A note on graphs whose signless Laplacian has three distinct eigenvalues

F. Ayoobi[a] G.R. Omidi[a,b,1] B. Tayfeh-Rezaie[b]

[a] *Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, 84156-83111, Iran*

[b] *School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran*

f.ayoobi@math.iut.ac.ir romidi@cc.iut.ac.ir tayfeh-r@ipm.ir

Abstract

We investigate graphs whose signless Laplacian matrix has three distinct eigenvalues. We show that the largest signless Laplacian eigenvalue of a connected graph G with three distinct signless Laplacian eigenvalues is noninteger if and only if $G = K_n - e$ for $n \geq 4$, where $K_n - e$ is the n vertex complete graph with an edge removed. Moreover, examples of such graphs are given.

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1 Introduction

In this paper, we are only concerned with undirected simple finite graphs. Let G be a graph of order n and with the vertex set $\{v_1, \ldots, v_n\}$. The *adjacency matrix* of G is an $n \times n$ matrix $A(G)$ whose (i, j)-entry is 1 if v_i is adjacent to v_j and is 0, otherwise. Assume that $D(G)$ is the $n \times n$ diagonal matrix whose (i, i)-entry is the degree of v_i (the number of vertices adjacent to v_i). The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the *Laplacian matrix* and *signless Laplacian matrix* of G, respectively. Since $A(G)$, $L(G)$ and $Q(G)$ are real symmetric matrices, their eigenvalues are real numbers. The eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are said to be *A-eigenvalues*, *L-eigenvalues* and *Q-eigenvalues* of G, respectively.

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Graphs with few distinct eigenvalues form an interesting class of graphs and possess nice combinatorial properties. Clearly, if all A-eigenvalues (L-eigenvalues, Q-eigenvalues) of a graph coincide, then it is trivial (i.e. a graph with no edges). It is also straightforward to see that connected graphs with only two distinct A-eigenvalues (L-eigenvalues, Q-eigenvalues) are complete graphs. Regular graphs with three distinct A-eigenvalues (L-eigenvalues, Q-eigenvalues) are precisely strongly regular graphs and therefore graphs with three distinct eigenvalues can be considered as a generalization of strongly regular graphs. For results on graphs with few distinct A-eigenvalues, we refer the reader to [1, 2, 3, 6, 7, 8, 10, 12] and on graphs with few distinct L-eigenvalues to [9, 13]. In this paper, we investigate graphs with three distinct Q-eigenvalues and show that the largest Q-eigenvalue of a connected graph G is noninteger if and only if $G = K_n - e$ for $n \geq 4$. Moreover, using the join operation of graphs, we give some infinite families of nonregular graphs with three distinct Q-eigenvalues.

Let us recall some definitions and notation to be used throughout the paper. For a graph G, the *complement* of G, denoted by $\overline{G}$, is the graph on the vertex set of G such that two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. If G_1 and G_2 are vertex disjoint graphs, then their *union* $G_1 + G_2$ is the graph whose vertex set (edge set) is the union of vertex sets (edge sets) of G_1 and G_2. We denote the star of order n, the complete graph of order n and the complete bipartite graph with two parts of sizes m and n, by S_n, K_n and $K_{m,n}$, respectively. The $n \times n$ identity matrix and the $m \times n$ all one matrix will be denoted by I_n and $J_{m\times n}$, respectively. We drop the subscripts whenever there is no danger of confusion.

2 Some preliminary results

In this section, we present some useful facts on graphs with three distinct Q-eigenvalues. The first lemma results in that the diameter of such graphs is 2.

Lemma 1 [4] *Let G be a graph with r distinct Q-eigenvalues and diameter d. Then $d \leq r - 1$.*

By the above lemma, a connected bipartite graph with three distinct Q-eigenvalues must be complete bipartite. Since $K_{m,n}$ has Q-spectrum $\{[0]^1, [m]^{n-1}, [n]^{m-1}, [m+n]^1\}$ (see Theorem 3 of Section 4), we have the following characterization.

Lemma 2 *A connected bipartite graph G has three distinct Q-eigenvalues if and only if it is S_n or $K_{n,n}$ for some n.*

By the Perron-Frobenius theorem, the largest Q-eigenvalue of a connected graph is simple and the entries of any corresponding eigenvector are positive (a Perron-Frobenius eigenvector) [5].

Theorem 1 *Let G be a connected graph with three distinct Q-eigenvalues $q_1 > q_2 > q_3$ and vertex set $\{v_1, v_2, \ldots, v_n\}$. Let d_i denote the degree of vertex v_i. Then there exists a Perron-Frobenius eigenvector $\alpha^t = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that*

(i) $(Q(G) - q_2 I)(Q(G) - q_3 I) = \alpha\alpha^t$,

(ii) $d_i^2 + d_i - (q_2 + q_3)d_i + q_2 q_3 = \alpha_i^2$,

(iii) $d_i + d_j + \lambda_{ij} - (q_2 + q_3) = \alpha_i \alpha_j$, *where λ_{ij} is the number of common neighbors of two adjacent vertices v_i and v_j,*

(iv) $\mu_{ij} = \alpha_i \alpha_j$ *is the number of common neighbors of two nonadjacent vertices v_i and v_j.*

Proof (i) Since the minimal polynomial of $Q(G)$ is $(x - q_1)(x - q_2)(x - q_3)$, we have $(Q(G) - q_1 I)B = 0$, where $B = (Q(G) - q_2 I)(Q(G) - q_3 I)$. Let $\beta^t = (\beta_1, \beta_2, \ldots, \beta_n)$ be a Perron-Frobenius eigenvector of $Q(G)$. Since q_1 is a simple eigenvalue, each column of B is a multiple of β. Let $C_j = t_j\beta$ be the j-th column of B. Since B is a symmetric matrix, there exists a real number c such that $t_i/\beta_i = c$ for each $1 \leq i \leq n$. By Lemma 1, the diameter of G is 2. If v_i and v_j are two nonadjacent vertices, then (i, j)-th entry of B is positive and so $t_j > 0$. This concludes that $c > 0$ and $B = c\beta\beta^t$. Now let $\alpha = \sqrt{c}\beta$. Then we have $(Q(G) - q_2 I)(Q(G) - q_3 I) = \alpha\alpha^t$. The remaining parts easily follow. □

A partition $\sigma = \{V_1, V_2\}$ of the vertex set of a graph G is called an *equitable partition*, if for any $v \in V_i$, $1 \leq i \leq 2$, the number $m_{ij} = |N_G(v) \cap V_j|$, $1 \leq j \leq 2$, depends only on i, j, where $N_G(v)$ is the set of neighbors of v. A graph whose vertices have only two distinct possibilities k_1 and k_2 for degree is said to be (k_1, k_2)*-regular*.

Lemma 3 *Let V_i, $1 \leq i \leq 2$, be the set of vertices of degree k_i of a (k_1, k_2)-regular graph G with three distinct Q-eigenvalues. Then $\sigma = \{V_1, V_2\}$ is an equitable partition for G.*

Proof We use the notation of Theorem 1. Suppose that t_i, $1 \leq i \leq 2$, is an entry corresponding to the vertices of degree k_i of α. Let $1 \leq i \leq 2$. Let v be a vertex of degree k_i and let $m_{ij} = |N_G(v) \cap V_j|$ for $1 \leq j \leq 2$. Then $m_{i1} + m_{i2} = k_i$. On the other hand, from $Q(G)\alpha = q_1\alpha$, we have $m_{i1}t_1 + m_{i2}t_2 + k_i t_i = q_1 t_i$. It follows that m_{ij} is independent of v. □

Example 1 Using Theorem 3 of Section 4, it is a straightforward task to compute the Q-eigenvalues of the graphs $K_n - e$, S_n and $\overline{K_1 + 2K_3}$. They have Q-spectra $\{[n-2]^{n-2}, [(3n - 6 \pm \sqrt{n^2 + 4n - 12})/2]^1\}$, $\{[0]^1, [1]^{n-2}, [n]^1\}$ and $\{[1]^1, [4]^5, [9]^1\}$, respectively, where the exponents represent multiplicities. These graphs are (k_1, k_2)-regular graphs with $(k_1, k_2) = (n - 1, n -$

2), $(n-1,1)$ and $(6,4)$, respectively. Let V_i, $1 \le i \le 2$, be the set of vertices of degree k_i. It is clear that $\sigma = \{V_1, V_2\}$ is an equitable partition for each of the mentioned graphs with

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} n-3 & 2 \\ n-2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & n-1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 6 \\ 1 & 3 \end{bmatrix},$$

respectively.

3 The largest Q-eigenvalue

In this section, we determine when the largest Q-eigenvalue of a connected graph G with three distinct Q-eigenvalues is noninteger. The main result is that such a graph is necessarily $K_n - e$ for $n \ge 4$.

Theorem 2 *Let G be a connected graph of order $n \ge 4$. Then G has a Q-eigenvalue c of multiplicity $n-2$ if and only if G is one of the graphs $K_n - e, S_n, K_{n/2,n/2}, \overline{K_3 + S_4}, \overline{K_1 + 2K_3}$.*

Proof Let G have a Q-eigenvalue c of multiplicity $n-2$. It is obvious that c is integer and G has three distinct eigenvalues. If G is regular, then it is a strongly regular graph and so by the absolute bound either $n \le 2$ or G is $\overline{tK_m}$ for some t and m. Since $n \ge 4$ and Q-spectrum of $\overline{tK_m}$ is $\{[tm-2m]^{t-1}, [tm-m]^{tm-t}, [2tm-2m]^1\}$, it follows that $G = K_{n/2,n/2}$.

Now let G be nonregular. Let u and v be two adjacent vertices of G whose degrees are d_1 and d_2, respectively such that $d_1 \ne d_2$. If $(d_1 - c)(d_2 - c) = 1$, then either $d_1 = d_2 = c+1$ or $d_1 = d_2 = c-1$, a contradiction. Hence $(d_1 - c)(d_2 - c) \ne 1$. By the fact that every symmetric matrix of rank k has a full rank principal submatrix of order k, we may assume that

$$Q(G) - cI = \begin{bmatrix} d_1 - c & 1 & J_{1\times r} & 0 & J_{1\times t} \\ 1 & d_2 - c & 0 & J_{1\times s} & J_{1\times t} \\ J_{r\times 1} & 0 & x_1 J_{r\times r} & x_2 J_{r\times s} & x_3 J_{r\times t} \\ 0 & J_{s\times 1} & x_2 J_{s\times r} & y_1 J_{s\times s} & y_2 J_{s\times t} \\ J_{t\times 1} & J_{t\times 1} & x_3 J_{t\times r} & y_2 J_{t\times s} & (x_3 + y_2) J_{t\times t} \end{bmatrix},$$

where

(a) $x_1 = \frac{d_2 - c}{(d_1-c)(d_2-c)-1}$, $x_2 = \frac{-1}{(d_1-c)(d_2-c)-1}$, $y_1 = \frac{d_1-c}{(d_1-c)(d_2-c)-1}$, $x_3 = x_1 + x_2$, $y_2 = x_2 + y_1$;

(b) $d_1 = 1 + r + t$, $d_2 = 1 + s + t$;

(c) $1 + (r-1)x_1 + sx_2 + tx_3 = x_1 + c$, if $r > 0$;

(d) $1 + rx_2 + (s-1)y_1 + ty_2 = y_1 + c$, if $s > 0$;

(e) $2 + rx_3 + sy_2 + (t-1)(x_3 + y_2) = x_3 + y_2 + c$, if $t > 0$.

Notice that if there exists a vertex w which is adjacent to none of the vertices u and v, then the row of $Q(G) - cI$ corresponding to w (which is a linear combination of the first two rows) should be zero vector, a contradiction to the connectedness of G. The equations in (a) are obtained by the fact that any row of $Q(G)-cI$ is a linear combination of the first two rows. For example, x_1, x_2, x_3 are easily computed by considering the third row as a linear combination of the first two rows. Since $d_1 \neq d_2$, with no loss of generality, we may let $r > 0$. Also note that we use the equations for y_1 and y_2 in (a) only when $s > 0$ or $t > 0$. Consider the following cases.

(i) $s > 0$. By (a), $x_2 = 1$ and so $(d_1 - c)(d_2 - c) = 0$. With no loss of generality, assume that $d_1 = c$. Then by (a), $y_1 = 0$, $x_1 = c - d_2$, $y_2 = 1$ and $x_3 = c - d_2 + 1$. First let $t = 0$. Then $r \neq s$ and by (b),(c), we obtain $(r - 3)(r - s) = 0$ which yields that $r = 3$. This in turn implies $x_1 = 1$, $c = d_1 = r + 1 = 4$ and so $s = d_2 - 1 = c - x_1 - 1 = 2$. Therefore, $G = \overline{K_3 + S_4}$. Next let $t > 0$. Then $x_3 = 0$ which gives $d_2 = c + 1$ and so $x_1 = -1$. This concludes that $r = 1$. From (b), we have $c = 2 + t$ and $c + 1 = 1 + s + t$ which imply $s = 2$. From (c), we have $c = 4$ and so $t = 2$. Again we find $G = \overline{K_3 + S_4}$.

(ii) $s = 0$ and $t > 0$. If $x_3 = 0$, then $d_2 = c + 1$ which implies $y_2 = 1$ by (a) and $c = t$ by (b). From $x_1 = 1/(d_1 - c - 1)$, it follows that $d_1 = c + 2$ and hence $r = 1$. By (c), we obtain $c = t = 0$, a contradiction. Therefore, $x_3 = 1$. We have $(d_2 - c) = (d_1 - c)(d_2 - c)$ and $x_1 = (d_2 - c)/(d_2 - c - 1)$. Hence, $d_2 - c = 0$ or $d_2 - c = 2$. However, the latter is impossible since otherwise $d_2 = c+2 < d_1 = c+1$, a contradiction. Therefore, $d_2 = c$. From this, we obtain $c = t + 1$ and $y_2 = 1 - r$. Then by (e), $c(1 - r) = 4(1 - r)$. If $r = 1$, then $G = K_n - e$. So let $c = 4$ and $r > 1$. Since $t = 3$, we necessarily have $r = 2$. It results in that $G = \overline{K_1 + 2K_3}$.

(iii) $s = 0$ and $t = 0$. Then G is S_n with Q-spectrum $\{[0]^1, [1]^{n-2}, [n]^1\}$ or G is $\overline{K_1 + S_{n-1}}$. In the latter case, by choosing another candidates for u and v, we have case (ii).

The converse is straightforward using the results in the above paragraphs and Example 1. □

Corollary 1 *The largest Q-eigenvalue of a connected graph G with three distinct Q-eigenvalues is noninteger if and only if $G = K_n - e$ for $n \geq 4$.*

Proof Let G be a connected graph of order n with three distinct Q-eigenvalues and let the largest Q-eigenvalue of G be noninteger. Obviously, $n \geq 4$ and G has a Q-eigenvalue c of multiplicity $n-2$. By Theorem 2, G is one of the graphs $K_n - e, S_n, K_{n/2,n/2}, \overline{K_3 + S_4}, \overline{K_1 + 2K_3}$. The Q-spectra of $K_n - e$, S_n, and $\overline{K_1 + 2K_3}$ are given in Example 1 and by Theorem 3 of Section 4, Q-spectra of $\overline{K_3 + S_4}$ and $K_{n/2,n/2}$ are $\{[1]^1, [4]^5, [9]^1\}$ and $\{[0]^1, [n/2]^{n-2}, [n]^1\}$, respectively. It follows that G must be $K_n - e$. The converse is trivial. □

4 Examples of three distinct Q-eigenvalue graphs

It seems impossible to give a complete characterization of nonregular graphs with three distinct Q-eigenvalues. Examples like $\overline{K_3 + S_4}, \overline{K_1 + 2K_3}$, $K_n - e$ and S_n were given in the previous sections. Here, we introduce more examples of such graphs. The *join* of graphs G_1 and G_2 is the graph $G_1 \vee G_2$ obtained from $G_1 + G_2$ by joining each vertex of G_1 with every vertex of G_2. Let $P(M, x)$ denote the characteristic polynomial of the matrix M.

Theorem 3 [11] *For $i = 1, 2$, let G_i be a r_i-regular graph on n_i vertices and let G_1 and G_2 be vertex disjoint. Then*

$$P(Q(G_1 \vee G_2), x) = \frac{P(Q(G_1), x - n_2)P(Q(G_2), x - n_1)}{(x - 2r_1 - n_2)(x - 2r_2 - n_1)} f(x),$$

where $f(x) = x^2 - (2(r_1 + r_2) + (n_1 + n_2))x + 2(2r_1r_2 + r_1n_1 + r_2n_2)$.

In the sequel, using Theorem 3, we give examples of nonregular graphs with three distinct Q-eigenvalues. Note that by Theorem 1, G_i must be strongly regular graph (see below for definition) or K_1 for $i = 1, 2$.

Example 2 Let $G_1 = 2K_n$ and $G_2 = K_n$. Then by Theorem 3, $G_1 \vee G_2$ has exactly 3 distinct Q-eigenvalues: $[5n - 2]^1, [3n - 2]^n, [2n - 2]^{2n-1}$.

Example 3 Let $G_1 = 2K_{n_1}$ $(n_1 > 1)$ and $G_2 = n_2K_1$. Then by Theorem 3, $G_1 \vee G_2$ has three distinct Q-eigenvalues if and only if $(n_1, n_2) = (4, 2)$ and in this case the Q-eigenvalues are $[12]^1, [8]^2, [4]^7$.

A *cone* over a graph G is defined as the graph $K_1 \vee G$. A *strongly regular graph* with parameters (n, k, λ, μ) is a k-regular graph of order n such that any two adjacent vertices have λ common neighbors and any two nonadjacent vertices have μ common neighbors. By Theorem 3, we have the following lemma.

Lemma 4 *Let G be a cone over a connected strongly regular graph with parameters (n, k, λ, μ). Then G has exactly 3 distinct Q-eigenvalues if and only if for $t = (\lambda - \mu)^2 + 4(k - \mu)$ we have*

$$n - \lambda + \mu - 1 \in \{\pm\sqrt{t} + \sqrt{(2k + n + 1)^2 - 8nk}\}.$$

Example 4 Using Lemma 4, the cones over the strongly regular graphs with parameters $(6, 3, 0, 3)$ (the Utility graph), $(9, 4, 1, 2)$, $(10, 6, 3, 4)$ (the 5-triangle graph) and $(10, 3, 0, 1)$ (the Petersen graph) have exactly 3 distinct Q-eigenvalues. Letting $0 \leq \lambda < k < n < 10000$ and $\mu = (k(k - \lambda - 1))/(n - k - 1)$, an easy computer search on parameters shows that there is no other set of parameters (n, k, λ, μ) satisfying the condition of the above lemma for $n < 10000$.

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