Interval minors of complete bipartite graphs

Bojan Mohar[*†]
Department of Mathematics
Simon Fraser University
Burnaby, BC, Canada
mohar@sfu.ca

Arash Rafiey
Department of Mathematics
Simon Fraser University
Burnaby, BC, Canada
arashr@sfu.ca

Behruz Tayfeh-Rezaie
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran
tayfeh-r@ipm.ir

Hehui Wu
Department of Mathematics
Simon Fraser University
Burnaby, BC, Canada
noshellwhh@gmail.com

Abstract

Interval minors of bipartite graphs were recently introduced by Jacob Fox in the study of Stanley-Wilf limits. We investigate the maximum number of edges in $K_{r,s}$-interval minor free bipartite graphs. We determine exact values when $r = 2$ and describe the extremal graphs. For $r = 3$, lower and upper bounds are given and the structure of $K_{3,s}$-interval minor free graphs is studied.

Keywords: interval minor, complete bipartite graph, forbidden configuration, forbidden pattern.

Mathematics Subject Classification (2010): 05C35, 05C83, 05B20.

[*]Supported in part by an NSERC Discovery Grant (Canada), by the Canada Research Chair program, and by the Research Grant P1–0297 of ARRS (Slovenia).

[†]On leave from: IMFM & FMF, Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.

1 Introduction

All graphs in this paper are simple, i.e. multiple edges and loops are not allowed. By an *ordered* bipartite graph $(G; A, B)$, we mean a bipartite graph G with independent sets A and B which partition the vertex set of G and each of A and B has a linear ordering on its elements. We call two vertices u and v *consecutive* in the linear order $<$ on A or B if $u < v$ and there is no vertex w such that $u < w < v$. By *identifying* two consecutive vertices u and v to a single vertex w, we obtain a new ordered bipartite graph such that the neighbourhood of w is the union of the neighbourhoods of u and v in G. All bipartite graphs in this paper are ordered and so, for simplicity, we usually say bipartite graph G instead of ordered bipartite graph $(G; A, B)$. Two ordered bipartite graphs G and G' are *isomorphic* if there is a graph isomorphism $G \to G'$ preserving both parts, possibly exchanging them, and preserving both linear orders. They are *equivalent* if G' can be obtained from G by reversing the orders in one or both parts of G and possibly exchange the two parts.

If G and H are ordered bipartite graphs, then H is called an *interval minor* of G if a graph isomorphic to H can be obtained from G by repeatedly applying the following operations:

(i) deleting an edge;

(ii) identifying two consecutive vertices.

If H is not an interval minor of G, we say that G *avoids* H as an interval minor or that G is *H-interval minor free.* Let $ex(p, q, H)$ denote the maximum number of edges in a bipartite graph with parts of sizes p and q avoiding H as an interval minor.

In classical Turán extremal graph theory, one asks about the maximum number of edges of a graph of order n which has no subgraph isomorphic to a given graph. Originated from problems in computational and combinatorial geometry, the authors in [2, 6, 7] considered Turán type problems for matrices which can be seen as ordered bipartite graphs. In the ordered version of Turán theory, the question is: what is the maximum number edges of an ordered bipartite graph with parts of size p and q with no subgraph isomorphic to a given ordered bipartite graph? More results on this problem and its variations are given in [1, 3, 4, 8, 9]. As another variation, interval minors were recently introduced by Fox in [5] in the study of Stanley-Wilf limits. He gave exponential upper and lower bounds for $ex(n, n, K_{\ell,\ell})$. In this paper, we are interested in the case when H is a complete bipartite graph. We determine the value of $ex(p, q, K_{2,\ell})$ and find bounds on $ex(p, q, K_{3,\ell})$. We note that our definition of interval minors for ordered bipartite graphs is slightly different from Fox's definition for matrices, since we allow exchanging parts of the bipartition, so for us a matrix and its transpose are the same. Of course, when the matrix of H is symmetric, the two definitions coincide.

2 $K_{2,\ell}$ as interval minor

For simplicity, we denote $ex(p, q, K_{2,\ell})$ by $m(p, q, \ell)$. In this section we find the exact value of this quantity. Let $(G; A, B)$ be an ordered bipartite graph where A has ordering $a_1 < a_2 < \cdots < a_p$ and B has ordering $b_1 < b_2 < \cdots < b_q$. The vertices a_1 and b_1 are called *bottom* vertices whereas a_p and b_q are said to be *top* vertices. The degree of a vertex v is denoted by $d(v)$.

Lemma 2.1. *For any positive integers p and q, we have*

$$m(p, q, \ell) \leqslant (\ell - 1)(p - 1) + q.$$

Proof. Let $(G; A, B)$ be a bipartite graph. Suppose that A has ordering $a_1 < a_2 < \cdots < a_p$ and B has ordering $b_1 < b_2 < \cdots < b_q$. For $1 \leqslant i \leqslant p-1$, let

$$A_i = \{b_j \mid \exists\ i_1 \leqslant i < i_2 \text{ such that } a_{i_1}b_j, a_{i_2}b_j \in E(G)\}.$$

Since G is $K_{2,\ell}$-interval minor free, $|A_i| \leqslant \ell - 1$. Each $b_j \in B$ appears in at least $d(b_j) - 1$ of sets A_i, $1 \leqslant i \leqslant p - 1$. It follows that

$$\sum_{i=1}^{q} (d(b_j) - 1) \leqslant \sum_{i=1}^{p-1} |A_i| \leq (\ell - 1)(p - 1).$$

This proves that $|E(G)| \leqslant (\ell - 1)(p - 1) + q$. □

If $(G; A, B)$ and $(G'; A', B')$ are disjoint ordered bipartite graphs and the bottom vertices x, y of G are adjacent and the top vertices x', y' of G' are adjacent, then we denote by $G \oplus G'$ the ordered bipartite graph obtained from $(G \cup G'; A \cup A', B \cup B')$ by identifying x with x' and y with y', where the linear orders of $A \cup A'$ and $B \cup B'$ are such that the vertices of G' precede those of G. The graph $G \oplus G'$ is called the *concatenation* of G and G'.

In the description of $K_{2,\ell}$-interval minor free graphs below, we shall use the following simple observation, whose proof is left to the reader. Let G and G' be vertex disjoint $K_{r,s}$-interval minor free bipartite graphs with $r \geq 2$ and $s \geq 2$ such that the bottom vertices in G are adjacent and the top vertices in G' are adjacent. Then $G \oplus G'$ is $K_{r,s}$-interval minor free.

Example 2.2. We introduce a family of $K_{2,\ell}$-interval minor free bipartite graphs which would turn out to be extremal. Let $\ell \geqslant 3$ and let p and q be positive integers and let $r = \lfloor (p-1)/(\ell-2) \rfloor$ and $s = \lfloor (q-1)/(\ell-2) \rfloor$. We can write $p = (\ell-2)r + e$ and $q = (\ell-2)s + f$, where $1 \leqslant e \leqslant \ell - 2$, $1 \leqslant f \leqslant \ell - 2$. Suppose now that $r < s$. Let H_0 be $K_{e,\ell-1}$ and let H_i be a copy of $K_{\ell-1,\ell-1}$ for $1 \leqslant i \leqslant r$. The concatenation $H = H_0 \oplus H_1 \oplus \cdots \oplus H_r$ is $K_{2,\ell}$-interval minor free by the above observation. It has parts of sizes p and $q' = (\ell-2)(r+1)+1$. It also has $r\ell(\ell-2)+e(\ell-1)$ edges. Finally, let $H^+ = K_{1,q-q'+1}$. The graph $\mathcal{H}_{p,q}(\ell) = H^+ \oplus H$ has parts of sizes p, q and has $(\ell-1)(p-1)+q$ edges. An example is depicted in Figure 1(b), where the identified top and bottom vertices used in concatenations are shown as square vertices.

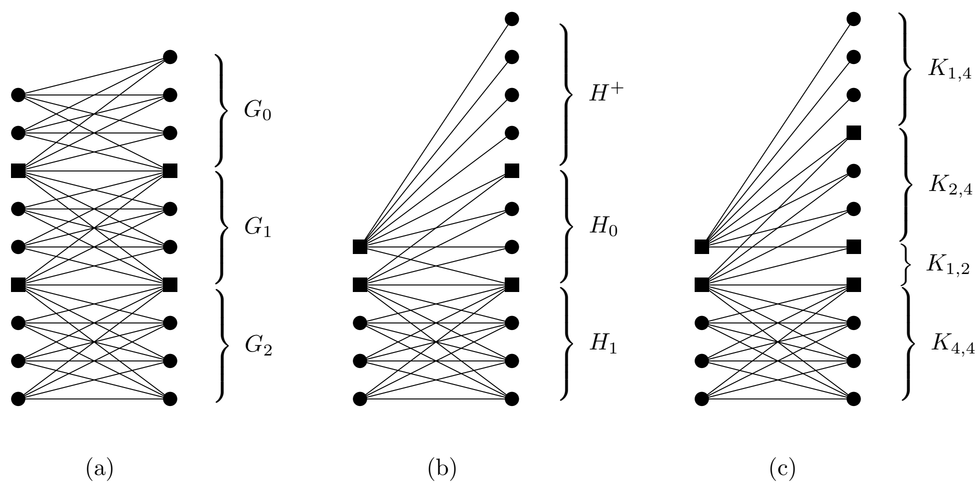

Figure 1: (a) $\mathcal{G}_{9,10}(5)$, (b) $\mathcal{H}_{5,11}(5)$, (c) $K_{1,4} \oplus K_{2,4} \oplus K_{1,2} \oplus K_{4,4}$

By Lemma 2.1 and Example 2.2, the following is obvious.

Theorem 2.3. *Let $\ell \geqslant 3$, $p = (\ell-2)r + e$ and $q = (\ell-2)s + f$, where $1 \leqslant e \leqslant \ell-2$, $1 \leqslant f \leqslant \ell-2$. If $r < s$, then*

$$m(p, q, \ell) = (\ell-1)(p-1) + q.$$

Extremal graphs for excluded $K_{2,\ell}$ given in Example 2.2 are of the form of a concatenation of r copies of $K_{\ell-1,\ell-1}$ together with $K_{e,\ell-1}$ and $K_{1,t}$ where $t = q - (\ell-2)(r+1)$. Note that the latter graph itself is a concatenation of copies of $K_{1,2}$ and that the constituents concatenated in another order than given in the example, are also extremal graphs. For an example, consider the graph in Figure 1(c), which is also extremal for $(p,q,\ell) = (5,11,5)$. Rearranging the order of concatenations is not the only way to obtain examples of extremal graphs. What one can do is also using the following operation. Delete a vertex in B of degree 1, replace it by a degree-1 vertex x adjacent to any vertex $a_i \in A$ which is adjacent to two consecutive vertices b_j and b_{j+1}, and put x between b_j and b_{j+1} in the linear order of B. This gives other extremal examples that cannot always be written as concatenations of complete bipartite graphs.

And there is another operation that gives somewhat different extremal examples. Suppose that G is an extremal graph for (p,q,ℓ) with $r < s$ as above. If A contains a vertex a_i of degree $\ell - 1$ (by Theorem 2.3, degree cannot be smaller since the deletion of that vertex would contradict the theorem), then we can delete a_i and obtain an extremal graph for $(p-1,q,\ell)$. The deletion of vertices of degrees $\ell - 1$ can be repeated. Or we can delete any set of k vertices from A if they are incident to precisely $k(\ell-1)$ edges.

We now proceed with the much more difficult case, in which we have $\lfloor (p-1)/(\ell-2) \rfloor = \lfloor (q-1)/(\ell-2) \rfloor$, i.e. $r = s$.

Example 2.4. Let $\ell \geqslant 3$, $p = (\ell-2)r + e$ and $q = (\ell-2)r + f$, where $1 \leqslant e \leqslant \ell - 2$ and $1 \leqslant f \leqslant \ell - 2$. Similarly as in Example 2.2, let G_0 be $K_{e,f}$ and let G_i be a copy of $K_{\ell-1,\ell-1}$ for $1 \leqslant i \leqslant r$. Let $\mathcal{G}_{p,q}(\ell)$ be the concatenation $G_0 \oplus G_1 \oplus \cdots \oplus G_r$. This graph is $K_{2,\ell}$-interval minor free. It has parts of sizes p, q and has $r\ell(\ell-2) + ef$ edges. An example is illustrated in Figure 1(a).

Theorem 2.5. *Let $\ell \geqslant 3$, $p = (\ell-2)r + e$ and $q = (\ell-2)r + f$, where $1 \leqslant e \leqslant \ell - 2$ and $1 \leqslant f \leqslant \ell - 2$. Then*

$$m(p,q,\ell) = r\ell(\ell-2) + ef.$$

Proof. Since the graphs in Example 2.4 attain the stated bound, it suffices to establish the upper bound, $m(p,q,\ell) \leqslant r\ell(\ell-2) + ef$. Let $(G; A, B)$ be a bipartite graph with parts of sizes p, q and with $m(p,q,\ell)$ edges. Let A have ordering $a_1 < a_2 < \cdots < a_p$ and B have ordering $b_1 < b_2 < \cdots < b_q$. Note that any two consecutive vertices of G have at least one common neighbour. Otherwise, by identifying two consecutive vertices with no common neighbour lying say in A, we obtain a graph with parts of sizes $p-1, q$ and with $m(p,q,\ell)$ edges. This is a contradiction since clearly $m(p,q,\ell) > m(p-1,q,\ell)$.

For $1 \leqslant i \leqslant p-1$, let

$$A_i = \{b_j \mid \exists\ i_1 \leqslant i < i_2 \text{ such that } a_{i_1}b_j, a_{i_2}b_j \in E(G)\}.$$

Also let $A'_i = A_i \setminus \{b_h\}$, where h is the smallest index for which $b_h \in A_i$. Since G is $K_{2,\ell}$-interval minor free, $|A'_i| \leqslant \ell - 2$. For each vertex $b_j \in B$, define

$$D(b_j) = \{a_i \mid j \text{ is the smallest index such that } a_i \text{ is adjacent to } b_j\},$$

and let $d'(b_j) = |D(b_j)|$. Every vertex in $N(b_j) \setminus D(b_j)$ is adjacent to b_j and also to some vertex $b_h \in B$ with $h < j$ and hence

$$d(b_j) - d'(b_j) \leqslant \ell - 1 \tag{1}$$

since G is $K_{2,\ell}$-interval minor free. Let h and h' be the smallest and largest indices such that $a_h, a_{h'} \in N(b_j) \setminus D(b_j)$, respectively. Observe that $h' - h \geqslant d(b_j) - d'(b_j) - 1$. We claim that b_j appears in sets $A'_h, A'_{h+1}, \ldots, A'_{h'-1}$. Let $h \leqslant i < h'$. Since b_j is adjacent to a_h and to $a_{h'}$, we have $b_j \in A_i$. We know that a_h is adjacent to some vertex b_{j_1} with $j_1 < j$. Also $a_{h'}$ is adjacent to some vertex b_{j_2} with $j_2 < j$. Suppose that $j_1 \leqslant j_2$. Now we use the property that every two consecutive vertices of G have at least one common neighbour for consecutive pairs of vertices b_t, b_{t+1} ($t = j_1, \ldots, j_2 - 1$). It follows that there is $j_1 \leqslant j_0 \leqslant j_2$ such that b_{j_0} is in A_i. If $j_2 < j_1$, the same property used for $t = j_2, \ldots, j_1 - 1$ shows that there exists j_0, $j_2 \leqslant j_0 \leqslant j_1$, such that $b_{j_0} \in A_i$. Since $j_0 < j$, from the definition of A'_i, we conclude that $b_j \in A'_i$. So we have proved the claim. We conclude that b_j appears in sets $A'_h, A'_{h+1}, \ldots, A'_{h+t-1}$ for some $1 \leqslant h \leqslant p-1$ and $t = d(b_j) - d'(b_j) - 1$.

Let $S = \{i \mid 1 \leqslant i \leqslant p-1, i \not\equiv 1, \ldots, e-1 \pmod{\ell - 2}\}$. We have $|S| = r(\ell - 1 - e)$. By the conclusion in the last paragraph, each $b_j \in B$ appears in at least $d(b_j) - d'(b_j) - 1$ consecutive sets A'_i. Combined with (1), we conclude that b_j appears in at least $d(b_j) - d'(b_j) - 1 - (e-1)$ of sets A'_i, where $i \in S$. Note that this number is negative for $j = 1$ since $d(b_1) = d'(b_1)$. Now it follows that

$$\sum_{i=2}^{q} (d(b_j) - d'(b_j) - e) \leqslant \sum_{i \in S} |A'_i|. \tag{2}$$

By adding $d(b_1) - d'(b_1)$ to the left side of (2) and noting that $\sum_j d(b_j) = |E(G)|$ and $\sum_j d'(b_j) = p$, we obtain therefrom that

$$|E(G)| - p - eq + e \leqslant r(\ell - 1 - e)(\ell - 2).$$

This in turn yields that $|E(G)| \leqslant r\ell(\ell-2) + ef$, which we were to prove. $\square$

Example 2.4 describes extremal graphs for Theorem 2.5. They are concatenations of complete bipartite graphs, all of which but at most one are copies of $K_{\ell-1,\ell-1}$. If $e = 1$ and $f > 1$, vertices of degree 1 can be inserted anywhere between two consecutive neighbors of their neighbor in A. But in all other cases, we believe that all extremal graphs are as in Example 2.4, except that the order of concatenations can be different.

3 $K_{2,2}$ as interval minor

In this section we determine the structure of $K_{2,2}$-interval minor free bipartite graphs. We first define two families of $K_{2,2}$-interval minor free graphs. For every positive integer $n \geqslant 3$, let $A = \{x, a_1, \ldots, a_{n-1}, z\}$ and $B = \{b_1, y, b'_2, b_2, \ldots, b_{n-1}, b'_{n-1}, t, b_n\}$ with ordering $x < a_1 < \cdots < a_{n-1} < z$ and $b_1 < y < b'_2 < b_2 < \cdots < b'_{n-1} < t < b_n$, respectively. Let R_n be the bipartite graph with parts A, B and edge set

$$E(G) = \{a_i b_i, a_i b_{i+1} \mid 1 \leqslant i \leqslant n-1\} \cup \{xy, a_1 b'_2, a_{n-1} b'_{n-1}, zt\}.$$

Similarly we define a graph S_n for every integer $n \geqslant 2$. Let $A = \{x, a_1, \ldots, a_{n-1}, a'_{n-1}, z, a_n\}$ and $B = \{b_1, y, b'_2, b_2, \ldots, b_n, t\}$ with ordering $x < a_1 < \cdots < a'_{n-1} < z < a_n$ and $b_1 < y < b'_2 < b_2 < \cdots < b_n < t$, respectively. Let S_n be the bipartite graph with parts A, B and edge set

$$E(G) = \{a_i b_i, a_i b_{i+1} \mid 1 \leqslant i \leqslant n-1\} \cup \{xy, a_1 b'_2, a'_{n-1} b_n, zt, a_n b_n\}.$$

For instance, R_5 and S_4 are shown in Figure 2.

Lemma 3.1. *For every positive integers p and q, we have $m(p, q, 2) = p + q - 1$.*

Proof. By Lemma 2.1, $m(p, q, 2) \leqslant p + q - 1$. We construct $K_{2,2}$-interval minor free bipartite graphs with parts of sizes p, q and with $p + q - 1$ edges. This is easy if $p \leqslant 4$. So let $5 \leqslant p \leqslant q$. Consider S_{p-3} and add edges $a_1 y, z b_{p-3}$. Also add $q - p$ vertices into the set B, all of them ordered between y and b'_2, and join each of them to a_1. The resulting graph has parts of size p, q and has $p + q - 1$ edges. □

In what follows we assume that $(G; A, B)$ is a bipartite graph without $K_{2,2}$ as an interval minor. Let A and B have the ordering $a_1 < a_2 < \cdots < a_p$ and $b_1 < b_2 < \cdots < b_q$, respectively. A vertex in G of degree 0 is said to be *reducible*. If $d(a_i) = 1$ and the neighbor b_j of a_i is adjacent to a_{i-1} if

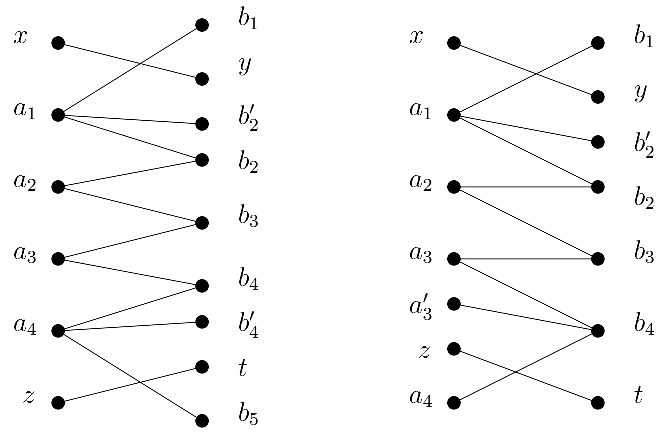

Figure 2: The graphs R_5 and S_4.

$i > 1$ and is adjacent to a_{i+1} if $i < p$, then a_i is also said to be *reducible*. Similarly we define when a vertex $b_j \in B$ is reducible. Clearly, if a_i (or b_j) is reducible, then G has a $K_{2,2}$-interval minor if and only if $G - a_i$ ($G - b_j$) has one. Therefore, we may assume that we remove all reducible vertices from G. When G has no reducible vertices, we say that G is *reduced*, which we assume henceforth.

Let $X = \{a_1, a_2\}$ if $d(a_1) = 1$ and $X = \{a_1\}$, otherwise. Similarly, let $Y = \{a_{p-1}, a_p\}$ if $d(a_p) = 1$ and $Y = \{a_p\}$, otherwise; $Z = \{b_1, b_2\}$ if $d(b_1) = 1$ and $Z = \{b_1\}$, otherwise; $T = \{b_{q-1}, b_q\}$ if $d(b_q) = 1$ and $T = \{b_q\}$, otherwise. We may assume that all these sets are mutually disjoint. Otherwise G has a simple structure – it is equivalent to a subgraph of a graph shown in Figure 3 and any such graph has no $K_{2,2}$ as interval minor. Note that each such subgraph becomes equivalent to a subgraph of R_2 after removing reducible vertices.

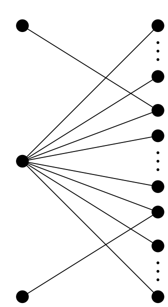

Figure 3: X and Y intersect only in special situations.

Claim 3.2. *There is an edge from X to b_1 or b_q.*

Proof. Suppose that there is no edge from X to $\{b_1, b_q\}$. Since G is reduced, there are two distinct vertices b_i and b_j $(1 < i < j < q)$ connected to X. Assume that b_1 and b_q are adjacent to a_k and a_l, respectively. Note that $a_k, a_l \notin X$. Consider the sets X, $A \setminus X, \{b_1, \ldots, b_i\}$ and $\{b_{i+1}, \ldots, b_q\}$ and identify them to single vertices to get $K_{2,2}$ as an interval minor, a contradiction. □

Note that Claim 3.2 also applies to Y, Z and T. Hence, considering an equivalent graph of G instead of G if necessary, we may assume that there is an edge from X to Z. If there is no edge from Y to T, then there are edges from Y to Z and from T to X. By reversing the order of B, we obtain an equivalent graph that has edges from X to Z and from Y to T. Thus we may assume henceforth that the following claim holds:

Claim 3.3. *The graph G has edges from X to Z and from Y to T.*

Claim 3.4. *Every vertex of G has degree at most 2, except possibly one of a_2, b_2 and/or one of a_{p-1}, b_{q-1}, which may be of degree 3. If $d(a_2) = 3$, then it has neighbors b_1, b_3, b_4, we have $d(a_1) = d(b_1) = d(b_2) = 1$ and $a_1b_2 \in E(G)$. Similar situations occur when b_2, a_{p-1}, or b_{q-1} are of degree 3.*

Proof. Suppose that $d(a_i) \geq 3$. We claim that a_i has at most one neighbour in Z. Otherwise, $|Z| \geq 2$ and hence $d(b_1) = 1$ and $a_ib_1, a_ib_2 \in E(G)$. This is a contradiction since G is reduced. Similarly we see that a_i has at most one neighbour in T.

Suppose now that a middle neighbor b_j of a_i is in $B \setminus (Z \cup T)$. Let b_{j_1} and b_{j_2} be neighbors of a_i with $j_1 < j < j_2$. If $d(b_j) > 1$, then an edge a_kb_j $(k \neq i)$, the edges joining X and Z and joining Y and T, and the edge $a_ib_{j_1}$ (if $k < i$) or $a_ib_{j_2}$ (if $k > i$) can be used to obtain a $K_{2,2}$-interval minor. Thus, $d(b_j) = 1$.

Let us now consider b_{j-1}. Suppose that b_{j-1} is not adjacent to a_i. Then $j_1 < j - 1$. If b_{j-1} is adjacent to a vertex a_k, where $k < i$, then the edges $a_ib_{j_1}, a_kb_{j-1}$ and the edges joining X with Z and Y with T give rise to a $K_{2,2}$-interval minor in G (which is excluded), unless the following situation occurs: the edge a_kb_{j-1} is equal to the edge joining X and Z. This is only possible if $j_1 = 1$, $j = 3$ and $|Z| = 2$, i.e., $d(b_1) = 1$. If a_1 is adjacent to b_1 or to some other b_t with $t > 2$, we obtain a $K_{2,2}$-interval minor again. So, it turns out that $k = 1$ and $d(a_1) = 1$. If $i > 2$, then we consider a neighbor of a_2. It cannot be b_2 since then a_1 would be reducible. It can neither be b_1 or b_t with $t > 2$ since this would yield a $K_{2,2}$-interval minor. Thus $i = 2$.

Similarly, a contradiction is obtained when $k > i$. (Here we do not have the possibility of an exception as in the case when $k = 1$.) Thus, we conclude that b_{j-1} is adjacent to a_i or we have the situation that $i = 2$, $j = 3$, etc. as described above. Similarly we conclude that b_{j+1} is adjacent to a_i unless we have $i = p - 1$, $j = q - 2$, etc. Note that we cannot have the exceptional situations in both cases at the same time since then we would have $i = 2 = p - 1$ and $X \cap Y$ would be nonempty. If $a_i b_{j-1}$ and $a_i b_{j+1}$ are both edges, then b_j would be reducible, a contradiction. Thus, the only possibility for a vertex of degree more than 2 is the one described in the claim. □

Claim 3.5. *We have a_1 adjacent to b_1 or we have a_1 adjacent only to b_2 and b_1 adjacent only to a_2.*

Proof. Suppose that $a_1 b_1 \notin E(G)$. By Claim 3.3, X is adjacent to Z and Y to T. If X is adjacent to a vertex $b_j \notin Z$ and Z is adjacent to a vertex $a_i \notin X$, then we have a $K_{2,2}$-interval minor in G. Thus, we may assume that X has no neighbors outside Z. Since $a_1 b_1 \notin E(G)$, we have that $a_1 b_2 \in E(G)$. In particular, $d(a_1) = 1$ and $d(b_1) = 1$. Then $a_2 \in X$ and $b_2 \in Z$. Since G is reduced, $a_2 b_2 \notin E(G)$. Since all neighbors of X are in Z, we conclude that a_2 is adjacent to b_1. This yields the claim. □

The same argument applies to the bottom vertices.

We can now describe the structure of $K_{2,2}$-interval minor free graphs. In fact, we have proved the following theorem.

Theorem 3.6. *Every reduced bipartite graph with no $K_{2,2}$ as an interval minor is equivalent to a subgraph of R_n or S_n for some positive integer n.*

A matching of size n is a 1-regular bipartite graph on $2n$ vertices. The following should be clear from Theorem 3.6.

Corollary 3.7. *For every integer $n \geq 4$, there are exactly eight $K_{2,2}$-interval minor free matchings of size n. They form three different equivalence classes.*

4 $K_{3,\ell}$ as interval minor

For $K_{3,\ell}$-interval minors in bipartite graphs, we start in a similar manner as when excluding $K_{2,\ell}$. We first establish a simple upper bound, which will later turn out to be optimal in the case when the sizes of the two parts are not very balanced.

Lemma 4.1. *For any integers $\ell \geq 1$ and $p, q \geq 2$, we have*

$$ex(p, q, K_{3,\ell}) \leqslant (\ell - 1)(p - 2) + 2q.$$

Proof. Let $(G; A, B)$ be a bipartite graph with parts of sizes p and q. Suppose that A has ordering $a_1 < a_2 < \cdots < a_p$ and B has ordering $b_1 < b_2 < \cdots < b_q$. For $2 \leqslant i \leqslant p - 1$, let

$$A_i = \{b_j \mid a_i b_j \in E(G), \exists\, i_1 < i < i_2 \text{ such that } a_{i_1} b_j, a_{i_2} b_j \in E(G)\}.$$

If G is $K_{3,\ell}$-interval minor free, we have $|A_i| \leqslant \ell - 1$. Each $b_j \in B$ of degree at least 2 appears in precisely $d(b_j) - 2$ of the sets A_i, $2 \leqslant i \leqslant p - 1$. It follows that

$$\sum_{j=1}^{q} (d(b_j) - 2) \leqslant \sum_{i=2}^{p-1} |A_i|.$$

This gives $|E(G)| \leqslant (\ell - 1)(p - 2) + 2q$, as desired. □

Let $(G; A, B)$ and $(G'; A', B')$ be disjoint ordered bipartite graphs. Let a_{p-1}, a_p be the last two vertices in the linear order in A and let b_{q-1}, b_q be the last two vertices in B. Denote by a'_1, a'_2 and b'_1, b'_2 the first two vertices in A' and B', respectively. Let us denote by $G \oplus_2 G'$ the ordered bipartite graph obtained from G and G' by identifying a_{p-1} with a'_1, a_p with a'_2, b_{q-1} with b'_1, and b_q with b'_2. The resulting ordered bipartite graph $G \oplus_2 G'$ is called the 2-*concatenation* of G and G'. We have a similar observation as used earlier for $K_{2,\ell}$-free graphs. If a_{p-1}, a_p and b_{q-1}, b_q form $K_{2,2}$ in G and a'_1, a'_2 and b'_1, b'_2 form $K_{2,2}$ in G', and $r \geq 3$ and $s \geq 3$, then $G \oplus_2 G'$ is $K_{r,s}$-interval minor free if and only if G and G' are both $K_{r,s}$-interval minor free.

Example 4.2. Let $\ell \geqslant 4$, $p = (\ell - 3)r + e$ and $q = (\ell - 3)s + f$ where $2 \leqslant e \leqslant \ell - 2$, $2 \leqslant f \leqslant \ell - 2$ and $r < s$. Let $\mathcal{K}_{p,q}(\ell)$ be the 2-concatenation of $K_{e,\ell-1}$, r copies of $K_{\ell-1,\ell-1}$ and $K_{2,q-(\ell-3)(r+1)}$. This graph has parts of sizes p and q and has $(\ell - 1)(p - 2) + 2q$ edges.

By Lemma 4.1 and Example 4.2, the following is clear.

Theorem 4.3. *Let $\ell \geqslant 4$, $p = (\ell - 3)r + e$ and $q = (\ell - 3)s + f$ where $2 \leqslant e \leqslant \ell - 2$, $2 \leqslant f \leqslant \ell - 2$ and $r < s$. Then*

$$ex(p, q, K_{3,\ell}) = (\ell - 1)(p - 2) + 2q.$$

We now consider the remaining cases, where both parts are "almost balanced", i.e., $\lfloor (p - 2)/(\ell - 3) \rfloor = \lfloor (q - 2)/(\ell - 3) \rfloor$.

Example 4.4. Let $\ell \geqslant 4$, $p = (\ell - 3)r + e$ and $q = (\ell - 3)r + f$ where $2 \leqslant e \leqslant \ell - 2$ and $2 \leqslant f \leqslant \ell - 2$. Let $\mathcal{K}_{p,q}(\ell)$ be the 2-concatenation of $K_{e,f}$ and r copies of $K_{\ell-1,\ell-1}$. This graph is $K_{3,\ell}$-interval minor free, has parts of sizes p and q, and has $r(\ell - 3)(\ell + 1) + ef$ edges. It follows that

$$ex(p, q, K_{3,\ell}) \geqslant r(\ell - 3)(\ell + 1) + ef.$$

We conjecture that this is in fact the exact value for $ex(p, q, K_{3,\ell})$. Unfortunately, we have not been able to adopt the proof of Theorem 2.5 for this case.

References

[1] M.H. Alberta, M. Elderb, A. Rechnitzerc, P. Westcottd, and M.Zabrockie, On the StanleyWilf limit of 4231-avoiding permutations and a conjecture of Arratia, Advances in Applied Mathematics 36 (2), (2006), 960–105.

[2] D. Bienstock and E. Györi, An extremal problem on sparse 0-1 matrices, SIAM J. Discrete Math. 4 (1991), 17–27.

[3] P. Brass, G. Karolyi and P. Valtr, A Turán-type extremal theory for convex geometric graphs, in: *Discrete and Computational Geometry. The Goodman-Pollack Festschrift*, Algorithms Combinatorics 25, Springer, Berlin, 2003, pp. 275–300.

[4] A. Claessona, V. Jelnekb, and E. Steingrmssona, Upper bounds for the Stanley-Wilf limit of 1324 and other layered patterns, J. Combin. Theory Ser. A 119 (8) (2012), 1680–1691.

[5] J. Fox, Stanley-Wilf limits are typically exponential, arXiv:1310.8378.

[6] Z. Füredi, The maximum number of unit distances in a convex n-gon, J. Combin. Theory Ser. A 55 (1990), 316–320.

[7] Z. Füredi and P. Hajnal, Davenport-Schinzel theory of matrices, Discrete Math. 103 (1992), 233–251.

[8] M. Klazar, Extremal problems for ordered (hyper) graphs: applications of Davenport-Schinzel sequences, European J. Combin. 25 (2004), 125–140.

[9] J. Pach and G. Tardos, Forbidden paths and cycles in ordered graphs and matrices, Israel J. Math. 155 (2006), 359–380.