Integral trees of odd diameters

E. Ghorbani A. Mohammadian B. Tayfeh-Rezaie*

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran

e_ghorbani@ipm.ir ali_m@ipm.ir tayfeh-r@ipm.ir

Abstract

A graph is called integral if all eigenvalues of its adjacency matrix consist entirely of integers. Recently, Csikvári proved the existence of integral trees of any even diameter. In the odd case, integral trees have been constructed with diameter at most 7. In this paper, we show that for every odd integer $n > 1$, there are infinitely many integral trees of diameter n.

AMS Mathematics Subject Classification: 05C50, 05C05, 15B36.

Keywords: Integral tree, Adjacency eigenvalue, Diameter.

1 Introduction

Let G be a graph with the vertex set $\{v_1, \ldots, v_n\}$. The *adjacency matrix* of G is an $n \times n$ matrix $A(G)$ whose (i, j)-entry is 1 if v_i is adjacent to v_j and 0 otherwise. The *characteristic polynomial* of G, denoted by $\varphi(G; x)$, is the characteristic polynomial of $A(G)$. We will drop the indeterminate x for simplicity of notation whenever there is no danger of confusion. The zeros of $\varphi(G)$ are called the *eigenvalues* of G. Note that $A(G)$ is a real symmetric matrix so that all eigenvalues of G are reals. The graph G is said to be *integral* if all eigenvalues of G are integers.

The notion of integral graphs was first introduced in [5]. The general characterization problem of integral graphs seems to be intractable. Therefore, it is natural to deal with the problem within specific classes of graphs such as trees, cubic graphs and so on. Here, we are concerned with finding integral trees. These objects are extremely rare and very difficult to find. For instance, among trees up to 50 vertices there are only 28 integral ones [1] and out of a total number of $2,262,366,343,746$ trees on 35 vertices only one tree is integral. For a long time, it has been an open question whether there exist integral trees of any diameter [13]. Many attempts by various authors led to the constructions of integral trees of diameters 2–8 and 10, see [6, 7, 8, 9, 10, 11, 12].

*Corresponding author.

Recently, Csikvári found a nice construction of integral trees for any even diameter [3]. In the odd case, all integral trees of diameter 3 have been characterized by a result from number theory given in [4]. Infinitely many integral trees of diameter 5 were first constructed in [9]. The existence of integral trees of diameter 7 was established in [7] where the authors found four such trees. In this paper, we prove the following theorem.

Theorem 1. *For every integer $n \geqslant 2$, there are infinitely many integral trees of diameter $2n+1$.*

2 Csikvári's trees

In this section, we revisit the trees constructed in [3] and compute their eigenvalues using a simple argument. We will make use of them to construct new integral trees of odd diameters.

A *rooted tree* is a tree with a specified vertex called the *root*. For a rooted tree T, we denote by T' the forest resulting from removing the root of T. Let n be a positive integer and T_1, T_2 be two rooted trees with disjoint vertex sets. Then $T_1 \sim nT_2$ is the rooted tree obtained from T_1 and n copies of T_2 by joining the root of T_1 to the roots of the copies of T_2. The root of T_1 is considered to be the root of the resulting tree. For positive integers $r_1 < r_2 < \cdots < r_n$, we define the rooted tree $C(r_1, \ldots, r_n)$ constructed in [3] recursively as

$$C(r_1, \ldots, r_n) = C(r_1, \ldots, r_{n-2}) \sim (r_n - r_{n-1}) C(r_1, \ldots, r_{n-1}),$$

for $n \geqslant 2$, with initial trees $C(\,)$ and $C(r_1)$ being the one-vertex tree and the star tree on $r_1 + 1$ vertices, respectively.

The following lemma is proved in [2, p. 59] for $n = 1$. The general case is straightforward by induction on n.

Lemma 2. *Let T_1, T_2 be two rooted trees and let $T = T_1 \sim nT_2$. Then*

$$\varphi(T) = \varphi(T_2)^{n-1} \left(\varphi(T_1)\varphi(T_2) - n\varphi(T_1')\varphi(T_2') \right).$$

The following lemma can be used to determine the eigenvalues of Csikvári's trees and their multiplicities.

Lemma 3. *Let $n \geqslant 2$ and $r_1, \ldots, r_n$ be positive integers. Then*

$$\varphi\big(C(r_1, \ldots, r_n)\big) = \varphi^{r_n - r_{n-1}}\big(C(r_1, \ldots, r_{n-1})\big)\varphi\big(C(r_1, \ldots, r_{n-2})\big)\frac{x^2 - r_n}{x^2 - r_{n-1}}.$$

Proof. For convenience, we set $P_k = \varphi(C(r_1, \ldots, r_k))$, $Q_k = \varphi(C'(r_1, \ldots, r_k))$ and $d_k = r_k - r_{k-1}$ for $k \geqslant 1$, with the convention $r_0 = 0$. Since $C'(r_1, \ldots, r_k) = C'(r_1, \ldots, r_{k-2}) \cup d_k C(r_1, \ldots, r_{k-1})$

for $k \geqslant 2$, we have $Q_k = P_{k-1}^{d_k} Q_{k-2}$. By Lemma 2,

$$\begin{aligned}
P_k &= P_{k-1}^{d_k-1}\big(P_{k-1}P_{k-2} - (r_k - r_{k-1})Q_{k-1}Q_{k-2}\big)\\
&= P_{k-1}^{d_k-1}\Big(P_{k-2}^{d_{k-1}}\big(P_{k-2}P_{k-3} - (r_{k-1} - r_{k-2})Q_{k-2}Q_{k-3}\big) - (r_k - r_{k-1})P_{k-2}^{d_{k-1}}Q_{k-2}Q_{k-3}\Big)\\
&= P_{k-1}^{d_k-1}P_{k-2}^{d_{k-1}}\big(P_{k-2}P_{k-3} - (r_k - r_{k-2})Q_{k-2}Q_{k-3}\big)\\
&\ \ \vdots\\
&= P_{k-1}^{d_k-1}P_{k-2}^{d_{k-1}}\cdots P_2^{d_3}\big(P_2P_1 - (r_k - r_2)Q_2Q_1\big)\\
&= P_{k-1}^{d_k-1}P_{k-2}^{d_{k-1}}\cdots P_1^{d_2}\big(P_1x - (r_k - r_1)Q_1\big)\\
&= P_{k-1}^{d_k-1}P_{k-2}^{d_{k-1}}\cdots P_1^{d_2}x^{d_1}(x^2 - r_k).
\end{aligned}$$

Note that $P_1 = x^{d_1-1}(x^2 - r_1)$ and so $P_k = P_{k-1}^{d_k-1}P_{k-2}^{d_{k-1}}\cdots P_1^{d_2}x^{d_1}(x^2 - r_k)$ holds for $k \geqslant 1$. To complete the proof, apply this equality for $k = n-1, n$ and then compute P_n/P_{n-1}. □

It is not hard to see that diameter of $C(r_1, \ldots, r_n)$ is $2n$ provided that $r_n - r_{n-1} > 1$. By Lemma 3, the multiplicity of r_{n-1} as an eigenvalue of $C(r_1, \ldots, r_n)$ is $r_n - r_{n-1} - 1$. So the following theorem readily follows from Lemma 3 which establishes the existence of infinitely many integral trees of any even diameter.

Theorem 4. [3] *If $r_n - r_{n-1} > 1$, then the set of distinct eigenvalues of the tree $C(r_1, \ldots, r_n)$ is $\{0, \pm\sqrt{r_1}, \ldots, \pm\sqrt{r_n}\}$.*

Let us introduce an alternative representation of $\varphi(C(r_1, \ldots, r_n))$ and $\varphi(C'(r_1, \ldots, r_n))$ to be used in sequel. For $C = C(r_1, \ldots, r_n)$, we let

$$f(C) = \prod_{i=1}^{\lceil \frac{n}{2} \rceil} \frac{\varphi^{d_{n-2i+2}}\big(C(r_1, \ldots, r_{n-2i+1})\big)}{x^2 - r_{n-2i+1}},$$

where $d_i = r_i - r_{i-1}$ with the convention $r_0 = 0$. By Lemma 3, $f(C)$ is a polynomial and clearly we have

$$\varphi(C) = xf(C) \prod_{i=1}^{\lceil \frac{n}{2} \rceil} (x^2 - r_{n-2i+2}) \tag{1}$$

and

$$\varphi(C') = f(C) \prod_{i=1}^{\lceil \frac{n}{2} \rceil} (x^2 - r_{n-2i+1}). \tag{2}$$

Notice by Lemma 3 that the multiplicity of r_{n-2} as an eigenvalue of C is $(r_n - r_{n-1})(r_{n-1} - r_{n-2} - 1) + 1$. Therefore, if $r_n - r_{n-1} > 1$ and $r_{n-1} - r_{n-2} > 1$, then the positive zeroes of $f(C)$ read as $\sqrt{r_1}, \ldots, \sqrt{r_{n-1}}$.

3 A class of trees

In this section, we introduce a class of trees which will be used to obtain integral trees of odd diameters. For positive integers $n, r, r_0, r_1, \ldots, r_n$ such that $n \geqslant 2$ and $\max\{r_0, r_1\} < r_2 < \cdots < r_n$, let $U = C(r_1, \ldots, r_n)$, $V = C(r_0, r_2, \ldots, r_{n-1})$, $W = C(r_2, \ldots, r_n)$, and define

$$T(r, r_0, r_1, \ldots, r_n) = U \sim (V \sim rW).$$

Note that for $n = 2$, we let $V = C(r_0)$. It is easily checked that the maximum distance between a vertex of $C(k_1, \ldots, k_n)$ and its root is n. So $T = T(r, r_0, r_1, \ldots, r_n)$ is a tree of diameter $2n + 1$. We proceed to determine $\varphi(T)$. Applying Lemma 2, we find that

$$\begin{aligned}\varphi(T) =& \varphi(U)\varphi^{r-1}(W)\left(\varphi(V)\varphi(W) - r\varphi(V')\varphi(W')\right) - \varphi(U')\varphi(V')\varphi^{r}(W)\\ =& \varphi^{r-1}(W)\left(\varphi(U)\varphi(V)\varphi(W) - r\varphi(U)\varphi(V')\varphi(W') - \varphi(U')\varphi(V')\varphi(W)\right).\end{aligned}$$

First assume that $n = 2m + 1$ is odd. By (1) and (2), we have

$$\varphi(U) = xf(U)(x^2 - r_1)(x^2 - r_n)\prod_{i=2}^{m}(x^2 - r_{2i-1}), \tag{3}$$

$$\varphi(V) = xf(V)\prod_{i=1}^{m}(x^2 - r_{2i}), \tag{4}$$

$$\varphi(W) = xf(W)(x^2 - r_n)\prod_{i=2}^{m}(x^2 - r_{2i-1}), \tag{5}$$

and

$$\varphi(U') = x^2 f(U)\prod_{i=1}^{m}(x^2 - r_{2i}), \tag{6}$$

$$\varphi(V') = f(V)(x^2 - r_0)\prod_{i=2}^{m}(x^2 - r_{2i-1}), \tag{7}$$

$$\varphi(W') = f(W)\prod_{i=1}^{m}(x^2 - r_{2i}). \tag{8}$$

Hence, by (3)–(8),

$$\varphi(T) = x(x^2 - r_n)\varphi^{r-1}(W)f(U)f(V)f(W)\prod_{i=2}^{m}(x^2 - r_{2i})\prod_{i=2}^{m}(x^2 - r_{2i-1})^2\psi_o(T),$$

where

$$\psi_o(T) = x^2(x^2 - r_1)(x^2 - r_n) - r(x^2 - r_0)(x^2 - r_1) - x^2(x^2 - r_0).$$

Next suppose that $n = 2m$ is even. By (1) and (2), we have

$$\varphi(U) = xf(U)(x^2 - r_n)\prod_{i=1}^{m-1}(x^2 - r_{2i}), \tag{9}$$

$$\varphi(V) = xf(V)(x^2 - r_0)\prod_{i=2}^{m}(x^2 - r_{2i-1}), \tag{10}$$

$$\varphi(W) = xf(W)(x^2 - r_n)\prod_{i=1}^{m-1}(x^2 - r_{2i}), \tag{11}$$

and

$$\varphi(U') = f(U)(x^2 - r_1)\prod_{i=2}^{m}(x^2 - r_{2i-1}), \tag{12}$$

$$\varphi(V') = x^2 f(V)\prod_{i=1}^{m-1}(x^2 - r_{2i}), \tag{13}$$

$$\varphi(W') = x^2 f(W)\prod_{i=2}^{m}(x^2 - r_{2i-1}). \tag{14}$$

Thus, by (9)–(14),

$$\varphi(T) = x^3(x^2 - r_n)\varphi^{r-1}(W)f(U)f(V)f(W)\prod_{i=2}^{m}(x^2 - r_{2i-1})\prod_{i=1}^{m-1}(x^2 - r_{2i})^2\psi_e(T),$$

where

$$\psi_e(T) = (x^2 - r_0)(x^2 - r_n) - rx^2 - (x^2 - r_1).$$

In summary, using the above notation, we have the following theorem.

Theorem 5. *Let n be odd (respectively, even). Then T is an integral tree of diameter $2n+1$ if and only if $r_0, r_1, \ldots, r_n$ are perfect squares and all the zeros of $\psi_o(T)$ (respectively, $\psi_e(T)$) are integers.*

4 Integral trees of diameter $4k+1$

Let n be even. It is not difficult to choose the parameters of $T = T(r, r_0, r_1, \ldots, r_n)$ in such a way that the zeros of $\psi_e(T)$ are all integers. For instance, let $r_0 = 1$, $r_1 = 4k^2$, $r_n = (k^2-1)^2$ and $r = 4k^2 - 1$. Then

$$\begin{aligned}\psi_e(T) &= (x^2-1)\big(x^2 - (k^2-1)^2\big) - (4k^2-1)x^2 - (x^2 - 4k^2)\\ &= (x^2-1)\big(x^2 - (k^2+1)^2\big).\end{aligned}$$

Clearly, if we choose k large enough, then we are able to take distinct prefect squares $r_2, \ldots, r_{n-1}$ in the interval $(4k^2, (k^2-1)^2)$. Hence, we have proved the following theorem.

Theorem 6. *For every even integer $n \geqslant 2$, there are infinitely many integral trees of diameter $2n+1$.*

5 Integral trees of diameter $4k+3$

Let n be odd. Our objective is to choose the parameters of $T = T(r, r_0, r_1, \ldots, r_n)$ in such a way that all the zeros of $\psi_o(T)$ are integers. This can be done in many ways. For instance, if we set $r_0 = r_1 = a^2$ and $r = r_n = 4(a-1)^2$ for some integer a with $|a| \geqslant 3$, then

$$\psi_o(T) = \big(x^2 - a^2\big)\Big(x^4 - \big(8(a-1)^2 + 1\big)x^2 + 4a^2(a-1)^2\Big).$$

The zeros of $\psi_o(T)$ are $\pm a$ and $\pm(a-\frac{3}{2})\pm\frac{1}{2}\sqrt{12a^2-20a+9}$. So the zeros of $\psi_o(T)$ are integers if and only if $12a^2-20a+9$ is a perfect square, say b^2. We have $(6a-5)^2-3b^2=-2$. From number theory, we know that the Pell-like equation $x^2-3y^2=-2$ has infinitely many integral solutions with $x\equiv\pm1 \pmod 6$. For example, one may take

$$a=\frac{1}{12}\Big(\big(1-\sqrt{3}\big)\big(-2+\sqrt{3}\big)^k+\big(1+\sqrt{3}\big)\big(-2-\sqrt{3}\big)^k+10\Big),$$

for arbitrary integer $k\geqslant 2$. Now, we are able to take the distinct prefect squares $r_2,\ldots,r_{n-1}$ in the interval $(a^2,4(a-1)^2)$. So we come up with the following theorem.

Theorem 7. *For every odd integer $n\geqslant 3$, there are infinitely many integral trees of diameter $2n+1$.*

In conclusion, combining Theorems 6 and 7, we obtain the main result of the paper which is formulated in Theorem 1.

Acknowledgments

The research of the first and second authors was in part supported by a grant from IPM. The authors are grateful to the referees for the helpful comments.

References

[1] A.E. Brouwer, Small integral trees, *Electron. J. Combin.* **15** (2008), Note 1.

[2] D.M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs—Theory and Applications*, Third edition, Johann Ambrosius Barth, Heidelberg, 1995.

[3] P. Csikvári, Integral trees of arbitrarily large diameters, *J. Algebraic Combin.* **32** (2010), 371–377.

[4] R.L. Graham, On a Diophantine equation arising in graph theory, *European J. Combin.* **1** (1980), 107–112.

[5] F. Harary and A.J. Schwenk, Which graphs have integral spectra? *Graphs and Combinatorics*, Lecture Notes in Math., Vol. 406, Springer, Berlin, 1974, 45–51.

[6] P. Híc and R. Nedela, Balanced integral trees, *Math. Slovaca* **48** (1998), 429–445.

[7] P. Híc and M. Pokorný, There are integral trees of diameter 7, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **18** (2007), 59–63.

[8] P. Híc and M. Pokorný, On integral balanced rooted trees of diameter 10, *Acta Univ. M. Belii Math.* **10** (2003), 9–15.

[9] R.Y. Liu, Integral trees of diameter 5, *J. Systems Sci. Math. Sci.* **8** (1988), 357–360.

[10] L. Wang, *Integral Trees and Integral Graphs*, Ph.D. Thesis, University of Twente, 2005.

[11] L. Wang, X. Li and X. Yao, Integral trees with diameters 4, 6 and 8, *Australas. J. Combin.* **25** (2002), 29–44.

[12] M. Watanabe, Note on integral trees, *Math. Rep. Toyama Univ.* **2** (1979), 95–100.

[13] M. Watanabe and A.J. Schwenk, Integral starlike trees, *J. Austral. Math. Soc. Ser. A* **28** (1979), 120–128.