Some indecomposable t-designs

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Abstract

The existence of large sets of 5-(14,6,3) designs is in doubt. There are five simple 5-(14,6,6) designs known in the literature. In this note, by the use of a computer program, we show that all of these designs are indecomposable and therefore they do not lead to large sets of 5-(14,6,3) designs. Moreover, they provide the first counterexamples for a conjecture on disjoint t-designs which states that if there exists a t-(v,k,λ) design (X,D) with minimum possible value of λ, then there must be a t-(v,k,λ) design (X,D') such that $D\cap D'=\varnothing$.

1 Introduction

Let t,k,v, and λ be integers such that $0\le t\le k\le v$ and $\lambda\ge 1$. A t-(v,k,λ) *design* (or briefly, t-design) is a pair $\mathcal{D}=(X,D)$ where X is a v-set and D is a collection of k-subsets of X such that every t-subset of X is exactly contained in λ elements of D. A simple counting argument shows that all the numbers $\lambda_i=\lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$ must be integers for $0\le i\le t$. For given t,k, and v, the minimum value of λ satisfying these necessary conditions is denoted by λ_{min}. If the elements of D are distinct, then $\mathcal{D}$ is called *simple*. Here we are concerned only with simple designs. Let $P_k(X)$ denote the set of all k-subsets of X. Then, it is easy to see $\mathcal{D}^s=(X,P_k(X)\setminus D)$ is also a t-design which is called the *supplement design* of $\mathcal{D}$. A t-(v,k,λ) design is called *decomposable* if it contains a t-(v,k,λ_1) design $\mathcal{D}'$ with $\lambda_1<\lambda$. Otherwise, it is *indecomposable*. A *large set* of t-(v,k,λ) designs of size N, denoted by $\mathrm{LS}[N](t,k,v)$, is a set of disjoint t-(v,k,λ) designs (X,D_i), $1\le i\le N$, such that D_i partition $P_k(X)$ and $N=\binom{v-t}{k-t}/\lambda$.

The family of 5-$(14, 6, \lambda)$ designs seems to be an interesting case to be studied. Apart from the trivial case with $\lambda = 9$, the only possible values of λ are 3 and 6. Note that the set of designs with $\lambda = 6$ is exactly the set of supplements of designs with $\lambda = 3$. There are five 5-(14,6,3) designs known in the literature, one of which was found by A. E. Brouwer some 17 years ago [1] (which is denoted by $\mathcal{D}_0$ in this note) and the others were recently constructed by M. M-Noori and B. Tayfeh-Rezaie [4] (which are denoted by $\mathcal{D}_1 - \mathcal{D}_4$ following the notation of [4]). On the other hand, there are no known 6-(15,7,3) and 7-(16,8,3) designs. These designs are necessarily extensions of some 5-(14,6,3) designs and therefore the classification of 5-(14,6,3) designs is an interesting and important problem. We already know that the designs $\mathcal{D}_i$ $(0 \leq i \leq 4)$ are not extendable. Also the existence of large sets of 5-(14,6,3) designs is in doubt and it seems to be a challenging problem for large set diggers. It is clear that the existence of large sets of 5-(14,6,3) designs is equivalent to the existence of decomposable 5-(14,6,6) designs. In this note, we show that the 5-(14,6,6) designs $\mathcal{D}_i^s$ $(0 \leq i \leq 4)$ are indecomposable. Finally, note that these designs furnish the first counterexamples for a conjecture on disjoint t-designs which states that if there exists a t-(v, k, λ_{min}) design (X, D), then there must be a t-(v, k, λ_{min}) design (X, D') such that $D \cap D' = \varnothing$ [3]. Despite of these counterexamples, it seems that the conjecture is true for a wide range of parameter sets. By the permutation lemma [2], one can see that the conjecture is true for t-(v, k, λ_{min}) designs with $\lambda_{min}^2 \binom{v}{k} < \binom{v-t}{k-t}^2$. Some examples satisfying this condition are t-$(v, k, 1)$ designs with $k \geq 2t$, Steiner triple systems, and symmetric designs. We believe that the conjecture is true at least for 2-designs.

2 Indecomposable 5-(14,6,6) designs

Given integers t, k, and v such that $0 \leq t \leq k \leq v$, the *inclusion matrix* $W_{tk}(v)$ is a $(0, 1)$ matrix whose rows are indexed by the t-subsets T and whose columns are indexed by the k-subsets K of a v-set X and $W_{tk}(v)(T, K) = 1$ if and only if $T \subseteq K$. We simply write W_{tk} instead of $W_{tk}(v)$ if there is no danger of confusion. It is clear that the existence of a t-(v, k, λ) design is equivalent to the existence of a $(0, 1)$ column vector $\mathbf{x}$ such that

$$W_{tk}\mathbf{x} = \lambda\mathbf{j},$$

where $\mathbf{j}$ is a column vector of all ones. Let $\mathcal{D} = (X, D)$ be a 5-(14,6,3) design. Then $\mathcal{D}^s$ is a 5-(14,6,6) design. Denote by $A_{\mathcal{D}}$ the square matrix of order 2002 obtained from $W_{5,6}(14)$ by deleting all columns whose indices are in D. Obviously, $\mathcal{D}^s$ is decomposable if and only if there exists a $(-1, 1)$ column vector $\mathbf{a}$ such that $A_{\mathcal{D}}\mathbf{a} = 0$. So to prove that $\mathcal{D}^s$ is indecomposable, we show that there is no vector with entries ± 1 in the null space of $A_{\mathcal{D}}$. To do this, we first need to determine a basis for the null space of $A_{\mathcal{D}}$. Because of the large size of $A_{\mathcal{D}}$ it is much easier to work with finite fields versus the rational field. Note that each row of $A_{\mathcal{D}}$ has exactly 6 nonzero elements which are all ones and so the following lemma is trivial.

Lemma 2.1 *Let $p > 3$ be a prime and let $\mathbf{a}$ be a $(-1, 1)$ column vector. $A_{\mathcal{D}}\mathbf{a} = 0$ holds on the rational field if and only if it holds on the finite field $GF(p)$.*

The rank of $W_{tk}(v)$ over the rational field is usually greater than its rank over finite fields (for rank formulas, see [5]). The same fact is expected to be true for $A_{\mathcal{D}}$. We now show that there is a suitable matrix whose rank is greater than the rank of $A_{\mathcal{D}}$ over any field. This is important because it helps us to deal with smaller search space. Consider the matrix $M_{tk}(v)$ defined as

$$M_{tk}(v) = \begin{bmatrix} W_{0k}(v) \\ W_{1k}(v) \\ \vdots \\ W_{tk}(v) \end{bmatrix},$$

and whose columns are indexed as the columns of $W_{tk}(v)$. Over any field, its rank is $\binom{v}{t}$ [5]. Instead of $A_{\mathcal{D}}$, we make use of the matrix $B_{\mathcal{D}}$ obtained from $M_{5,6}(14)$ by deleting all columns whose indices are in D. Since every t-design is also an i-design for $0 \leq i < t$, it is clear that the set of solutions of $A_{\mathcal{D}}\mathbf{x} = 0$ and $B_{\mathcal{D}}\mathbf{x} = 0$ over the rational field are identical. This fact together with Lemma 2.1 show that the set of $(-1, 1)$ solutions of $A_{\mathcal{D}}\mathbf{x} = 0$ and $B_{\mathcal{D}}\mathbf{x} = 0$ over the finite field $GF(p)$ are identical for any prime $p > 3$. The ranks of $A_{\mathcal{D}_i}$ and $B_{\mathcal{D}_i}$ $(0 \leq i \leq 4)$ over some finite fields are given in Table 1 for a comparison.

Table 1 The ranks of $A_{\mathcal{D}_i}$ and $B_{\mathcal{D}_i}$

	$A_{\mathcal{D}_0}$	$B_{\mathcal{D}_0}$	$A_{\mathcal{D}_1}$	$B_{\mathcal{D}_1}$	$A_{\mathcal{D}_2}$	$B_{\mathcal{D}_2}$	$A_{\mathcal{D}_3}$	$B_{\mathcal{D}_3}$	$A_{\mathcal{D}_4}$	$B_{\mathcal{D}_4}$
$GF(2)$	1287	1999	1287	1988	1287	1989	1287	1966	1287	1975
$GF(3)$	1728	1994	1728	1990	1728	1992	1728	1982	1728	1994
$GF(5)$	1989	2001	1978	1990	1980	1992	1980	1986	1983	1995
$GF(7)$	2001	2001	1990	1990	1990	1990	1982	1982	1993	1993
$GF(11)$	2001	2001	1990	1990	1986	1986	1983	1983	1995	1995

A basis for the null space of $B_{\mathcal{D}}$ over $GF(p)$ is obtained in the following way. By a number of row operations and a permutation σ of columns and removing zero rows, $B_{\mathcal{D}}$ can be reduced to a standard form $B'_{\mathcal{D}} = [I\ C]$. The null space of $B'_{\mathcal{D}}$ is equal to the row space of $[-C^t\ I]$. Therefore, after applying the permutation σ^{-1} to the columns of $[-C^t\ I]$, the set of its rows provide a basis of the null space of $B_{\mathcal{D}}$. Once a basis is obtained, the null space can be searched for $(-1, 1)$ vectors by taking $(-1, 1)$ combinations of elements of the basis. For each of the designs $\mathcal{D}_i$ $(0 \le i \le 4)$, this approach is applied over $GF(5)$. A simple computer program is employed to find the bases and then to search the spaces for $(-1, 1)$ vectors. Note that by Table 1, the ranks of null spaces are rather small and an exhaustive search is possible. The results show that there are no $(-1, 1)$ vectors in the null spaces of $B_{\mathcal{D}_i}$. Therefore, all designs $\mathcal{D}_i^s$ $(0 \le i \le 4)$ are indecomposable.

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