Integral trees with given nullity

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Abstract

A graph is called integral if all eigenvalues of its adjacency matrix consist entirely of integers. We prove that for a given nullity more than 1, there are only finitely many integral trees. Integral trees with nullity at most 1 were already characterized by Watanabe and Brouwer. It is shown that integral trees with nullity 2 and 3 are unique.

Keywords: adjacency eigenvalue, eigenvalue multiplicity, nullity, integral tree.

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1 Introduction

Throughout this article, all graphs are assumed to be finite and without loops or multiple edges. For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. The *order* of G is defined as $|V(G)|$. The *adjacency matrix* of G, denoted by $A(G)$, is a matrix whose entries indexed by $V(G)\times V(G)$ and the (u, v)-entry is 1 if u and v are adjacent and 0 otherwise. The *characteristic polynomial* of G, denoted by $\varphi(G; x)$, is the characteristic polynomial of $A(G)$. We will drop the indeterminate x for the simplicity of notation. The zeros of $\varphi(G)$ are called the *eigenvalues* of G. Note that $A(G)$ is a real symmetric matrix so that all eigenvalues of G are real numbers. We denote the eigenvalues of G in non-increasing order as $\lambda_1(G) \geqslant \cdots \geqslant \lambda_n(G)$, where n is the order of G. The graph G is said to be *integral* if all eigenvalues of G are integers. The *nullity* of G is defined as the nullity of $A(G)$, which is equal to the multiplicity of 0 as an eigenvalue of G. A large number of articles on nullity of graphs have been published. We refer the reader to see [9] and references therein for a survey on this topic.

The notion of integral graphs was first introduced in [10]. A lot of articles deal with integral graphs. We refer the reader to [1] for a comprehensive but rather old survey on the subject. Here, we are concerned with integral trees. These objects are extremely rare and hence very difficult to find. For a long time, it was an open question whether there exist integral trees with arbitrarily large diameter [12]. Recently, this question was affirmatively answered in [4, 8], where the authors constructed integral trees for any diameter. It is well known that the tree on two vertices is the only integral tree with nullity zero [13]. Thereafter, Brouwer proved that any integral tree with nullity 1 is a subdivision of a star graph where the order of the star graph is a perfect square [2]. The latter result has motivated us to investigate integral trees from the 'nullity' point of view.

In this article, we prove that with a fixed nullity more than 1, there are only finitely many integral trees. We also characterize integral trees with nullity 2 and 3 showing that there is a unique integral tree with nullity 2 as well as a unique integral tree with nullity 3.

2 Reduced trees

In this section we introduce 'reduced trees' and derive some properties of their spectrum. We shall use these properties in the next section to prove our finiteness result.

As usual, the *degree* of a vertex v of a graph G is the number of edges of G incident on v. A vertex of degree 1 is called *pendant* and a vertex adjacent to a pendant vertex is said to be *quasi-pendant*. Write P_n for the path graph of order n. For a vertex v of a graph G, we say that there are k *pendant* P_2 *at* v if removing v from G increases the number of P_2 components by k. A graph G is called *reduced* if there exists at most one pendant P_2 at each vertex of G.

We denote the multiplicity of λ as an eigenvalue of a graph G by $\mathrm{mult}(G;\lambda)$. We also denote the number of eigenvalues of G in the interval $(-1, 1)$ by $m(G)$. It is worth to mention that the eigenvalue spectrum of any bipartite graph is symmetric with respect to the origin [3, p. 6]. The following folklore fact, which is stated in [6, p. 49] as an exercise, shows that a reduced graph obtained from a graph G by removing some pendant P_2 has the same nullity as G.

Lemma 1. *Let G be a graph and $v \in V(G)$ be a pendant vertex. If u is the neighbor of v, then the nullities of G and $G - \{u, v\}$ are the same.*

The following result is immediately deduced from Lemma 1 and is proved in [7, Theorem 2].

Corollary 2. *The size of the maximum matching in a tree of order n with nullity h is $\frac{n-h}{2}$.*

The first and second statements of the following theorem are respectively obtained from the Cauchy interlacing theorem for symmetric matrices [3, Corollary 2.5.2] and the Perron–Frobenius theory of nonnegative matrices [3, Theorem 2.2.1].

Theorem 3. *If G is a graph of order n and H is an induced subgraph of G of order m, then $\lambda_{n-m+i}(G) \leqslant \lambda_i(H) \leqslant \lambda_i(G)$ for $i = 1, \ldots, m$. Moreover, if G is a connected graph and $G \neq H$, then $\lambda_1(H) < \lambda_1(G)$.*

As a consequence of Theorem 3, one readily deduces that $\lambda_1(G) > \lambda_2(G)$ for any connected graph G of order at least 2.

Lemma 4. *Let G be a graph and $v \in V(G)$ be a pendant vertex. If u is the neighbor of v, then $m(G - \{u, v\}) \leqslant m(G)$.*

Proof. Note that $m(G - \{u, v\}) = m(G - u) - 1$. Applying Theorem 3 for G and $G - u$, we see that $m(G - u) - 1 \leqslant m(G)$, implying the result. □

The following lemma generalizes a result in [13].

Lemma 5. *The tree P_2 is the only tree with no eigenvalue in $(-1, 1)$.*

Proof. We have $m(P_1) = 1$. By induction on n, we will show for any tree T of order $n \geqslant 3$ that $m(T) \geqslant 1$. Let v be a pendant vertex in a tree T and v' be its neighbor. If $T_v = T - \{v, v'\}$ has a connected component other than P_2, then it follows from Lemma 4, $m(P_1) = 1$, and the induction hypothesis that $m(T) \geqslant m(T_v) \geqslant 1$, as desired. Otherwise, all the connected components of T_v must be P_2. Indeed, we may assume that this property holds for each pendant vertex v of T. This forces that $T = P_4$. But $m(P_4) = 2$ by [5, Table 2], completing the proof. □

Theorem 6. *For any nonnegative integer k, there are finitely many reduced trees with exactly k eigenvalues in $(-1, 1)$.*

Proof. We prove the assertion by induction on k. By Lemma 5, we may assume that $k \geqslant 1$. Let T be a reduced tree with $m(T) = k$. First suppose that there exists $v \in V(T)$ such that three of the connected components $T_1, \ldots, T_d$ of $T - v$ are not P_2. From Theorem 3, $m(T - v) \leqslant m(T) + 1$. Since T is reduced, at most one of $T_1, \ldots, T_d$ is P_2. Hence, Lemma 5 yields that $d - 1 \leqslant \sum_{i=1}^{d} m(T_i) \leqslant k + 1$ and $m(T_i) + 2 \leqslant \sum_{i=1}^{d} m(T_i) \leqslant k + 1$ for $i = 1, \ldots, d$. It follows that $d \leqslant k + 2$ and $m(T_i) \leqslant k - 1$ for $i = 1, \ldots, d$. Note that if some T_i is not reduced, then it has exactly one vertex with more than one pendant P_2 and such a vertex has exactly two pendant P_2. By the induction hypothesis, the number of reduced trees F with $m(F) \leqslant k-1$ is finite and thus there are only finitely many ways of choosing $T_1, \ldots, T_d$. Since $d \leqslant k+2$, the result follows. Now suppose otherwise. This means that any vertex of T is of degree at most 3 and all vertices of degree 3 of T have a pendant P_2. Hence, T is obtained from a path graph P_t by attaching one pendant P_2 at some vertices of degree 2 in P_t. Moreover, it follows from Lemma 4 that $m(P_t) \leqslant m(T)$. We know from [3, p. 9] that $\lambda_i(P_t) = 2\cos\frac{\pi\ell}{t+1}$ for $\ell = 1, \ldots, t$. Therefore, $m(P_t) \geqslant \frac{t-2}{3}$ and so $t \leqslant 3k+2$. This completes the proof. □

For later use, we need the following refinement of Lemma 4.

Lemma 7. *Let T be a tree with at least one pendant P_2 at $v \in V(T)$. Then increasing the number of pendant P_2 at v by one, leaves the number of eigenvalues in $(-1,1)$ unchanged and increases the multiplicity of 1 by one.*

Proof. Suppose that T' is the resulting tree from T by adding two new vertices a and b where a is joined to both b and v. Let c and d be the vertices of a pendant P_2 of T at v. Let $k = \mathrm{mult}(T;1)$ and assume that $\{x_1, \ldots, x_k\}$ is a basis for the eigenspace $\mathscr{E}$ of T corresponding to eigenvalue 1 when $k \geqslant 1$. Since each vector $x \in \mathscr{E}$ takes the same value on c and d, we conclude that x vanishes on v. For each i with $1 \leqslant i \leqslant k$, extend x_i to find the vector y_i defined on $V(T')$ with value 0 on $\{a,b\}$. Also, define the vector y_{k+1} so that $y_{k+1}(a) = y_{k+1}(b) = 1$, $y_{k+1}(c) = y_{k+1}(d) = -1$, and 0 elsewhere. It is now readily verified that $y_1, \ldots, y_{k+1}$ are linearly independent eigenvectors of T' corresponding to eigenvalue 1. By Theorem 3, $\mathrm{mult}(T';1) - 1 \leqslant \mathrm{mult}(T'-a;1) = k$. Hence, $\mathrm{mult}(T';1) = k+1$, as desired. Furthermore, from $\mathrm{mult}(T';1) = \mathrm{mult}(T'-a;1)+1$ and by applying Theorem 3 for T' and $T'-a$, one concludes that $m(T'-a) = m(T')+1$. Since $m(T'-a) = m(T)+1$, we deduce that $m(T) = m(T')$. This completes the proof. □

3 Finiteness of integral trees with a given nullity

In this section we present our main result which states that for every integer $h \geqslant 2$, there are finitely many integral trees with nullity h.

Definition 8. By considering a tree T as a connected bipartite graph, one finds a unique pair $\{A, B\}$ for which A and B are disjoint independent subsets of T with $V(T) = A \cup B$ and $B \neq \varnothing$. Define $\mathcal{S}(T;A)$ to be the tree obtained from T by attaching a pendant vertex to each vertex in B.

Definition 9. Let T be a tree. For every distinct vertices $v_1, \ldots, v_k \in V(T)$ and nonnegative integers $s_1, \ldots, s_k$, we denote by $T(v_1, \ldots, v_k; s_1, \ldots, s_k)$ the resulting tree from T by attaching s_i pendant P_2 at v_i for $i = 1, \ldots, k$.

Definition 10. Let p, q, and r be nonnegative integers and let T_1, T_2, T_3, and T_4 be the trees depicted in Figure 1 with some specified vertices. We will denote by $S(p)$, $S(p,q)$, $S(p,q,r)$, and $S'(p,q,r)$ the trees $T_1(u;p)$, $T_2(u,v;p,q)$, $T_3(u,v,w;p,q,r)$, and $T_4(u,v,w;p,q,r)$, respectively.

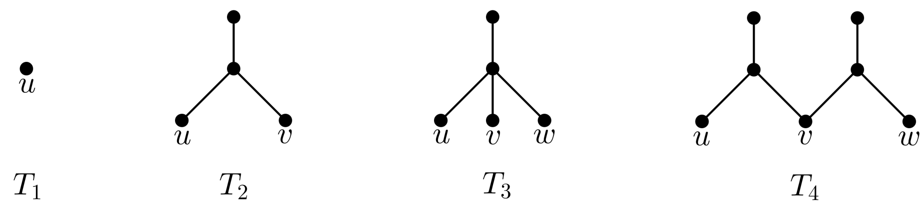

Figure 1. The trees of Definition 10.

Note that all trees introduced in Definition 10 are of the form described in Definition 8. The following lemma is established in [3, Proposition 5.1.1(i)] for $k = 1$. The general case is straightforwardly proved by induction on k as mentioned in [3, p. 90].

Lemma 11. *Let T_1 and T_2 be two vertex disjoint trees with specified vertices $v_1 \in V(T_1)$ and $v_2 \in V(T_2)$. For a positive integer k, assume that T is the tree obtained from T_1 and k copies of T_2 by joining v_1 to the k copies of v_2. Then*

$$\varphi(T) = \varphi(T_2)^{k-1}\big(\varphi(T_1)\varphi(T_2) - k\varphi(T_1 - v_1)\varphi(T_2 - v_2)\big).$$

Using Lemma 11, one obtains that

$$\varphi\big(S(p)\big) = x(x^2 - p - 1)(x^2 - 1)^{p-1}, \tag{1}$$

for any nonnegative integer p. In the rest of the article, we will use frequently the next lemma which is obtained from [4, Lemma 2.8].

Lemma 12. *Let G be a bipartite graph with bipartition $\{X, Y\}$ and with k positive eigenvalues. Let G' be the graph obtained from G by joining r new pendant vertices to each vertex of X for some positive integer r. Then $\lambda_i^2(G') = \lambda_i^2(G) + r$ for $i = 1, \ldots, k$.*

Remark 13. Let T be a tree of order n and $k \geqslant 1$. For every distinct vertices $v_1, \ldots, v_k \in V(T)$ and nonnegative integers $s_1, \ldots, s_k$ with $s_1 \geqslant \cdots \geqslant s_k$, the tree $T' = T(v_1, \ldots, v_k; s_1, \ldots, s_k)$ satisfies

$$\lambda_i\big(T'\big) \geqslant \sqrt{s_i + 1} + \lambda_n(T), \tag{2}$$

for $i = 1, \ldots, k$. So, when all values $s_1, \ldots, s_k$ go to infinity, then the k largest eigenvalues of T' tend to infinity. In order to obtain (2), let L be the subgraph of T' which is isomorphic to the vertex disjoint union of $S(s_1), \dots, S(s_k)$ and $E(L) \cap E(T) = \varnothing$. If we denote the spanning subgraphs of T' with the edge sets $E(L)$ and $E(T)$ respectively by F_1 and F_2, then we may write $A(T') = A(F_1) + A(F_2)$ by a suitable labeling of the vertices. Now (2) follows from the Courant–Weyl inequalities [3, Theorem 2.8.1(ii)] and (1).

Lemma 14. *Let T be a tree, k, t be positive integers, and $v_1, \dots, v_k$ be distinct vertices of T. Suppose that there exists a polynomial $f(x)$ such that for every integers $s_1, \dots, s_k \geqslant t$, the tree $T' = T(v_1, \dots, v_k; s_1, \dots, s_k)$ satisfies*

$$\varphi(T') = (x^2-1)^{s_1+\cdots+s_k-k} f(x) \prod_{i=1}^{k} \big(x^2 - \alpha_i(s_1, \dots, s_k)\big), \tag{3}$$

where $\alpha_i(s_1, \dots, s_k), \dots, \alpha_k(s_1, \dots, s_k)$ are positive-valued functions in terms of $s_1, \dots, s_k$. Then $T = \mathcal{S}(R; \{v_1, \dots, v_k\})$ for some tree R.

Proof. We prove the assertion by induction on k. First assume that $k = 1$. For convenience in notation, let $v = v_1$, $s = s_1$, and $\alpha = \alpha_1$. We have $\varphi(T') = (x^2-1)^{s-1}((x^2-1)\varphi(T) - sx\varphi(T-v))$ by Lemma 11. Hence, we deduce from (3) that $(x^2-1)\varphi(T) - sx\varphi(T-v) = f(x)(x^2-\alpha(s))$ for any integer $s \geqslant t$. In particular,

$$(x^2-1)\varphi(T) - tx\varphi(T-v) = f(x)\big(x^2 - \alpha(t)\big) \tag{4}$$

and

$$(x^2-1)\varphi(T) - (t+1)x\varphi(T-v) = f(x)\big(x^2 - \alpha(t+1)\big). \tag{5}$$

Using (4) and (5), one obtains that $f(x) = x\varphi(T-v)/(\alpha(t+1) - \alpha(t))$. It is clear from (3) that $f(x)$ is a monic polynomial, implying $\alpha(t+1) - \alpha(t) = 1$. Therefore, $f(x) = x\varphi(T-v)$. It follows from (4) that

$$(x^2-1)\varphi(T) = x(x^2-\mu)\varphi(T-v), \tag{6}$$

for some real number μ. Hence, $\mathrm{mult}(T; 0) = \mathrm{mult}(T-v; 0) + 1$ and so it follows from Lemma 1 that v is not a quasi-pendant vertex in T. Consequently, T contains a copy of $S(r)$ as an induced subgraph with the central vertex v, where r is the degree of v. We know that the sum of squares of all eigenvalues of a graph equals twice the number of edges of the graph [3, Proposition 1.3.1]. Applying this fact to T and $T-v$, we obtain from (6) that

$$r = E(T) - E(T-v) = \tfrac{1}{2}\left(\sum_{i=1}^{|V(T)|} \lambda_i^2(T) - \sum_{i=1}^{|V(T)|-1} \lambda_i^2(T-v)\right) = \mu - 1. \tag{7}$$

Further, it follows from (6) and Theorem 3 that $\lambda_1(T) = \mu$. Also, we find from (1) and (7) that $\lambda_1(S(r)) = \mu$ and therefore, $\lambda_1(T) = \lambda_1(S(r))$. Since T contains a copy of $S(r)$ as a subgraph, Theorem 3 yields that $T = S(r)$, as desired.

Now assume that $k \geqslant 2$. Let $T'' = T(v_1, \ldots, v_{k-1}; s_1, \ldots, s_{k-1})$. By Lemma 11, we have

$$\varphi(T') = (x^2-1)^{s_k-1}\big((x^2-1)\varphi(T'') - s_k x\varphi(T''-v_k)\big). \tag{8}$$

Combining (3) with (8) and setting $\rho = s_1 + \cdots + s_{k-1} - k + 1$, we conclude that

$$(x^2-1)\varphi(T'') - s_k x\varphi(T''-v_k) = (x^2-1)^{\rho} f(x) \prod_{i=1}^{k} \big(x^2 - \alpha_i(s_1, \ldots, s_k)\big),$$

for every integers $s_1, \ldots, s_k \geqslant t$. In particular, we have

$$(x^2-1)\varphi(T'') - tx\varphi(T''-v_k) = (x^2-1)^{\rho} f(x) \prod_{i=1}^{k} \big(x^2 - \alpha_i(s_1, \ldots, s_{k-1}, t)\big) \tag{9}$$

and

$$(x^2-1)\varphi(T'') - (t+1)x\varphi(T''-v_k) = (x^2-1)^{\rho} f(x) \prod_{i=1}^{k} \big(x^2 - \alpha_i(s_1, \ldots, s_{k-1}, t+1)\big), \tag{10}$$

for every integers $s_1, \ldots, s_{k-1} \geqslant t$. It is easily obtained from (9) and (2) that

$$x\varphi(T''-v_k) = (x^2-1)^{\rho} f(x) \prod_{i=1}^{k-1} \big(x^2 - \beta_i(s_1, \ldots, s_{k-1})\big), \tag{11}$$

where $\beta_i(s_1, \ldots, s_{k-1}), \ldots, \beta_{k-1}(s_1, \ldots, s_{k-1})$ are positive-valued function in terms of $s_1, \ldots, s_{k-1}$. It follows from Remark 13 that $k-1$ of the roots of $\varphi(T''-v_k)$ tend to infinity as $s_1, \ldots, s_{k-1}$ grow and hence $\prod_{i=1}^{k-1}(x^2 - \beta_i(s_1, \ldots, s_{k-1}))$ is not divisible by x for some integers $s_1, \ldots, s_{k-1}$. So, we find from (11) that

$$f(x) = xg(x), \tag{12}$$

for some polynomial $g(x)$, and thus we can rewrite (11) as

$$\varphi(T''-v_k) = (x^2-1)^{\rho} g(x) \prod_{i=1}^{k-1} \big(x^2 - \beta_i(s_1, \ldots, s_{k-1})\big). \tag{13}$$

Let $W = \{v_1, \ldots, v_k\}$. By (13), Lemma 7, and the induction hypothesis, we deduce that each connected component H of $T - v_k$ with $V(H) \cap W \neq \varnothing$ is of the form $\mathcal{S}(R; V(H) \cap W)$. By replacing v_k with any of $v_1, \ldots, v_{k-1}$ in the above argument, we find that this property also holds for $T - v_1, \ldots, T - v_{k-1}$. From this, we conclude that each connected component H of $T - v_k$ with $V(H) \cap W = \varnothing$ must be P_2, since if not, then for an index $i \neq k$ the connected component H of $T - v_i$ containing v_k does not have the form $\mathcal{S}(R; V(H) \cap W)$, a contradiction.

Denote by $L_1 = \mathcal{S}(F_1; A_1), \ldots, L_\ell = \mathcal{S}(F_\ell; A_\ell)$ the connected components of $T - v_k$ which are not P_2. We claim that the neighbor of v_k in $V(L_i)$ is contained in $V(F_i) \setminus A_i$ for $i = 1, \ldots, \ell$.

This evidently shows that T is of the form $\mathcal{S}(R;W)$. Aiming for a contradiction, suppose for some i that the neighbor of v_k in $V(L_i)$ either is contained in A_i or is a pendant vertex in L_i. This implies that there exists $v_p \in A_i$ of distance 1 or 3 from v_k. If $k=2$, then $\ell=1$ and T is isomorphic to one of the trees depicted in Figure 2.

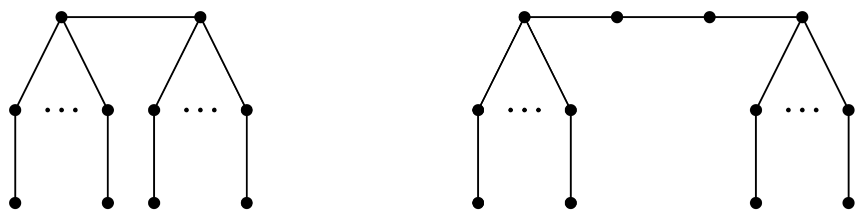

Figure 2.

From (3) and (12), we deduce that T' has eigenvalue 0 and so Corollary 2 yields that T' and T have no perfect matching. But both trees in Figure 2 have perfect matchings, so we get a contradiction. If $k \geqslant 3$, then for an index $q \notin \{k,p\}$ the connected component H of $T - v_q$ containing v_k does not have the form $\mathcal{S}(R;V(H)\cap W)$, since v_k and v_p have odd distance in H. This contradiction establishes the claim and completes the proof. □

The following lemma is a special case of [6, Theorem 8.1.7].

Lemma 15. *Let e be an edge of a tree T. Let T' be the tree obtained from T by contracting e to a vertex u and attaching a pendant vertex to u. Then $\lambda_1(T') \geqslant \lambda_1(T)$.*

Lemma 16. *Let T be a tree of order n and $v \in V(T)$ be of the degree k. For any nonnegative integer m, define $T_v(m)$ as the tree obtained from T by attaching m pendant vertices to v. Then $\lambda_1^2(T_v(m)) < m+k+1$ if $m > (k+1)(n-k-2)$.*

Proof. By applying the operation described in Lemma 15 iteratively on all the edges of $T_v(m)$ not incident with v, we reach at a tree $T'_v(m)$ depicted in Figure 3.

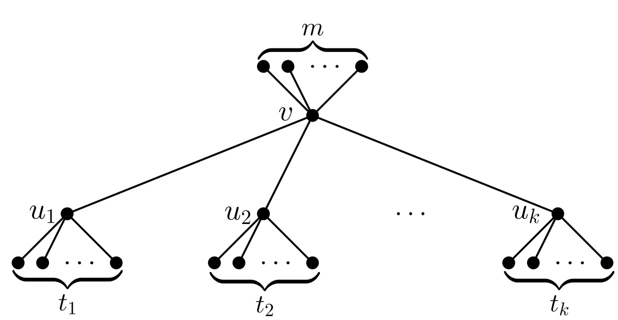

Figure 3. The tree $T'_v(m)$.

It follows from Lemma 15 and Theorem 3 that $\lambda_1(T_v(m)) \leqslant \lambda_1(T'_v(m)) \leqslant \lambda_1(T''_v(m))$, where $T''_v(m)$ is the tree obtained for $T'_v(m)$ by increasing the number of pendant vertices attached to each of $u_1, \ldots, u_k$ to t, where $t = \max\{t_1, \ldots, t_k\}$. The characteristic polynomial of $T''_v(m)$ can be computed by applying Lemma 11. So, an easy calculation shows that

$$\lambda_1^2\big(T''_v(m)\big) = \frac{m+k+t+\sqrt{(m+k+t)^2-4mt}}{2}.$$

Hence, $\lambda_1^2(T_v(m)) < m+k+1$ if $m > (k+1)(t-1)$. Since $n \geqslant k+t+1$, the result follows. □

Now we are in a position to present our main result.

Theorem 17. *For every integer $h \geqslant 2$, there are finitely many integral trees with nullity h.*

Proof. Arguing toward a contradiction, suppose that there are infinitely many integral trees with nullity h for some $h \geqslant 2$. Therefore, by Lemma 7 and Theorem 6, we find a reduced tree T with $V(T) = \{v_1, \ldots, v_n\}$ and nullity h such that $T(v_1, \ldots, v_n; s_{i1}, \ldots, s_{in})$ is integral for an infinite set $\{(s_{i1}, \ldots, s_{in})\}_{i\in\mathbb{N}}$ of n-tuples of nonnegative integers. If for some fixed integers j and s, the set $\{i \,|\, s_{ij} = s\}$ is infinite, then we replace T by $T(v_j; s)$. Repeating this operation, we may assume that there is a tree T of order n with specified vertices $v_1, \ldots, v_k$ and an infinite set $\{(s_{i1}, \ldots, s_{ik})\}_{i\in\mathbb{N}}$ of k-tuples of nonnegative integers such that $s_{ij} < s_{(i+1)j}$ for $j = 1, \ldots, k$, and $T_i = T(v_1, \ldots, v_k; s_{i1}, \ldots, s_{ik})$ is integral for all i.

By Remark 13, the set $\{\lambda_j(T_i) \,|\, i \in \mathbb{N}\}$ is not bounded for $j = 1, \ldots, k$, and by Theorem 3, the set $\{\lambda_{k+1}(T_i) \,|\, i \in \mathbb{N}\}$ is bounded above by $\max\{1, \lambda_1(T - \{v_1, \ldots, v_k\})\}$. This clearly implies that there exists an integer i_0 such that $\lambda_j(T_i)$ is fixed for $j = k+1, \ldots, k+\frac{n-h}{2}$ and each $i \geqslant i_0$. Furthermore, by Lemma 7, we have $\lambda_j(T_i) = 1$ for $j = k+\frac{n-h}{2}+1, \ldots, s_{i1}+\cdots+s_{ik}+\frac{n-h}{2}$ and all $i \geqslant i_0$. By Theorem 3, it is not hard to see that $T' = T(v_1, \ldots, v_k; s_1, \ldots, s_k)$ satisfies in (3) for all integers $s_1, \ldots, s_k \geqslant t$, where $t = \max\{s_{i_01}, \ldots, s_{i_0k}\}$. Therefore, it follows from Lemma 14 that $T = \mathcal{S}(R; \{v_1, \ldots, v_k\})$ for some tree R.

We proceed to obtain a contradiction by showing that for large enough i, $\lambda_1(T_i)$ is not an integer. For a fixed i, we may relabel $v_1, \ldots, v_k$ such that $s_{i1} \geqslant \cdots \geqslant s_{ik}$. Define the tree E_i as follows: start with the tree R, attach s_{i1} pendant vertices to each of $v_1, \ldots, v_k$, and then in the resulting tree attach a pendant vertex to each vertex outside of $\{v_1, \ldots, v_k\}$. Obviously, T_i is a subgraph of E_i and so, using Lemma 12 twice and by Theorem 3, we find that $\lambda_1^2(T_i) \leqslant 1+s_{i1}+\lambda_1^2(R)$. Since T_i contains vertex disjoint copies of $S(s_{i1})$ and $S(s_{i2})$, Theorem 3 and (1) imply that $\lambda_2^2(T_i) \geqslant \lambda_1^2(S(s_{i2})) = 1+s_{i2}$. It follows that $\lambda_1^2(T_i) - \lambda_2^2(T_i) - \lambda_1^2(R) \leqslant s_{i1} - s_{i2}$. Since $\lambda_1^2(T_i) - \lambda_2^2(T_i)$ is the difference of two distinct perfect squares, $\lambda_1^2(T_i) - \lambda_2^2(T_i)$ and so $s_{i1} - s_{i2}$ tend to infinity when i grows. Define the tree F_i as follows: start with the tree $R_{v_1}(s_{i1} - s_{i2})$ in the sense which is defined in Lemma 16, attach s_{i2} pendant vertices to each of $v_1, \ldots, v_k$, and then in the resulting tree attach a pendant vertex to each vertex outside of $\{v_1, \dots, v_k\}$. Evidently, T_i is a subgraph of F_i and so, using Lemma 12 twice and by Theorem 3, we find that

$$\lambda_1^2(T_i) \leqslant 1 + s_{i2} + \lambda_1^2\big(R_{v_1}(s_{i1} - s_{i2})\big). \tag{14}$$

Applying Lemma 16 and assuming i is large enough, we obtain that

$$\lambda_1^2\big(R_{v_1}(s_{i1} - s_{i2})\big) < s_{i1} - s_{i2} + \ell + 1, \tag{15}$$

where ℓ is the degree of v_1 in R. Clearly, it follows from $h \geqslant 2$ and Lemma 1 that $k \geqslant 2$. From this and by Theorem 3 and (1), one deduces that

$$\lambda_1^2(T_i) > \lambda_1^2\big(S(s_{i1} + \ell)\big) = s_{i1} + \ell + 1. \tag{16}$$

It follows from (14)–(16) that $s_{i1} + \ell + 1 < \lambda_1^2(T_i) < s_{i1} + \ell + 2$ for large enough i. This contradiction completes the proof. □

4 Integral trees with nullity 2 and 3

Integral trees with nullity 0 and 1 are respectively classified in [2] and [13]. In this section we characterize integral trees with nullity 2 and 3. Before that, we determine all integral trees among the trees introduced in Definition 10. From (1), we find that $S(p)$ is integral if and only if $p + 1$ is a perfect square.

Lemma 18. *Let p and q be nonnegative integers. Then $S(p, q)$ is not integral.*

Proof. Towards a contradiction, suppose that $T = S(p, q)$ is integral. We first assume that $p = q$. Using Lemma 12 twice, we find that $\lambda_1^2(T) = p + 3$. Since T has two vertex disjoint copies of $S(p)$, we obtain from Theorem 3 and (1) that $\lambda_2^2(T) \geqslant \lambda_1^2(S(p)) = p + 1$. Therefore, $\lambda_1^2(T) - \lambda_2^2(T) \leqslant 2$. This is a contradiction, since no two distinct non-zero perfect squares have difference at most 2. We now assume without loss of generality that $p > q$. Again, using Lemma 12 twice and by Theorem 3, we find that $\lambda_1^2(T) < p + 3$. Since T contains a copy of $S(p + 1)$ as a subgraph, Theorem 3 and (1) yield that $\lambda_1^2(T) > \lambda_1^2(S(p + 1)) = p + 2$. Hence, $p + 2 < \lambda_1^2(T) < p + 3$ which implies that $\lambda_1(T)$ is not an integer, a contradiction. □

Lemma 19. *Let p, q, r be nonnegative integers and let $T \in \{S(p, q, r), S'(p, q, r)\}$. Then either $T = S(0, 0, 0)$ or T is not integral.*

Proof. Assume that T is integral and let t and t' be the largest and second largest number among p, q, r, respectively. We know from [5, Table 2] that $S(0, 0, 0)$ is integral while $S'(0, 0, 0)$

is not integral. Hence, towards a contradiction, we suppose that $t \geqslant 1$. Since T contains a copy of $S(t+1)$ as a subgraph, Theorem 3 and (1) imply that

$$\lambda_1^2(T) > \lambda_1^2(S(t+1)) = t+2. \tag{17}$$

Assume that R is one of the star graph of order 4 or P_5. Using Lemma 12 twice and by Theorem 3, one obtains that $\lambda_1^2(T) \leqslant 1+t+\lambda_1^2(R)$, where the equality occurs if and only if $p=q=r$. From [2, pp. 8–9], we find that $\lambda_1(R) = \sqrt{3}$ and so $\lambda_1^2(T) \leqslant t+4$. In the case of equality, T has three vertex disjoint copies of $S(t)$ and thus Theorem 3 and (1) yield that $\lambda_2^2(T) \geqslant \lambda_1^2(S(t)) = t+1$ which in turn implies that $\lambda_1^2(T) - \lambda_2^2(T) \leqslant 3$. This is impossible, since $\lambda_1(T)$ and $\lambda_2(T)$ are two distinct integers more than 1. Thus, in view of (17), one deduces that $\lambda_1^2(T) = t+3$. We know from [5, Table 2] that $\lambda_1(S(1,0,0))$, $\lambda_1(S'(1,0,0))$, and $\lambda_1(S'(0,1,0))$ are greater than 2. This implies that $t \geqslant 2$ and therefore $\lambda_1^2(T) - \lambda_2^2(T) \geqslant 5$. On the other hand, T contains two vertex disjoint copies of $S(t')$, so Theorem 3 and (1) yield that $\lambda_2^2(T) \geqslant \lambda_1^2(S(t')) = t'+1$. This follows that $t - t' \geqslant \lambda_1^2(T) - \lambda_2^2(T) - 2 \geqslant 3$. Assume that R' is one of the trees R'_1, R'_2, R'_3 depicted in Figure 4.

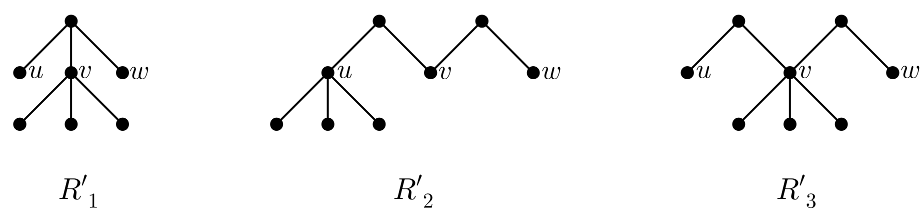

Figure 4. The tree R'.

Define the tree T' as follows: start with the tree R', attach $t-3$ pendant vertices to each of u, v, w, and then in the resulting tree attach a pendant vertex to each vertex outside of $\{u, v, w\}$. Since $t' \leqslant t-3$, T is a subgraph of T' and so, by Theorem 3 and using Lemma 12 twice, one deduces that $t+3 = \lambda_1^2(T) \leqslant 1+t-3+\lambda_1^2(R')$, implying $\lambda_1(R') \geqslant \sqrt{5}$. We know from [5, Table 2] that $\lambda_1(R'_1)$ and $\lambda_1(R'_2)$ are less than $\sqrt{5}$ and thus $R' = R'_3$. Since T is obtained from T' by deleting some pendant P_2 at u and w, T contains a copy of $S(t+2)$ as a subgraph. Hence, Theorem 3 and (1) imply that $\lambda_1^2(T) > \lambda_1^2(S(t+2)) = t+3$, a contradiction. □

We are now ready to characterize integral trees with nullity 2 and 3. In order to do this in a simple manner, we use the following interesting result which is called the Parter–Wiener theorem [11, 14].

Theorem 20. *If T is a tree and* $\mathrm{mult}(T;\lambda) \geqslant 2$ *for some λ, then there exists $v \in V(T)$ such that* $\mathrm{mult}(T-v;\lambda) = \mathrm{mult}(T;\lambda)+1$.

The integral trees with nullity 1 are characterized in [2, Theorem 3]. In the next theorem, we will generalize this result by a short and simple proof. We start with the following easy lemma.

Lemma 21. *Let T be a tree with no eigenvalue in $(0,1)\cup(1,2)$. Then the order of T is at most $2\,\mathrm{mult}(T;0)+3\,\mathrm{mult}(T;1)-1$.*

Proof. Since the spectrum of eigenvalues of T is symmetric around the origin [3, p. 6] and the sum of squares of all eigenvalues of T equals twice the number of its edges [3, Proposition 1.3.1], we obtain that

$$2\,\mathrm{mult}(T;1)+4\big(n-\mathrm{mult}(T;0)-2\,\mathrm{mult}(T;1)\big)\leqslant 2(n-1).$$

This follows the assertion. □

Theorem 22. *Let T be a tree with nullity 1 and no eigenvalue in $(0,1)\cup(1,2)$. Then $T=S(p)$ for some $p\geqslant 0$.*

Proof. If $\mathrm{mult}(T;1)\leqslant 1$, then Lemma 21 implies that T is of order at most 4. We know from [5, Table 2] that, among the trees of order at most four, $S(0)$ is the only tree satisfying the assumption of the theorem. So, assume that $\mathrm{mult}(T;1)\geqslant 2$. From Theorem 20, there exists a vertex v such that $\mathrm{mult}(T-v;1)=\mathrm{mult}(T;1)+1$. Hence, Theorem 3 implies that $m(T-v)=0$. It follows from Lemma 5 that $T-v$ is a vertex disjoint union of some copies of P_2, yielding the result. □

The following conclusion, which is first appeared in [2], should be clear from (1) and Theorem 22.

Corollary 23. *Each integral tree with nullity 1 is of the form $S(p^2-1)$ for some $p\geqslant 1$.*

Theorem 24. *Let T be a tree with nullity 2 and no eigenvalue in $(0,1)\cup(1,2)$. Then either T is the tree depicted in Figure 5 or $T=S(p,q)$ for some nonnegative integers p,q.*

Proof. If $\mathrm{mult}(T;1)\leqslant 1$, then Lemma 21 yields that the order of T is at most 6. We know from [5, Table 2] that, among the trees of order at most 6, the only tree satisfying the assumption of the theorem is the tree depicted in Figure 5. So, assume that $\mathrm{mult}(T;1)\geqslant 2$. By Theorem 20, there exists a vertex v such that $\mathrm{mult}(T-v;1)=\mathrm{mult}(T;1)+1$. Employing Theorem 3, one concludes that $T-v$ has nullity 1 and has no eigenvalue in $(0,1)\cup(1,2)$. Thus, in view of Lemma 5 and Theorem 22, $T-v$ is of the form $S(p)\cup qP_2$ for some nonnegative integers p,q. If the neighbor of v in $S(p)$ is not quasi-pendant, then T would have a perfect matching and so Corollary 2 yields that the nullity of T would be 0, a contradiction. Hence, v is adjacent to a quasi-pendant vertex in $S(p)$. This means that $p\geqslant 1$ and $T=S(p-1,q)$, the result follows. □

By combining Lemma 18 and Theorem 24, the following is obtained.

Corollary 25. *There is only one integral tree with nullity 2; namely, the tree depicted in Figure 5.*

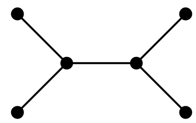

Figure 5. The unique integral tree with nullity 2.

Theorem 26. *The star graph of order 5 is the only integral tree with nullity 3.*

Proof. Let T be an integral tree with nullity 3. If $\mathrm{mult}(T;1) \leqslant 1$, then it follows from Lemma 21 that T has at most 8 vertices. We know from [5, Table 2] that, among the trees of order at most 8, there is only one integral tree with nullity 3 that is the star graph of order 5, we are done. Towards a contradiction, suppose that $\mathrm{mult}(T;1) \geqslant 2$. From Theorem 20, there exists a vertex v such that $\mathrm{mult}(T-v;1) = \mathrm{mult}(T;1)+1$. Moreover, by Theorem 3, $T-v$ has nullity 2 and has no eigenvalue in $(0,1)\cup(1,2)$. From Theorems 22 and 24, it follows for some nonnegative integers p,q,r that $T-v$ is of one of the following forms:

(i) $S(p)\cup S(q)\cup rP_2$;

(ii) $S(p,q)\cup rP_2$;

(iii) $Y\cup rP_2$, where Y is the tree depicted in Figure 5.

If (i) is the case, then by Corollary 2, v is necessarily adjacent to two quasi-pendant vertices in $S(p)$ and $S(q)$. This means that $p,q\geqslant 1$ and $T=S'(p-1,r,q-1)$, which contradicts Lemma 19. In the case (ii), it follows from Corollary 2 that the neighbor of v in $S(p,q)$ is quasi-pendant. This implies that $T\in\{S(p,q,r), S'(r,p-1,q), S'(p,q-1,r)\}$ and so by Lemma 19, we find that $T=S(0,0,0)$. This is a contradiction, since $\mathrm{mult}(T;1)\geqslant 2$. For the case (iii), using Corollary 2, v is necessarily adjacent to one of the two vertices of degree 3 in Y. By applying Lemma 11, we find that $\varphi(T)=x^3(x^2-1)^r(x^4-(r+6)x^2+4r+6)$. From the intermediate value theorem, it is easily seen that $\varphi(T)$ has a zero in $(1,2)$, a contradiction. The proof is now complete. □

We mention here that one can apply a similar method to find all integral trees with other small nullities which of course would be an elaborate task. By [2], among trees up to fifty vertices, there is no integral tree with nullities 4, 6, or 9. Therefore, one may ask if there exist integral trees with nullity 4. Further, one may ask a more general question: Does there exist arbitrarily large integer h such that there is no integral tree with nullity h? Eventually, we pose the question: For given integers $m,k\geqslant 1$, is the number of integral trees with eigenvalue m of multiplicity k finite?

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