On a family of diamond-free strongly regular graphs

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Abstract

The existence of a partial quadrangle $\mathsf{PQ}(s, t, \mu)$ is equivalent to the existence of a diamond-free strongly regular graph $\mathsf{SRG}(1+s(t+1)+s^2t(t+1)/\mu, s(t+1), s-1, \mu)$. Let $\mathcal{S}$ be a $\mathsf{PQ}(3, (n+3)(n^2-1)/3, n^2+n)$ such that for every two non-collinear points p_1 and p_2, there is a point q non-collinear with p_1, p_2, and all points collinear with both p_1 and p_2. In this article, we establish that $\mathcal{S}$ exists only for $n \in \{-2, 2, 3\}$ and probably $n = 10$.

Key words and phrases: adjacency matrix, eigenvalue multiplicity, automorphism group, diamond-free graph, negative Latin square graph, partial quadrangle, strongly regular graph, transitive graph.

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I. Introduction

A *strongly regular graph* with parameters (ν, k, λ, μ), denoted by $\mathsf{SRG}(\nu, k, \lambda, \mu)$, is a regular graph of order ν and valency k such that (i) it is not complete or edgeless, (ii) every two adjacent vertices have λ common neighbors, and (iii) every two non-adjacent vertices have μ common neighbors. The concept of strongly regular graphs was first introduced by Bose and Shimamoto in [4]. Strongly regular graphs form an important class of graphs and lie somewhere between highly structured graphs and apparently random graphs. They often appear in different areas such as coding theory, design theory, discrete geometry, group theory, and so on. Obviously, complete multipartite graphs with equal part sizes and their complements are trivial examples of strongly regular graphs. To exclude these examples, we assume that a strongly regular graph and its complement are connected; or equivalently, $0 < \mu < k < \nu - 1$.

The *adjacency matrix* of a graph G, denoted by $\mathcal{A}_G$, has its rows and columns indexed by the vertex set of G and its (i, j)-entry is 1 if the vertices i and j are adjacent and 0 otherwise. The zeros of the characteristic polynomial of $\mathcal{A}_G$ are called the *eigenvalues* of G. The statement that G is an $\mathsf{SRG}(\nu, k, \lambda, \mu)$ is equivalent to

$$\mathcal{A}_G J_\nu = k J_\nu \quad \text{and} \quad \mathcal{A}_G^2 + (\mu - \lambda)\mathcal{A}_G + (\mu - k) I_\nu = \mu J_\nu,$$

where I_t and J_t are the $t \times t$ identity matrix and the $t \times t$ all one matrix, respectively. It is easy to verify that the eigenvalues of an $\mathsf{SRG}(\nu, k, \lambda, \mu)$ are

$$\begin{aligned} &k, \text{ with the multiplicity } 1;\\ &r = \frac{\lambda - \mu + \sqrt{\varDelta}}{2}, \text{ with the multiplicity } f = \frac{\nu - 1}{2} - \frac{2k + (\nu - 1)(\lambda - \mu)}{2\sqrt{\varDelta}};\\ &s = \frac{\lambda - \mu - \sqrt{\varDelta}}{2}, \text{ with the multiplicity } g = \frac{\nu - 1}{2} + \frac{2k + (\nu - 1)(\lambda - \mu)}{2\sqrt{\varDelta}}, \end{aligned}$$

where $\varDelta = (\lambda - \mu)^2 + 4(k - \mu)$. It is well known that the second largest eigenvalue of a graph G is non-positive if and only if the non-isolated vertices of G form a complete multipartite graph. Also, it is a known fact that the smallest eigenvalue of a graph G is at least -1 if and only if G is a disjoint union of some complete graphs. So, for any $\mathsf{SRG}(\nu, k, \lambda, \mu)$, we necessarily have $r > 0$ and $s < -1$.

The *diamond* is the graph on four vertices with five edges. A graph with no diamond as an induced subgraph is called *diamond-free*. It is straightforward to see that a graph is diamond-free if and only if the neighborhood of any vertex is a disjoint union of some complete graphs. Furthermore, an $\mathsf{SRG}(\nu, k, \lambda, \mu)$ is diamond-free if and only if $\lambda + 1 \mid k$ and the neighborhood of each vertex is $\frac{k}{\lambda+1} K_{\lambda+1}$.

A *partial quadrangle* with parameters (s, t, μ), denoted by $\mathsf{PQ}(s, t, \mu)$, is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{L}$ are disjoint non-empty sets of elements called points and lines, respectively, and $\mathcal{I} \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ is a symmetric incidence relation satisfying the following conditions:

(i) Each line is incident with $s + 1$ points and each point is incident with $t + 1$ lines.

(ii) Every two distinct points are incident with at most one line.

(iii) For each non-incident pair $(p, \ell) \in \mathcal{P} \times \mathcal{L}$, there is at most one pair $(p', \ell') \in \mathcal{P} \times \mathcal{L}$ such that the both p, p' are incident with ℓ' and p' is incident with ℓ.

(iv) For every two non-collinear points, there are exactly μ points collinear with both of them.

Partial quadrangles were firstly introduced by Cameron in [5]. Clearly, for any $\mathsf{PQ}(s, t, \mu)$, we necessarily have $\mu \leqslant t + 1$. In the literature, a $\mathsf{PQ}(s, t, t + 1)$ is called a *generalized quadrangle* and is denoted by $\mathsf{GQ}(s, t)$. The *collinearity graph* of a $\mathsf{PQ}(s, t, \mu)$ is the graph whose vertices are the points and two vertices are adjacent if they are collinear. It is straightforward to verify that the collinearity graph of a $\mathsf{PQ}(s, t, \mu)$ is a diamond-free

$$\mathsf{SRG}\left(1 + s(t + 1) + \frac{s^2t(t + 1)}{\mu}, s(t + 1), s - 1, \mu\right).$$

Inversely, a diamond-free strongly regular graph is the collinearity graph of a partial quadrangle whose points are vertices of the graph and lines are maximal cliques of the graph. So, an $\mathsf{SRG}(\nu, k, \lambda, \mu)$ with $\lambda \leqslant 1$ or $\mu = 1$ is the collinearity graph of a partial quadrangle.

Recently, Bondarenko and Radchenko showed in [3] that a $\mathsf{PQ}(2, (n^3 + 3n^2 - 2)/2, n^2 + n)$, or equivalently, an $\mathsf{SRG}((n^2 + 3n - 1)^2, n^2(n + 3), 1, n(n + 1))$, exists if and only if $n \in \{1, 2, 4\}$. Let $\mathcal{S}$ be a $\mathsf{PQ}(3, (n + 3)(n^2 - 1)/3, n^2 + n)$ such that for every two non-collinear points p_1 and p_2, there is a point q non-collinear with p_1, p_2, and all points collinear with both p_1 and p_2. In this article, we will show that if $\mathcal{S}$ exists, then $n \in \{-2, 2, 3, 10\}$. Equivalently, we will establish the following theorem.

Theorem 1. *If there exists a diamond-free* $\mathsf{SRG}((n^2 + 3n - 2)^2, n(n^2 + 3n - 1), 2, n(n + 1))$, *for some integer* n, *satisfying the following condition:*

For every two non-adjacent vertices u and v, there is a vertex that is not adjacent to u, v, and all common neighbors of u and v, (1)

then $n \in \{-2, 2, 3, 10\}$.

For $n = -2$, there are exactly two non-isomorphic $\mathsf{SRG}(16, 6, 2, 2)$; these are the lattice graph $\mathscr{L}_{4,4}$, that is the Cartesian product of two copies of K_4, and the Shrikhande graph. Only the first one is diamond-free. For details of these facts, see [6] and the references therein. In case $n = 2$, it is demonstrated in [6] that there exist precisely 167 non-isomorphic $\mathsf{SRG}(64, 18, 2, 6)$ and only one of them is diamond-free. For $n = 3$, we are aware of a diamond-free $\mathsf{SRG}(256, 51, 2, 12)$ which is found in [7]. The uniqueness of such a graph seems to be unknown. Note that all these three diamond-free strongly regular graphs satisfy (1). For $n = 10$, the question whether there exists a diamond-free $\mathsf{SRG}(16384, 1290, 2, 110)$ is left as an open problem. Finally, we believe that Theorem 1 holds without assuming the condition (1).

II. Notation and Preliminaries

We first recall some notation from graph theory. For a graph G, the vertex set of G is denoted by $V(G)$. We employ the notation $u \sim v$ when two vertices $u, v \in V(G)$ are adjacent. For any vertices $v_1, \ldots, v_t \in V(G)$, we let

$$N(v_1, \ldots, v_t) = \{x \in V(G) \,|\, x \sim v_i, \text{ for } i = 1, \ldots, t\}.$$

For every two subsets S and T of $V(G)$, we denote by $\langle S, T\rangle$ the induced subgraph of G on all edges with one endpoint in S and the other endpoint in T. For simplicity, we will use the notation $N[v]$, $\overline{N}(v)$, and $\langle S\rangle$ instead of $N(v) \cup \{v\}$, $V(G) \setminus (N(v) \cup \{v\})$, and $\langle S, S\rangle$, respectively.

It is a simple and well known fact that a strongly regular graph whose valency is equal to the multiplicity of a non-principal eigenvalue is either a conference graph, that is an $\mathsf{SRG}(n, (n-1)/2, (n-5)/4, (n-1)/4)$, or an

$$\mathsf{SRG}((n^2 + 3n - \lambda)^2, n(n^2 + 3n - \lambda + 1), \lambda, n(n + 1)), \tag{2}$$

for some integer n; depending on $f = g$ or not. Let G be a graph of the family given by (2). The eigenvalues of G are n with the multiplicity $\nu - 1 - k$ and $\lambda - n^2 - 2n$ with the multiplicity k. Traditionally, if $n > 0$, then $g = k$ and G is called a *negative Latin square* graph and if $n < 0$, then $f = k$ and G is called a *pseudo Latin square* graph. Note that if $n < 0$, then $\lambda - n^2 - 2n > 0$ and so $n > -1 - \sqrt{1 + \lambda}$. This means that, for a fixed parameter λ, there are only finitely many strongly regular graphs with $f = k$. In this article, we only deal with strongly regular graphs with $f \neq g$ and $g = k$.

Let G be a diamond-free $\mathsf{SRG}(\nu, k, \lambda, \mu)$ in the family (2) with $0 \leqslant \lambda \leqslant n - 1$. Fix a vertex $u \in V(G)$ and assume that $\langle N(u)\rangle = sK_{\lambda+1}$, where $s = k/(\lambda + 1)$. Letting $H = \langle N[u]\rangle$, we may write

$$\mathcal{A}_G = \begin{bmatrix} X & Y \\ Y^\top & \mathcal{A}_H \end{bmatrix}, \tag{3}$$

for some matrices X and Y. Since $\lambda \leqslant n-1$, n is not an eigenvalue of H. With an easy calculation, we find that

$$n(n+1)^2(n-\lambda)(nI_{k+1}-\mathcal{A}_H)^{-1} = \left[\begin{array}{c|c} (aI_{\lambda+1}+\mu J_{\lambda+1})\otimes I_s & b\boldsymbol{j}_k \\ \hline b\boldsymbol{j}_k^\top & c \end{array}\right] - J_{k+1}, \tag{4}$$

where $a = \mu(n-\lambda)$, $b = \lambda+1-n$, $c = (\lambda+1-n)(n+1-\lambda)$, and $\boldsymbol{j}_k$ is the all one column vector of length k. For every two vertices $v, w \in \overline{N}(u)$, let $p_u(v,w) = |N(u,v,w)|$ and $q_u(v,w)$ be the number of pairs $x \sim y$ with $x \in N(u,v)$ and $y \in N(u,w)$. Since $g = k$, we have $rank\,(nI_\nu - \mathcal{A}_G) = rank\,(nI_{k+1} - \mathcal{A}_H)$, which implies by (3) that

$$nI_{\nu-k-1} - X = Y(nI_{k+1} - \mathcal{A}_H)^{-1}Y^\top. \tag{5}$$

Using (4) and (5), it is not hard to see that

$$(n-\lambda+1)p_u(v,w) + q_u(v,w) = \begin{cases} \lambda(n+1), & \text{if } v \sim w; \\ \mu, & \text{otherwise,} \end{cases} \tag{6}$$

for every two vertices $v, w \in \overline{N}(u)$.

Now, fix a vertex $v \in \overline{N}(u)$ and set $t = \lfloor \mu/(n-\lambda+1) \rfloor$. For $i = 0, 1, \ldots, t$, let $M_i(u,v)$ be the set of all vertices $x \notin N[u] \cup N[v]$ with $p_u(v,x) = i$, and put $m_i(u,v) = |M_i(u,v)|$. By a double counting argument, it is straightforward to find that

$$\begin{cases} \displaystyle\sum_{i=0}^{t} m_i(u,v) = \nu - 2k + \mu - 2; \\ \displaystyle\sum_{i=0}^{t} i m_i(u,v) = \mu(k - 2\lambda - 2); \\ \displaystyle\sum_{i=0}^{t} \binom{i}{2} m_i(u,v) = (\mu-2)\binom{\mu}{2}. \end{cases} \tag{7}$$

Notice that G satisfies (1) if and only if $m_0(u,v) \neq 0$ for every two non-adjacent vertices $u, v \in V(G)$.

III. The Proof of Theorem 1

In this section, we give a proof of Theorem 1. Let $\mathbb{G}$ be a diamond-free $\mathsf{SRG}((n^2+3n-2)^2, n(n^2+3n-1), 2, n(n+1))$, for some integer $n \geqslant 3$, satisfying (1). We will demonstrate that either $n = 3$ or $n = 10$. In the following lemma, we solve the system (7) for each pair $u \nsim v$ of vertices of $\mathbb{G}$. For any vertex $u \in V(\mathbb{G})$, we denote by $\Phi(u)$ the partition of $N(u)$ into cliques of size 3.

Lemma 2. *For every two non-adjacent vertices $u, v \in V(\mathbb{G})$, the system (7) has the unique solution*

$$\begin{cases} m_0(u,v) = 2; \\ m_1(u,v) = \cdots = m_{n-1}(u,v) = 0; \\ m_n(u,v) = n(n+2)(n^2-1); \\ m_{n+1}(u,v) = 2n(n^2-4); \\ m_{n+2}(u,v) = n(n+1). \end{cases} \tag{8}$$

Moreover, if $M_0(u,v) = \{a, b\}$, for some vertices $a, b \in V(\mathbb{G})$, then $a \nsim b$, $p_u(a,b) = 0$, and any element of $\Phi(u)$ which meets $N(v)$, also meets both $N(a)$ and $N(b)$.

Proof. Fix two non-adjacent vertices $u, v \in V(\mathbb{G})$. Since $\mathbb{G}$ satisfies (1), there exists a vertex $a \in M_0(u,v)$. We first establish the following steps.

Step 1. $\langle M_0(u,v), M_{n+2}(u,v)\rangle$ is complete bipartite.

By contrary, suppose that $x \in M_0(u,v)$ is not adjacent to $y \in M_{n+2}(u,v)$. Since $q_u(v,x) = \mu$, $p_u(v,y) = 2$, and $q_u(v,y) = n+2$, one can easily deduce that $q_u(x,y) \geqslant n+2$ and $p_u(x,y)+q_u(x,y) = n+4$. Further, we have from (6) that $(n-1)p_u(x,y)+q_u(x,y) = \mu$. These two equalities yield that $q_u(x,y) = 2$, a contradiction.

Step 2. $\langle N(u,a), N(v,a)\rangle$ is 1-regular.

Consider an arbitrary vertex $x \in N(v,a)$. Since $\langle N[v]\rangle$ is a disjoint union of triangles, $p_u(v,x) = 1$ and so (6) implies that $q_u(v,x) = n+3$. This shows that $p_u(a,x)+q_u(a,x) = n+4$. Again, (6) yields that $p_u(a,x) = 1$, as required.

Step 3. $m_{n+2}(u,v) \leqslant \mu$.

Consider an arbitrary vertex $x \in M_{n+2}(u,v)$. Since $q_u(v,a) = \mu$, $p_u(v,x) = n+2$, and $q_u(v,x) = 2$, we conclude that $p_u(a,x) + q_u(a,x) = n+4$. By Step 1 and (6), we find that $p_u(a,x) = 1$ and similarly, $p_v(a,x) = 1$. Let $N(u,a,x) = \{u'\}$ and $N(v,a,x) = \{v'\}$. Since $\mathbb{G}$ is diamond-free, $u' \sim v'$. It follows from Step 2 that $m_{n+2}(u,v) \leqslant \mu$, as desired.

Step 4. $m_0(u,v) \leqslant 2$ and the 'Moreover' statement holds.

For every two vertices $x, y \in M_0(u,v)$, we have $p_u(x,y) + q_u(x,y) = \mu$ and by (6), $(n-1)p_u(x,y) + q_u(x,y) = \epsilon(n+1)$, where $\epsilon \in \{2, n\}$. This yields that $p_u(x,y) = 0$ and $x \nsim y$. Since $\langle N[u]\rangle$ is a disjoint union of triangles, we must have $m_0(u,v) \leqslant 2$. If $M_0(u,v) = \{a,b\}$, then (6) forces that $q_u(v,a) = q_u(v,b) = \mu$. This shows clearly that the 'Moreover' statement is valid.

Step 5. Let $\{u, v_1, w_1\}$ be an independent set with $p_u(v_1,w_1) \neq 0$. Then $p_u(v_1,w_1) \geqslant n$.

Let $v_2 \in M_0(u, v_1)$ and $w_2 \in M_0(u, w_1)$. Since $p_u(v_1, w_1) \neq 0$, Step 4 shows that $v_2 \neq w_2$. Let t denote the number of elements in $\Phi(u)$ meeting both $N(v_1)$ and $N(w_1)$. Using Step 4 and (6), we have

$$(n-1)p_u(v_i, w_j) + \big(t - p_u(v_i, w_j)\big) = \epsilon_{ij}(n+1), \qquad \text{for } i, j \in \{1, 2\}, \tag{9}$$

where $\epsilon_{ij} = 2$, if $v_i \sim w_j$ and $\epsilon_{ij} = n$, otherwise. Since $n \geqslant 3$ and $p_u(v_1, w_1) + p_u(v_1, w_2) + p_u(v_2, w_1) + p_u(v_2, w_2) = t$, summing up the four formulae given in (9), we obtain that $t \leqslant 4\mu/(n+2)$. The equality (9) for $i = j = 1$ yields that $p_u(v_1, w_1) \geqslant \mu/(n+2) > n-1$, as we wanted to prove.

We are now prepared to solve the system (7) for $\mathbb{G}$. Obviously, Step 5 means that $m_1(u, v) = \cdots = m_{n-1}(u, v) = 0$. Solving the system (7) in terms of $m_n(u, v)$, $m_{n+1}(u, v)$, $m_{n+2}(u, v)$, we obtain that

$$\begin{cases} m_n(u, v) = (n+1)(n+2)(n^2 - n + 1) - \binom{n+2}{2} m_0(u, v); & (10) \\ m_{n+1}(u, v) = 2n(n+2)(n-3) + n(n+2)m_0(u, v); & (11) \\ m_{n+2}(u, v) = 2n(n+1) - \binom{n+1}{2} m_0(u, v). & (12) \end{cases}$$

From (12) and using Steps 3 and 4, we deduce that $m_0(u, v) = 2$ and $m_{n+2}(u, v) = n(n+1)$. Now, the solution (8) is clearly obtained from (10) and (11). □

Consider a vertex $u \in V(\mathbb{G})$. Obviously, Lemma 2 shows that $\overline{N}(u)$ has a partition $\Psi(u)$ into independent sets of size 3 such that $p_u(x, y) = 0$, for every two distinct vertices x and y belonging to an element of $\Psi(u)$. Notice that for every subsets $\phi \in \Phi(u)$ and $\psi \in \Psi(u)$, $\langle \phi, \psi \rangle$ is either edgeless or 1-regular. In the latter case, we say that ϕ and ψ are *matched* together.

Lemma 3. *Let $u \in V(\mathbb{G})$ and let ψ, ψ' be two distinct elements of $\Psi(u)$. Then $\langle \psi, \psi' \rangle$ is r-regular with $r \in \{0, 1, 2\}$. Moreover, for every two vertices $v \in \psi$ and $w \in \psi'$,*

$$p_u(v, w) = \begin{cases} \max\{0, r-1\}, & \text{if } v \sim w; \\ n + r, & \text{otherwise.} \end{cases}$$

Proof. Let $v \in \psi$, $\psi' = \{w_1, w_2, w_3\}$, and $t = p_u(v, w_1) + p_u(v, w_2) + p_u(v, w_3)$. By Lemma 2, t is independent of the choice of v in ψ and $q_u(v, w_i) = t - p_u(v, w_i)$, for $i = 1, 2, 3$. Applying (6), we find for each i that $(n-2)p_u(v, w_i) = \epsilon_i(n+1) - t$, where $\epsilon_i = 2$, if $v \sim w_i$

and $\epsilon_i = n$, otherwise. Summing up these three formulae, we obtain that $\epsilon_1 + \epsilon_2 + \epsilon_3 = t$. It follows from $n \geqslant 3$ that the degrees of the elements in ψ as some vertices of $\langle \psi, \psi' \rangle$ are the same. Clearly, a similar property holds for the elements of ψ'. This shows that $\langle \psi, \psi' \rangle$ is r-regular, for some r. By Lemma 2, $m_2(u, v) = 0$ and so $r \in \{0, 1, 2\}$. The rest of the proof is straightforward. $\square$

Lemma 4. *Let $u \in V(\mathbb{G})$ and let $\psi = \{v_1, v_2, v_3\}$, $\psi' = \{w_1, w_2, w_3\}$ be two distinct elements of $\Psi(u)$ in which $\langle \psi, \psi' \rangle$ is 2-regular and $v_i \nsim w_i$, for $i = 1, 2, 3$. Then for any element $\{a_1, a_2, a_3\} \in \Phi(u)$ matched to both ψ and ψ', there is an permutation $\pi \in \langle (1\,2\,3) \rangle$ such that $a_i \sim v_i$ and $a_i \sim w_{\pi(i)}$, for $i = 1, 2, 3$.*

Proof. By the contrary and with no loss of generality, suppose that there is an element $\{a_1, a_2, a_3\} \in \Phi(u)$ with $a_1 \in N(v_1, w_1)$, $a_2 \in N(v_2, w_3)$, and $a_3 \in N(v_3, w_2)$. Since the neighborhood of each vertex of $\mathbb{G}$ is a disjoint union of triangles, there is a vertex $x \in N(a_2, v_2, w_3)$. Since $\{a_2, w_3, x\} \in \Phi(v_2)$, $\{u, v_1, v_3\} \in \Psi(v_2)$, $a_2 \sim u$, and $w_3 \sim v_1$, we deduce that $x \sim v_3$. Also, since $\{a_2, v_2, x\} \in \Phi(w_3)$, $\{u, w_1, w_2\} \in \Psi(w_3)$, $a_2 \sim u$, and $v_2 \sim w_1$, we conclude that $x \sim w_2$. Thus $\langle \{a_3, v_3, w_2, x\} \rangle$ contains a diamond as a subgraph, which forces that $x \sim a_3$. However, this is impossible, since $\{u, a_1, x\} \subseteq N(a_2, a_3)$. $\square$

Lemma 5. *Let $u \in V(\mathbb{G})$ and let ϕ, ϕ' be two distinct elements of $\Phi(u)$. Then there is a suitable labeling $\phi = \{a_1, a_2, a_3\}$ and $\phi' = \{b_1, b_2, b_3\}$ such that for any element $\{v_1, v_2, v_3\} \in \Psi(u)$ matched to both ϕ and ϕ', the relations $a_i \sim v_i$ and $b_i \sim v_{\pi(i)}$ hold, for any $i \in \{1, 2, 3\}$ and some permutation $\pi \in \langle (1\,2\,3) \rangle$.*

Proof. Let $\mathcal{R}_{ij\ell} = \{\{v_1, v_2, v_3\} \in \Psi(u) \mid v_1 \in N(a_1, b_i), v_2 \in N(a_2, b_j), v_3 \in N(a_3, b_\ell)\}$, for every i, j, ℓ with $\{i, j, \ell\} = \{1, 2, 3\}$. Since each pair $a_i \nsim b_j$ has $\mu - 1$ common neighbors except u, it is easily seen that $|\mathcal{R}_{123}| = |\mathcal{R}_{231}| = |\mathcal{R}_{312}|$ and $|\mathcal{R}_{132}| = |\mathcal{R}_{321}| = |\mathcal{R}_{213}| = \mu - 1 - |\mathcal{R}_{123}|$. Let $\mathcal{S} = \mathcal{R}_{123} \cup \mathcal{R}_{231} \cup \mathcal{R}_{312}$ and $\mathcal{T} = \mathcal{R}_{132} \cup \mathcal{R}_{321} \cup \mathcal{R}_{213}$. The assertion of the lemma is equivalent to that either $\mathcal{S} = \varnothing$ or $\mathcal{T} = \varnothing$. By contrary, suppose that both $\mathcal{S}$ and $\mathcal{T}$ are not empty. We show that the degree of each vertex of $\langle \mathcal{S} \rangle$ is at least $2n$. With no loss of generality, consider $x \in \mathcal{S} \cap N(a_1, b_1)$. It is easily checked by Lemmas 3 and 4 that $\langle \mathcal{S}, \mathcal{T} \rangle$ is edgeless. Since $b_2 \sim b_3$, at least one set in each of pairs $\{N(x, a_2, b_2), N(x, a_2, b_3)\}$ and $\{N(x, a_3, b_2), N(x, a_3, b_3)\}$ is not empty. On the other hand, it follows from (8) that either $p_x(a_i, b_j) = 0$ or $p_x(a_i, b_j) \geqslant n$, for every indices $i, j \in \{2, 3\}$. This clearly means that the degree of x as a vertex of $\langle \mathcal{S} \rangle$ is at least $2n$, as desired. Obviously, the similar property holds for $\langle \mathcal{T} \rangle$. So, the second largest eigenvalue of $\langle \mathcal{S}, \mathcal{T} \rangle = \langle \mathcal{S} \rangle \cup \langle \mathcal{T} \rangle$ would be at least $2n$. This is a contradiction by the interlacing theorem, since the second largest eigenvalue of $\mathbb{G}$ is $r = n$. $\square$

We now proceed to define a permutation σ_u on $V(\mathbb{G})$ of order 3 and then demonstrate that σ_u is in fact an automorphism of $\mathbb{G}$. Put $\sigma_u(u) = u$. Fix an element $\zeta = \{z_1, z_2, z_3\}$ of $\Phi(u)$ and define $\sigma_u(z_1) = z_2$, $\sigma_u(z_2) = z_3$, and $\sigma_u(z_3) = z_1$. We repeatedly do the following process until σ_u is defined on the whole $V(\mathbb{G})$:

> Assume that $\{a_1, a_2, a_3\} \in \Phi(u)$ and $\{v_1, v_2, v_3\} \in \Psi(u)$ form a matched pair with $a_i \sim v_i$, for $i = 1, 2, 3$. If σ_u is already defined on only one of the two triples, then we define σ_u on the other one such that σ_u induces the same permutation on indices of elements of the two triples.

Note that we may first define σ_u on the all elements of $\Psi(u)$ matched with ζ and then we can proceed to define σ_u on each element of $\Phi(u)$, since $\mu > 1$. Finally, σ_u is defined on each element of $\Psi(u)$. We show that σ_u is a well defined permutation. For this, it suffices to demonstrate that

(i) if σ_u is defined on two elements $\psi = \{v_1, v_2, v_3\}$ and $\psi' = \{w_1, w_2, w_3\}$ in $\Psi(u)$ and $\phi = \{a_1, a_2, a_3\} \in \Phi(u)$ is matched to ψ and ψ', then the definitions of σ_u forced by ψ and ψ' on ϕ are the same;

(ii) if σ_u is defined on two elements $\phi = \{a_1, a_2, a_3\}$ and $\phi' = \{b_1, b_2, b_3\}$ of $\Phi(u)$ and $\psi = \{v_1, v_2, v_3\} \in \Psi(u)$ is matched to ϕ and ϕ', then the definitions of σ_u forced by ϕ and ϕ' on ψ are the same.

The assertions (i) and (ii) are direct consequences of Lemmas 4 and 5, respectively. For (i), note that we may assume that ζ is matched to ψ and ψ'. For (ii), note that $z_1 \in M_i(a_1, b_1)$, for some $i \geqslant 1$, and so there is a vertex $w \in N(z_1, a_1, b_1)$. This shows that there is an element in $\Psi(u)$ containing w which matches to ζ, ψ, and ψ'.

The above discussion implies that σ_u is well defined. Also, from the definition of σ_u, we easily see that the subgraphs $\langle N[u] \rangle$ and $\langle N[u], \overline{N}(u) \rangle$ are fixed by σ_u. Therefore, applying (5), $\langle \overline{N}(u) \rangle$ is fixed by σ_u and hence σ_u is an automorphism of $\mathbb{G}$.

As we saw in the above, for each vertex $u \in V(\mathbb{G})$, we can associate to u two automorphisms of $\mathbb{G}$ of order 3, that are the inverse of each other. Fix a vertex $z \in V(\mathbb{G})$ and also fix σ_z to be one of the two automorphisms associated to z. Now, for any arbitrary vertex $u \in V(\mathbb{G})$, let σ_u be that automorphism associated to u satisfying $\sigma_u(z) = \sigma_z^{-1}(u)$.

Lemma 6. *For every two vertices $u, v \in V(\mathbb{G})$, $\sigma_u(v) = \sigma_v^{-1}(u)$.*

Proof. In order to prove the lemma, we need to establish a more general result. For any vertex $u \in V(\mathbb{G})$, fix ${}^u\tau_u$ to be one of the two automorphisms which perviously defined at u. Also, for each other vertex $v \in V(\mathbb{G})$, let ${}^u\tau_v$ be that automorphism defined at v satisfying ${}^u\tau_v(u) = {}^u\tau_u^{-1}(v)$. Consequently, we have ${}^u\tau_v^{-1}(u) = {}^u\tau_u(v)$, for every vertices $u, v \in V(\mathbb{G})$. We claim that ${}^a\tau_b(c) = {}^a\tau_c^{-1}(b)$, for every vertices $a, b, c \in V(\mathbb{G})$. This clearly implies the assertion of the lemma, if we consider z instead of a. We will just prove the claim when a, b, c are mutually distinct, since otherwise the claim follows from the definition. We consider the following seven cases.

Case 1. $a \sim b$, $a \sim c$, $b \sim c$.

In this case, the claim is easily checked from the definition.

Case 2. $a \sim b$, $a \sim c$, $b \nsim c$.

Let $\{b, u, u'\}, \{c, v, v'\} \in \Phi(a)$ and $M_0(b, c) = \{w, w'\}$. From $N(b, c, w) = N(b, c, w') = \varnothing$, we find that $a \nsim w$ and $a \nsim w'$. Also, from $a \notin M_0(w, w') = \{b, c\}$, one concludes that $M_0(a, w)$ and $M_0(a, w')$ are disjoint. Let $M_0(a, w) = \{x, x'\}$ and $M_0(a, w') = \{y, y'\}$. Since $\{a, u, u'\} \in \Phi(b)$, $\{c, w, w'\} \in \Psi(b)$, and $a \sim c$, we may, with no loss of generality, assume that $u \sim w$ and $u' \sim w'$. Similarly, let $v \sim w$ and $v' \sim w'$. Without loss of generality, assume that ${}^a\tau_a(b) = u$ and $b \sim x$. Then ${}^a\tau_b(a) = {}^a\tau_a^{-1}(b) = u'$, which yields that ${}^a\tau_b(c) = w'$. Consider two elements $\{a, x, x'\}, \{b, c, w'\} \in \Psi(w)$. Since $a \in N(b, c)$, Lemma 3 yields that $\langle \{a, x, x'\}, \{b, c, w'\} \rangle$ is 2-regular and so we conclude from $b \sim x$ that $c \sim x'$. Therefore, $x \sim v'$. Since ${}^a\tau_a(b)$ has cycle $(b\, u\, u')$, it also has cycles $(x\, w\, x')$ and $(v'\, v\, c)$. Hence ${}^a\tau_a(c) = v'$, which in turn implies that ${}^a\tau_c^{-1}(a) = {}^a\tau_a(c) = v'$. So ${}^a\tau_c$ has cycle $(v'\, a\, v)$ and so it also has cycle $(w'\, b\, w)$. Thus ${}^a\tau_c^{-1}(b) = w'$, as desired.

Case 3. $a \sim b$, $a \nsim c$, $b \sim c$.

By the definition, either ${}^b\tau_a = {}^a\tau_a$ or ${}^b\tau_a = {}^a\tau_a^{-1}$. We only consider the first equality. The argument is similar, if the second equality occurs. We have ${}^a\tau_b(a) = {}^a\tau_a^{-1}(b) = {}^b\tau_a^{-1}(b) = {}^b\tau_b(a)$. Since ${}^a\tau_b$ and ${}^b\tau_b$ are coincide on $\{a, b\}$, we conclude from the definition that ${}^a\tau_b = {}^b\tau_b$. Also, Case 2 implies that ${}^b\tau_c(a) = {}^b\tau_a^{-1}(c) = {}^a\tau_a^{-1}(c) = {}^a\tau_c(a)$, which yields that ${}^b\tau_c = {}^a\tau_c$. Therefore, ${}^a\tau_b(c) = {}^b\tau_b(c) = {}^b\tau_c^{-1}(b) = {}^a\tau_c^{-1}(b)$, as required.

Case 4. $N(a, b, c) \neq \varnothing$.

Consider a vertex $x \in N(a, b, c)$. We assume that ${}^x\tau_a = {}^a\tau_a$. The argument is similar when ${}^x\tau_a = {}^a\tau_a^{-1}$. Using Cases 1 and 2, we can write ${}^x\tau_b(a) = {}^x\tau_a^{-1}(b) = {}^a\tau_a^{-1}(b) = {}^a\tau_b(a)$. Hence ${}^x\tau_b = {}^a\tau_b$, and similarly, ${}^x\tau_c = {}^a\tau_c$. Therefore, by Cases 1 and 2, we find that ${}^a\tau_b(c) = {}^x\tau_b(c) = {}^x\tau_c^{-1}(b) = {}^a\tau_c^{-1}(b)$, as we wanted to prove.

Case 5. $a \nsim b$, $a \nsim c$, $b \nsim c$.

If $a \in M_0(b,c)$, then the claim is easily checked from the definition. So, let $a \notin M_0(b,c)$, which means that there exists a vertex $x \in N(a,b,c)$. Now we are done by Case 4.

Case 6. $a \nsim b$, $a \nsim c$, $b \sim c$.

It suffices by Case 4 to assume that $N(a,b,c) = \varnothing$. Let $y, y' \in N(a,b)$ and $z \in N(b,y')$. Since $a \nsim b$, we have $y \nsim y'$. We assume that ${}^a\tau_y = {}^y\tau_y$. The argument is similar when ${}^a\tau_y = {}^y\tau_y^{-1}$. By Case 3, we obtain that ${}^a\tau_b(y) = {}^a\tau_y^{-1}(b) = {}^y\tau_y^{-1}(b) = {}^y\tau_b(y)$, which yields that ${}^a\tau_b = {}^y\tau_b$. Since $\langle N(b)\rangle$ and $\langle N(y')\rangle$ are disjoint unions of triangles, $z \notin N(a) \cup N(c) \cup N(y)$. It follows from $y' \in N(a,b,z)$ and Cases 3 and 4 that ${}^y\tau_z(b) = {}^y\tau_b^{-1}(z) = {}^a\tau_b^{-1}(z) = {}^a\tau_z(b)$ and thus ${}^y\tau_z = {}^a\tau_z$. Moreover, it follows from $b \in N(c,y,z)$ and Cases 4 and 5 that ${}^y\tau_c(z) = {}^y\tau_z^{-1}(c) = {}^a\tau_z^{-1}(c) = {}^a\tau_c(z)$ and hence ${}^y\tau_c = {}^a\tau_c$. Since $N(a,b,c) = \varnothing$, we have $c \nsim y$, which together Case 3 imply that ${}^a\tau_b(c) = {}^y\tau_b(c) = {}^y\tau_c^{-1}(b) = {}^a\tau_c^{-1}(b)$, as desired.

Case 7. $a \sim b$, $a \nsim c$, $b \nsim c$.

We assume that ${}^c\tau_a = {}^a\tau_a$. The argument for the case ${}^c\tau_a = {}^a\tau_a^{-1}$ is similar. We have ${}^a\tau_c(a) = {}^a\tau_a^{-1}(c) = {}^c\tau_a^{-1}(c) = {}^c\tau_c(a)$, which implies that ${}^a\tau_c = {}^c\tau_c$. Using Case 6, ${}^c\tau_b(a) = {}^c\tau_a^{-1}(b) = {}^a\tau_a^{-1}(b) = {}^a\tau_b(a)$ and so ${}^c\tau_b = {}^a\tau_b$. Now, we find that ${}^a\tau_b(c) = {}^c\tau_b(c) = {}^c\tau_c^{-1}(b) = {}^a\tau_c^{-1}(b)$, as required.

The proof of the claim is now completed and so the assertion of the lemma follows. □

In order to continue, we need the following result.

Theorem 7. [1, Theorem 3.2] *If π is a non-trivial automorphism of an $\mathsf{SRG}(\nu,k,\lambda,\mu)$ with the second largest eigenvalue r, then the number of fixed points of π is at most*

$$\frac{\nu}{k-r}\max(\lambda,\mu).$$

Corollary 8. *Each non-trivial automorphism of $\mathbb{G}$ has at most $\nu/4$ fixed points.*

Lemma 9. *For every two vertices $u_1, u_2 \in V(\mathbb{G})$, $(\sigma_{u_1}\sigma_{u_2}^{-1})^2$ is equal to the identity.*

Proof. For four distinct vertices $a,b,c,d \in V(\mathbb{G})$, we call the set $\{a,b,c,d\}$ to be *related* if either it is a clique or it is an independent set with $M_0(a,b) = \{c,d\}$. Note that every two distinct vertices of $\mathbb{G}$ is contained in a unique related set. Let $U = \{u_1,u_2,u_3,u_4\}$ be a related set and let $\rho_{ij} = \sigma_{u_i}\sigma_{u_j}^{-1}$, for every $i,j \in \{1,2,3,4\}$. Consider a vertex $x \in V(\mathbb{G}) \setminus U$. By Lemma 6, we find that $\sigma^{-1}_{\sigma^{-1}_{u_i}(x)}(U) = \{\rho_{1i}(x), \rho_{2i}(x), \rho_{3i}(x), \rho_{4i}(x)\}$ and

$\sigma_{u_j}\sigma_x(U) = \{\rho_{j1}(x), \rho_{j2}(x), \rho_{j3}(x), \rho_{j4}(x)\}$ are related, for every $i, j \in \{1, 2, 3, 4\}$. Since every two distinct vertices of $\mathbb{G}$ is contained in a unique related set, it is easily seen that $\sigma^{-1}_{\sigma^{-1}_{u_i}(x)}(U) = \sigma_{u_j}\sigma_x(U)$, for every indices $i \neq j$. It follows that the eight sets which we associated to x in the above are the same. Denote the common set by $\mathcal{H}_x$. Note that if $y \in \mathcal{H}_x$, then $\mathcal{H}_x = \mathcal{H}_y$. Therefore, $\mathscr{P} = \{\mathcal{H}_x \,|\, x \in V(\mathbb{G}) \setminus U\}$ is clearly a partition of $V(\mathbb{G}) \setminus U$ into related sets.

Working towards a contradiction, suppose that $\rho_{12} \neq \rho_{21}$. Consider an arbitrary element $\mathcal{H}_x \in \mathscr{P}$. Since ρ_{12}^2 is a permutation on $\mathcal{H}_x$, $|\mathcal{H}_x| = 4$, and $\rho_{12}(x) \neq x$, we obviously deduce that either ρ_{12}^2 has no fixed point in $\mathcal{H}_x$ or ρ_{12}^2 is the identity on $\mathcal{H}_x$. Thus, Corollary 8 shows that ρ_{12}^2 has no fixed point in at least $\frac{3}{16}\nu - 1$ elements of $\mathscr{P}$.

Assume that ρ_{12}^2 fixes no element of $\mathcal{H}_x = \{x, \rho_{12}(x), \rho_{13}(x), \rho_{14}(x)\}$. So $\rho_{12}(x) \neq \rho_{21}(x)$. We claim that one of $\rho_{12}\rho_{13}$ or $\rho_{12}\rho_{14}$ is the identity on $\mathcal{H}_x$. Note that by Lemma 6, $\rho_{ij}(x) \neq \rho_{ij'}(x)$ and $\rho_{ij}(x) \neq \rho_{i'j}(x)$ whenever $i \neq i'$ and $j \neq j'$. We clearly have $\rho_{21}(x) \in \{\rho_{13}(x), \rho_{14}(x)\}$. Suppose that $\rho_{21}(x) = \rho_{13}(x)$. Since the eight sets which we associated to x in the first paragraph of the proof are equal, one concludes that the elements of $\mathcal{H}_x \setminus \{x\}$ are

$$\begin{cases} \rho_{12}(x) = \rho_{24}(x) = \rho_{31}(x), \\ \rho_{13}(x) = \rho_{21}(x) = \rho_{34}(x), \\ \rho_{14}(x) = \rho_{23}(x) = \rho_{32}(x). \end{cases}$$

It is then easy to check that $\rho_{12}\rho_{13}$ is the identity on $\mathcal{H}_x$. With a similar argument, one deduces that if $\rho_{21}(x) = \rho_{14}(x)$, then $\rho_{12}\rho_{14}$ is the identity on $\mathcal{H}_x$. This establishes the claim.

Note that none of $\rho_{12}\rho_{13}$ and $\rho_{12}\rho_{14}$ are trivial. For instance, if $\rho_{12}\rho_{13}(u_1) = u_1$, then $\sigma_{u_2}^{-1}\sigma_{u_1}\sigma_{u_3}^{-1}(u_1) = u_1$ and so by Lemma 6, we find that $\sigma_{u_2}(u_1) = \sigma_{u_1}\sigma_{u_3}^{-1}(u_1) = \sigma_{u_1}^{-1}(u_3) = \sigma_{u_3}(u_1)$, which means that $u_2 = u_3$, a contradiction. Therefore, one of $\rho_{12}\rho_{13}$ or $\rho_{12}\rho_{14}$ is a non-trivial automorphism of $\mathbb{G}$ which is the identity on at least $\frac{3}{32}\nu - \frac{1}{2}$ elements of $\mathscr{P}$. It follows from Corollary 8 that $\frac{3}{8}\nu - 2 \leqslant \frac{1}{4}\nu$, which it contradicts $n \geqslant 3$. □

Lemma 10. *The group Γ generated by $\{\sigma_u\sigma_v^{-1} \,|\, u, v \in V(\mathbb{G})\}$ is Abelian and it acts transitively on $V(\mathbb{G})$.*

Proof. Consider the arbitrary vertices $u, v, x, y \in V(\mathbb{G})$. By Lemma 6, $\sigma_v\sigma^{-1}_{\sigma_u^{-1}(v)}(u) = v$, meaning that Γ acts transitively on $V(\mathbb{G})$. Applying Lemma 9, we have $(\sigma_u\sigma_v^{-1})(\sigma_x\sigma_y^{-1}) = \sigma_u\sigma_x^{-1}\sigma_v\sigma_y^{-1} = \sigma_x\sigma_u^{-1}\sigma_y\sigma_v^{-1} = (\sigma_x\sigma_y^{-1})(\sigma_u\sigma_v^{-1})$. So, Γ is Abelian. □

Lemma 11. *The order of* $\mathbb{G}$ *is either* 256 *or* 16384.

Proof. Applying Lemmas 9 and 10, we find that $\mathbb{G}$ admits a transitive automorphism group whose order is a power of 2. It follows from the orbit-stabilizer theorem that $n^2 + 3n - 2 = 2^t$, for some integer t. We have $(2n+3)^2 = 2^{t+2} + 17$. Using a result in [2, p. 401], we obtain that $(n, t) \in \{(1, 1), (2, 3), (3, 4), (10, 7)\}$. Since $n \geqslant 3$, we conclude that $(n, \nu) \in \{(3, 256), (10, 16384)\}$. □

Now, the proof of Theorem 1 is finally completed after proving Lemma 11. Notice that we employed the assumption (1) only in the proof of Lemma 2. As mentioned before, we believe that (1) automatically holds for any diamond-free $\mathsf{SRG}((n^2+3n-2)^2, n(n^2+3n-1), 2, n(n+1))$.

IV. Partial Quadrangle PQ(3, 35, 20)

In the following, we demonstrate that there exists no $\mathsf{PQ}(3, 35, 20)$, or equivalently, there is no diamond-free $\mathsf{SRG}(676, 108, 2, 20)$. Notice that this strongly regular graph belongs to the family (2) with $n = 4$ and $\lambda = 2$.

Theorem 12. *There exists no diamond-free* $\mathsf{SRG}(676, 108, 2, 20)$.

Proof. Suppose, toward a contradiction, that G is a diamond-free $\mathsf{SRG}(676, 108, 2, 20)$. Consider two non-adjacent vertices $u, v \in V(G)$. Since G is diamond-free, there are vertices $w \in N(u)$ and $v', v'' \in \overline{N}(u)$ such that $\{v, v', v'', w\}$ is a clique. For $i = 0, 1, 2, 3$, assume that s_i is the number of cliques $\varOmega$ in $N(u)$ of size 3 such that $\langle \varOmega, \{v, v', v''\}\rangle$ has i edges. By a double counting argument, we find that

$$\begin{cases} s_0 + s_1 + s_2 + s_3 = 35; \\ s_1 + 2s_2 + 3s_3 = 57; \\ s_2 + 3s_3 = 21, \end{cases}$$

which gives $s_0 = -s_3 - 1$, a contradiction. □

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