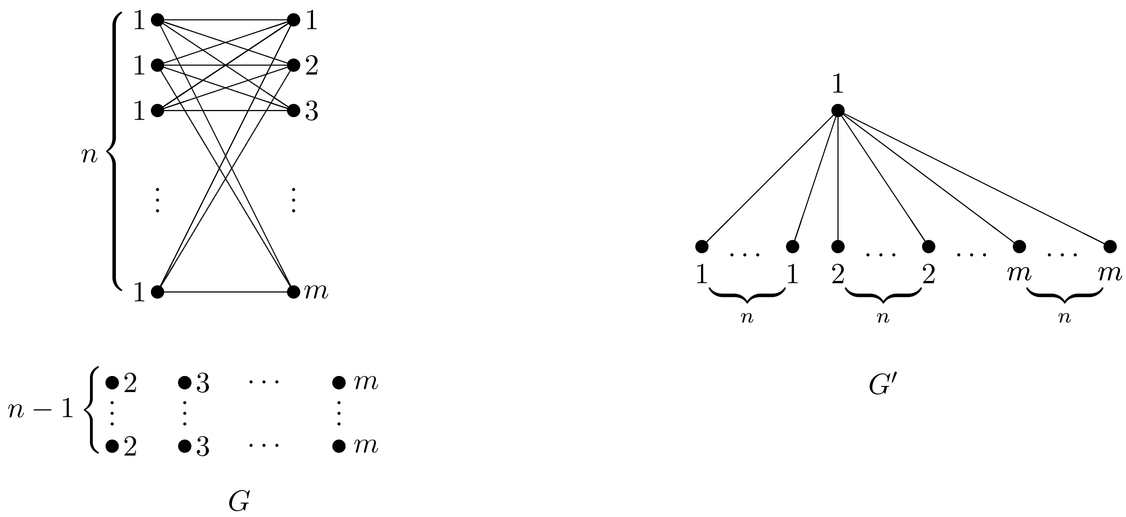

Figure 4: m-cospectral graphs

Let $\mathcal{H}$ be the sequence $P_{r_1}, P_{r_2}, \dots, P_{r_m}$, where r_i are positive integers. Using Theorems 2 and 3, we find a cospectral mate for any starlike tree of the form

$$\mathrm{S}(r_1 - 1, \underbrace{r_1, \dots, r_1}_{n}, \underbrace{r_2, \dots, r_2}_{n}, \dots, \underbrace{r_m, \dots, r_m}_{n}).$$

This suggests that there is probably no simple characterization of DS starlike trees.

3 Path dividing starlike tree

All known examples of cospectral mates of starlike trees have path as a component and so it is natural to consider the following question: When the characteristic polynomial of a path divides the characteristic polynomial of a starlike tree? We try to find necessary and sufficient conditions. The characteristic polynomial of P_n will be denoted by $p_n = p_n(\lambda)$.

3.1 Some useful lemmas

First we recall two well known results from the literature.

Theorem 4 [1, page 78] *Let N denote the set of vertices adjacent to vertex x in a tree T. Then*

$$\chi(T) = \lambda\chi(T - x) - \sum_{y\in N} \chi(T - x - y).$$

Theorem 5 [1, page 59] *Let G be the graph obtained by joining vertex x of a graph G_1 to vertex y of a graph G_2 by an edge. Then*

$$\chi(G) = \chi(G_1)\chi(G_2) - \chi(G_1 - x)\chi(G_2 - y).$$

Lemma 1 $p_{k(m+1)+r} \equiv (-1)^k p_{m-1}^k p_r \pmod{p_m}$ *for any $m \geq 1$, $k \geq 0$ and $r \geq 0$.*

Proof. We give a proof by induction on k. For $k = 0$ there is nothing to prove. For $k = 1$, by Theorem 5, we have

$$\begin{aligned} p_{m+1+r} &= p_m p_{r+1} - p_{m-1} p_r \\ &\equiv -p_{m-1} p_r \pmod{p_m}. \end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned} p_{(k+1)(m+1)+r} &= p_{k(m+1)+m+1+r} \\ &\equiv (-1)^k p_{m-1}^k p_{m+r+1} \pmod{p_m} \\ &\equiv (-1)^{k+1} p_{m-1}^{k+1} p_r \pmod{p_m}. \end{aligned}$$

□

Lemma 2 *Let $m \geq 1$ and $s \geq 3$ and let $m_i = k_i(m + 1) + r_i$, where $1 \leq r_i \leq m + 1$ for $1 \leq i \leq s$. Then*

$$p_m|\chi(\mathrm{S}(m_1, \ldots, m_s)) \Leftrightarrow p_m|\chi(\mathrm{S}(r_1, r_1, \ldots, r_s)).$$

Proof. By Theorem 4, we have

$$\chi(\mathrm{S}(m_1,m_2,\ldots,m_s)) = \lambda p_{m_1}p_{m_2}\cdots p_{m_s} - p_{m_1-1}p_{m_2}\cdots p_{m_s} - p_{m_1}p_{m_2-1}\cdots p_{m_s} - \cdots - p_{m_1}p_{m_2}\cdots p_{m_s-1}.$$

Hence by Lemma 1 and by letting $K=\sum_{i=1}^{s} k_i$, we have

$$\begin{aligned}\chi(\mathrm{S}(m_1,m_2,\ldots,m_s)) &= (-1)^K p_{m-1}^K(\lambda p_{r_1}p_{r_2}\cdots p_{r_s} - p_{r_1-1}p_{r_2}\cdots p_{r_s} \\ &\quad - p_{r_1}p_{r_2-1}\cdots p_{r_s} - \cdots - p_{r_1}p_{r_2}\cdots p_{r_s-1}) \pmod{p_m} \\ &= (-1)^K p_{m-1}^K \chi(\mathrm{S}(r_1,r_2,\ldots,r_s)) \pmod{p_m}.\end{aligned}$$

Since $\gcd(p_m,p_{m-1})=1$, the assertion follows. □

Lemma 3 *Let $m\geq 1$, $s\geq 3$ and $k\geq 1$. Then*

$$p_m|\chi(\mathrm{S}(m_1,\ldots,m_s)) \Leftrightarrow p_m|\chi(\mathrm{S}(m_1,\ldots,m_s,k(m+1))).$$

Proof. By Lemma 2, we may assume that $k=1$. By Theorem 5, we have

$$\chi(\mathrm{S}(m_1,m_2,\ldots,m_s,m+1)) = p_{m+1}\chi(\mathrm{S}(m_1,m_2,\ldots,m_s)) - p_m p_{m_1}p_{m_2}\cdots p_{m_s}.$$

Since $\gcd(p_m,p_{m+1})=1$, the assertion follows. □

Lemma 4 *Let $m\geq 1$, $s\geq 3$ and suppose $p_m|\chi(\mathrm{S}(r_1,\ldots,r_s))$, where $1\leq r_i\leq m+1$ for $1\leq i\leq s$. Then $r_1=m$ implies $r_i=m$ for some $2\leq i\leq s$.*

Proof. Let $r_1=m$. The largest eigenvalue of p_m is $2\cos\frac{\pi}{m+1}$. On the other hand by Theorem 4, we have

$$\chi(\mathrm{S}(m,r_2,\ldots,r_s)) = \lambda p_m p_{r_2}\cdots p_{r_s} - p_{m-1}p_{r_2}\cdots p_{r_s} - p_m p_{r_2-1}\cdots p_{r_s} - \cdots - p_m p_{r_2}\cdots p_{r_s-1}.$$

Therefore,

$$p_m \mid p_{m-1}p_{r_2}\cdots p_{r_s}.$$

Hence the result easily follows. □

The proof of the following lemma is straightforward by Theorem 4.

Lemma 5 *Let $m \geq 1$ and $s \geq 3$. Then*

$$p_m|\chi(\mathrm{S}(m, m, m_3, \ldots, m_s)).$$

Lemma 6 *Let $m \geq 1$, $s \geq 3$ and $1 \leq r_i < m$ for $1 \leq i \leq s$. Then p_m does not divide $\chi(\mathrm{S}(r_1, r_2, \ldots, r_s))$.*

Proof. Assume that $p_m \mid \chi(\mathrm{S}(r_1, r_2, \ldots, r_s))$. First suppose that $\lambda_1(\mathrm{S}(r_1, r_2, \ldots, r_s)) \geq 2$. By interlacing theorem, we have

$$\lambda_2(\mathrm{S}(r_1, r_2, \ldots, r_s)) \leq 2\cos\frac{\pi}{r_i + 1},$$

where $r_i = \max\{r_j | j = 1, \ldots, s\}$. Therefore, $2\cos\pi/(m+1) \leq 2\cos\pi/(r_i + 1)$ and hence $r_i \geq m$, which is a contradiction.

Now let $\lambda_1(\mathrm{S}(r_1, r_2, \ldots, r_s)) < 2$. Then our graph T is $\mathrm{S}(1, 2, 2)$, $\mathrm{S}(1, 2, 3)$, $\mathrm{S}(1, 2, 4)$ or $\mathrm{S}(1, 1, l)$ [13] (see also [1, page 78]). Therefore, $\chi(T, 2)$ is 3, 2, 1 or 4, respectively (note that $\chi(\mathrm{S}(a, b, c), 2) = 2 + a + b + c - abc$). If $p_m|\chi(T)$, then $m + 1$ divides 3, 2, 1 or 4, respectively. Hence the only possibility is that $T = \mathrm{S}(1, 1, 1)$ or $T = \mathrm{S}(1, 1, 2)$ and $m = 3$. But it is easy to see that in any case p_3 does not divide $\chi(T)$. □

The following lemma is also trivial.

Lemma 7 *Let $m, r \geq 1$. Then*

$$p_m \mid p_r \Leftrightarrow r \equiv -1 \pmod{m+1}.$$

3.2 The main result

We are now ready to state the main theorem.

Theorem 6 *Let $m \geq 1$, $s \geq 3$ and $m_i \geq 1$ for $1 \leq i \leq s$. Then $p_m \mid \chi(\mathrm{S}(m_1, m_2, \ldots, m_s))$ if and only if (without loss of generality) one of the following holds:*

(*i*) $m_1, m_2 \equiv -1 \pmod{m+1}$,

(*ii*) $m_3, m_4, \ldots, m_s \equiv 0 \pmod{m+1}$ *and* $m_1 + m_2 \equiv -2 \pmod{m+1}$.

Proof. Let $m_i = k_i(m+1) + r_i$, $1 \leq r_i \leq m+1$ for $1 \leq i \leq s$. By Lemma 2, it suffices to prove the theorem for $\mathrm{S}(r_1, r_2, \ldots, r_s)$.

If (i) holds, then the result follows from Lemma 5. Now assume that (ii) holds. Then by Lemma 3, it is sufficient to show that $p_m \mid \chi(\mathrm{S}(r_1, r_2, m+1))$. By Theorem 4 and Lemma 7, we have

$$\begin{aligned}\chi(\mathrm{S}(r_1, r_2, m+1)) &= \lambda p_{r_1} p_{r_2} p_{m+1} - p_{r_1-1} p_{r_2} p_{m+1} - p_{r_1} p_{r_2-1} p_{m+1} - p_{r_1} p_{r_2} p_m \\ &\equiv p_{m+1}(p_{r_1+r_2+1}) \pmod{p_m} \\ &\equiv 0 \pmod{p_m}.\end{aligned}$$

Now let $p_m \mid \chi(\mathrm{S}(r_1, r_2, \ldots, r_s))$ and assume that (i) does not hold. Then by Lemma 4, $r_i \neq m$ for any $1 \leq i \leq s$. So from Lemmas 3 and 6, it follows that $r_j = m+1$ for $3 \leq j \leq s$ (we assume that $r_1 \leq r_2 \leq \cdots \leq r_s$). Hence $p_m \mid \chi(\mathrm{S}(r_1, r_2, m+1))$. But as above, we have

$$\chi(\mathrm{S}(r_1, r_2, m+1)) \equiv p_{m+1}(p_{r_1+r_2+1}) \pmod{p_m}.$$

Therefore by Lemma 7, $r_1 + r_2 + 1 \equiv -1 \pmod{m+1}$ and we are done. □

Corollary 1 *Let* $m \geq 1$, $k > 1$ *and* T *be a starlike tree. Then* $p_m^k \mid \chi(T)$ *if and only if* T *has at least* $k+1$ *branches of lengths* $-1 \pmod{m+1}$.

Proof. If T has at least $k+1$ branches whose lengths are $-1 \pmod{m+1}$, then by Lemma 7, it is easy to see that $p_m^k \mid \chi(T)$. We now prove the converse by induction on k.

First let $k = 2$. Let λ be an eigenvalue of P_m. Since λ is a multiple eigenvalue of T, there is a corresponding eigenvector which is zero at the central vertex of T. Consequently, by Lemma 7, T has two branches A and B of lengths $-1 \pmod{m+1}$. Now T has an eigenvector which is zero on A. Removing A from T, we obtain a starlike tree T' such that $p_m \mid \chi(T')$. Therefore, by Theorem 6, T' has another branch (apart from B) of length -1 $\pmod{m+1}$.

Now let $k > 2$. Fix a branch A of T. Let λ be an eigenvalue of P_m. There are at least $k-1$ independent eigenvectors (corresponding to λ) which are zero on A. This yields that $p_m^{k-1} \mid \chi(T')$, where T' is obtained from T by removing A. Now the assertion follows from the induction hypothesis. □

From Theorem 6 and the above corollary, we also have the following result.

Corollary 2 *The multiplicity of zero as an eigenvalue of* $\mathrm{S}(m_1, m_2, \ldots, m_s)$ *is* $|\sum_{i=1}^{s} t_i - 1|$*, where* $t_i = 0, 1$ *is the parity of* m_i*.*

3.3 Application

We present an application of Theorem 6 to cospectral graphs. We show that how Theorem 6 can be used to find the cospectral mates of starlike trees.

Let G and H be two cospectral graphs. Then the degrees of vertices satisfy certain equations. Let x_i and y_i denote the numbers of vertices of degree i in G and H, respectively. By counting the number of vertices, edges and closed walks of length 4 in G and H, we have the following relations:

$$\sum x_i = \sum y_i,$$
$$\sum i x_i = \sum i y_i,$$
$$\sum \binom{i}{2} x_i + 2n_4 = \sum \binom{i}{2} y_i + 2n_4',$$

where n_4 and n_4' are the numbers of cycles of length 4 in G and H, respectively. When one of the graphs is starlike, then by adding up these equations with coefficients $2, -2$ and 2, respectively, we obtain the following.

Lemma 8 *Let G be cospectral with a starlike tree T with the maximum degree Δ. Then*

$$2n_4 + \sum \binom{i-1}{2} x_i = \binom{\Delta - 1}{2},$$

where x_i is the number of vertices of degree i and n_4 is the number of cycles of length 4 in G.

The following theorem characterizes all starlike threes with maximum degree three which are determined by their spectrum. We denote by $D(m, n)$ the graph obtained by joining a vertex of degree one of path P_n to a vertex of cycle C_m by a new edge.

Theorem 7 [14] *Let $T = \mathrm{S}(l_1, l_2, l_3)$ $(l_1 \le l_2 \le l_3)$. Then T has a cospectral mate if and only if $(l_1, l_2, l_3) = (l, l, 2l-2)$ for some $l \ge 2$ which in this case it is cospectral with $G = P_{l-1} + D(2l+2, l-2)$.*

Let G be cospectral with $T = \mathrm{S}(l_1, l_2, l_3)$. Let x_i denote the number of vertices of degree i and n_4 denote the number of cycles of length 4 in G. Then by Lemma 8,

$$\sum (i-1)(i-2)x_i + 4n_4 = 2.$$

This yields $x_i = 0$ for $i \ge 4$, $n_4 = 0$ and $x_0 + x_3 = 1$. Therefore, $G = P_{l-1} + D(2n, m)$ for some n and m.

Note that if $l = 2$ or $m = 0$, then $T = \mathrm{S}(2, 2, 2)$ and $n = 3$. So we may assume that $l \ge 3$ and $m \ge 1$. Also note that $p_{n-1}p_{l-1} \mid \chi(G)$. Hence

$$p_{n-1}p_{l-1} \mid \chi(T).$$

This suggests that a new proof of Theorem 7 may be given by the use of Theorem 6. By the Theorem, we know that the lengths of branches in T are of special forms. In fact, there are seven cases for the lengths of branches. All these cases are dealt with elementary arguments. Here for example we illustrate one of these cases.

In our case, without loss of generality, we assume that $l_1 = kl = k_1 n - 1$, $l_2 = k_1' n - 1$ and $l_3 + k_1' n = k'' l - 1$. First note that $m < l_i$ for $i = 1, 2, 3$ (this follows from a theorem of Hoffman and Smith on subdividing internal paths, see [6]). This implies $k_1' = 1$. Now since $k_1 n - 1 + k'' l - 1 = 2n + m + l - 1$, we have $k_1 < 3$. If $k_1 = 2$, then $m = (k'' - 1)l - 1$ and on the other hand $l_3 + n = m + l$ gives $n < l$ and since $m < n$ we have $m < l - 1$, a contradiction. Therefore $k_1 = 1$. Let $l_3 = k' l - 2$. We have $m = (k' - 1)l - 2$ and so $k' \le 1 + k$ (since $m < l_3$). Now evaluating the characteristic polynomials of T and G at 2 gives:

$$\begin{aligned}
&-n^2(k'l-1)+2n(k'l-1)+n^2=-2nl((k'-1)l-2)\\
\Rightarrow\ &-(kl+1)(k'l-1)+2(k'l-1)+kl+1=-2l((k'-1)l-2)\\
\Rightarrow\ &-kk'l+2k+k'=-2k'l+2l+4\\
\Rightarrow\ &-kk'l+3k>-2k'l+2l\\
\Rightarrow\ &(-kk'+k)l>l(-2k'+2)\\
\Rightarrow\ &kk'<k+2k'-2\\
\Rightarrow\ &k'<3.
\end{aligned}$$

Hence $k'=2$ and consequently $k=1$. Therefore, T must be of the type $\mathrm{S}(l,l,2l-2)$.

Finally, we note that a similar approach can be applied to find cospectral mates of starlike trees with four branches. However this procedure is too long and laborious in this case since there are a lot of different possibilities to be considered. This enforces us to wait until new ideas are developed for this problem.

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