A short proof of a theorem of Bang and Koolen

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Abstract

Let a graph Γ be locally disjoint union of three copies of complete graphs K_{q-1} and let Γ be cospectral with the Hamming graph $H(3,q)$. Bang and Koolen [Asian-Eur. J. Math. **1** (2008), 147–156] proved that if $q > 3$, then Γ is isomorphic to $H(3,q)$. We present a short proof of this result.

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1 Introduction

The Hamming graphs $H(d,q)$ $(d,q \geqslant 2)$, the Cartesian product of d copies of the complete graph K_q on q vertices, constitute an important family of distance-regular graphs. In [3], a question was posed whether the Hamming graphs are uniquely determined by the adjacency spectrum. It is known that $H(3,2)$ and $H(2,q)$ $(q \neq 4)$ are uniquely determined by the adjacency spectrum and $H(2,4)$, $H(3,4)$, $H(d,2)$ $(d \geqslant 4)$ and $H(d,q)$ $(d \geqslant q \geqslant 3)$ have cospectral mates (see [3] and the references therein).

Let G and H be two graphs. The graph G is called *locally* H if for any vertex x of G, the graph induced on the neighborhood of x is isomorphic to H. Obviously, $H(d,q)$ is locally disjoint union of d copies of K_{q-1}. In [2], it is shown that for $q > 3$, if a graph Γ is cospectral with $H(3,q)$ and locally disjoint union of three copies of K_{q-1}, then Γ is isomorphic to $H(3,q)$. In this note, we give a short proof of this theorem. We remark that in [1], using this result, $H(3,q)$ is shown to be uniquely determined by the adjacency spectrum for $q \geqslant 36$.

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2 The proof

We give a short proof of the following theorem from [2].

Theorem. *Let Γ be a graph cospectral with $H(3,q)$ for $q>3$ and let Γ be locally disjoint union of three copies of K_{q-1}. Then Γ is isomorphic to $H(3,q)$.*

Proof. For any vertex x of Γ, let $\Gamma_i(x)$ denote the set of vertices in Γ at distance i from x. By [4], a distance-regular graph cospectral with $H(3,q)$ is $H(3,q)$ if $q\neq 4$, and either $H(3,4)$ or the Doob graph of the diameter 3, otherwise. Since the Doob graph of the diameter 3 is not locally disjoint union of three copies of K_3, it suffices to show that Γ is distance-regular. In order to establish the distance-regularity of Γ, using [5, Lemma 1.2], we only need to prove that $|\Gamma_2(x)|=3(q-1)^2$ for any vertex x of Γ.

For two vertices x and y of Γ at distance 2, let $\mu(x,y)$ denote the number of common neighbors of x and y. Since Γ is locally disjoint union of three complete graphs, $1\leqslant\mu(x,y)\leqslant 3$. For any vertex x of Γ and any $y\in\Gamma_1(x)$, the vertex sets of two disjoint complete graphs induced on $\Gamma_1(x)\setminus(\Gamma_1(y)\cup\{y\})$ are denoted by $\omega_1(x;y)$ and $\omega_2(x;y)$. Since Γ is cospectral with $H(3,q)$ and $H(3,q)$ is connected and regular, so is Γ. Recall that the distinct eigenvalues of $H(3,q)$ are $3q-3, 2q-3, q-3, -3$ (see [3]). Let A be the adjacency matrix of Γ. Using the Hoffman polynomial [6], we have

$$A^3-3(q-3)A^2+(2q^2-18q+27)A+3(2q-3)(q-3)I=6J, \tag{1}$$

where I and J are the identity and all one matrices, respectively. For any vertex x of Γ and any $z\in\Gamma_2(x)$, by (1), we have

$$A^3_{(x,z)}=3(q-3)\mu(x,z)+6, \tag{2}$$

where $A^3_{(x,z)}$, the (x,z)-entry of A^3, is equal to the number of walks of the length 3 from x to z in Γ.

Fix a vertex x of Γ and let $\Omega=\{\omega_i(y;x)\,|\,y\in\Gamma_1(x) \text{ and } i=1,2\}$. For any $\omega\in\Omega$ and $1\leqslant i\leqslant 3$, let $a_i(\omega)$ be the number of vertices $z\in\omega$ such that $\mu(x,z)=i$. For $1\leqslant i\leqslant 3$, define $S_i=\{z\in\Gamma_2(x)\,|\,\mu(x,z)=i\}$ and $s_i=|S_i|$. By counting the number of edges between $\Gamma_1(x)$ and $\Gamma_2(x)$ in two ways, we find that

$$s_1+2s_2+3s_3=6(q-1)^2. \tag{3}$$

Let c be the number of 4-cycles passing through x and some vertex of $\Gamma_2(x)$. It is not hard to see that c is determined from the local structure of Γ and $A^4_{(x,x)}$ which the latter is computable by (1). On the other hand, for every vertex α of $H(3,q)$, the number of 4-cycles passing through α and some vertex in distance 2 from α, is equal to $3(q-1)^2$. Therefore, by the hypothesis of theorem on Γ and using the relation $c=\sum_{i=1}^{3}\binom{i}{2}s_i$, we obtain that

$$s_2+3s_3=3(q-1)^2. \tag{4}$$

Assume that W_i is the set of all walks of the length 3 from x to some vertex in S_i. By (2), $|W_i| = 3i(q-3)+6$. Using the local structure of Γ, the number of those elements of W_i containing two vertices of $\Gamma_1(x)$ is $i(q-2)s_i$. Now, by considering those elements of W_i containing two vertices of some $\omega \in \Omega$, we conclude that

$$2(q-2)s_2 + \sum_{\omega\in\Omega} a_2(\omega)\Big(a_1(\omega) + 2\big(a_2(\omega)-1\big) + 3a_3(\omega)\Big) \leqslant s_2\big(6(q-3)+6\big)$$

and

$$3(q-2)s_3 + \sum_{\omega\in\Omega} a_3(\omega)\Big(a_1(\omega) + 2a_2(\omega) + 3\big(a_3(\omega)-1\big)\Big) = s_3\big(9(q-3)+6\big).$$

By $a_1(\omega)+a_2(\omega)+a_3(\omega) = q-1$, $\sum_{\omega\in\Omega} a_i(\omega) = is_i$ and (4), we obtain that

$$\sum_{\omega\in\Omega}\big(a_2(\omega)+2a_3(\omega)\big)^2 = \sum_{\omega\in\Omega} a_2(\omega)\big(a_2(\omega)+2a_3(\omega)\big) + 2\sum_{\omega\in\Omega} a_3(\omega)\big(a_2(\omega)+2a_3(\omega)\big) \leqslant 6(q-1)^3.$$

Therefore, by (4) and the Cauchy-Schwarz inequality, we have

$$6(q-1)^3 \geqslant \sum_{\omega\in\Omega}\big(a_2(\omega)+2a_3(\omega)\big)^2 \geqslant \frac{\left(\sum_{\omega\in\Omega} a_2(\omega)+2a_3(\omega)\right)^2}{6(q-1)} = \frac{(2s_2+6s_3)^2}{6(q-1)} = 6(q-1)^3.$$

Since equality occurs in the above inequalities, it follows that $a_2(\omega)+2a_3(\omega) = q-1$ for every $\omega \in \Omega$. Thus, if $\omega \in \{\omega_1(y;x), \omega_2(y;x)\}$ for some $y \in \Gamma_1(x)$, then the number of edges between ω and $\omega_1(x;y)\cup\omega_2(x;y)$ is $a_1(\omega)+2a_2(\omega)+3a_3(\omega)-(q-1) = q-1$. This establishes the following property of Γ:

(∗) For every two adjacent vertices u and v of Γ, the number of edges between $\omega_i(u;v)$ and $\omega_1(v;u)\cup\omega_2(v;u)$ is equal to $q-1$, where $i = 1, 2$.

Now we show that $s_1 = 0$. By contrary, assume that $z \in S_1$ and $\omega = \omega_1(y;x)$ is the unique element of Ω containing z for some $y \in \Gamma_1(x)$. We know that the number of walks of the length 3 from x to z containing two vertices of $\Gamma_1(x)$ is $q-2$ and the number of such walks passing through two vertices of ω is $a_1(\omega)-1+2a_2(\omega)+3a_3(\omega) = 2q-3$. Furthermore, applying (∗) for y and z, we find $q-1$ walks of the length 3 from x to z containing some vertex of $\omega_1(z;y)\cup\omega_2(z;y)$. Hence, by (2), we obtain that $(q-2)+(2q-3)+(q-1) \leqslant A^3_{(x,z)} = 3(q-3)+6$. This yields that $q \leqslant 3$, a contradiction. Thus, $s_1 = 0$ which in turn implies that $s_2 = 3(q-1)^2$ and $s_3 = 0$ by (3) and (4). This shows that $|\Gamma_2(x)| = 3(q-1)^2$, as required. □

Remark. In [2], it is proven that a graph cospectral with $H(3,3)$ which is locally disjoint union of three copies of K_2, is either $H(3,3)$ or its dual. We can also show this assertion using the property (∗).

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