LECTURE NOTES

ON

Large sets of t-designs

G. B. Khosrovshahi B. Tayfeh-Rezaie

Institute for studies in theoretical Physics and Mathematics (IPM)
P.O. Box 19395-5746, *Tehran, Iran*

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Preface

The method of partitionable sets for constructing large sets of t-designs have now been used for nearly a decade. The method has resulted in some powerful recursive constructions and also existence results especially for large sets of prime sizes. Perhaps the main feature of the approach is its simplicity. In these notes, we describe the method and show how it is employed to obtain large sets. We will present almost all of the existence results and recursive constructions which have been found by this method.

Contents

1 Introduction

The problem of partitioning a set of subsets of a finite set into parts with some special regularity conditions has been dealt with extensively in combinatorics and graph theory. Large sets of t-designs which are about partitioning the set of all k-subsets of a v-set into block sets of t-(v, k, λ) designs are examples of these kinds of problems. Large sets by themselves are not only interesting combinatorial arrangements, but also they provide a correct settings for the study of the existence problem of t-designs. The celebrated theorem of Teirlink on the existence of t-designs for all t involves constructing large sets of t-designs. Large sets also have applications in cryptography [42].

The known existence results on large sets have been obtained by various methods which are very different in nature. In 1975, Baranyai settled the existence of large sets of Steiner 1-designs [7]. Later, Hartman using this result established the existence of large sets of 1-designs in general [20]. During seventies of the last century, many combinatorialists worked on the problem of large sets of Steiner triple systems. But it was Lu who finally solved the problem in 1984 [33] with a few exceptions which later on were completed by Teirlink [46]. Later, large sets of triple systems were completely answered [33, 34, 39, 40, 46, 50]. The next great achievement was obtained by Teirlink who showed that large sets of t-designs exist for all t [48]. In 1987, an important conjecture by Hartman (also known as halving conjecture) which asserts that large sets of size 2 exist for all parameter sets satisfying the trivial necessary conditions appeared [20]. This conjecture inspired the researchers in this field and initiated many new results on the existence problem of large sets. A new approach sprouted out from these efforts now known as the method of *partitionable sets*. The best result found by this method is due to Ajoodani-Namini who showed that halving conjecture is true for 2-designs [1]. After that, the method was used for constructing large sets of prime sizes. At present most of the results obtained by the approach of partitionable sets is for large sets of prime sizes, although some important recursive constructions have also been found for large sets in the general case. One of the main features of this approach is its simplicity. For example, Teirlink's long and complicated proof of the existence of t-designs for all t can be established in less than a page by the use of partitionable sets. The approach has also provided some extension theorems which are unique in design theory in the sense that no further conditions are imposed on the parameters.

In these notes, after definitions and review of the known results by other methods, we first describe the approach and review the results which have been found for large sets of any sizes. Then we pay our attention to large sets of prime sizes. There are nice results on large sets of prime sizes including the notion of root cases. Large sets of sizes 2 and 3 are of special interest and there are more comprehensive results for them. We devote particular sections for these cases. Coming to the end, the existence results obtained by the approach are reviewed. We will also present some open problems. Finally, an updated table on the existence of small large sets with at most 18 points is provided.

2 Definitions and Preliminaries

Let t, k, v and λ be integers such that $0 \le t \le k \le v$ and $\lambda > 0$. Let X be a v-set and $P_k(X)$ denote the set of all k-subsets of X. A t-(v, k, λ) *design* (briefly a t-design) is a pair $\mathbf{D} = (X, \mathcal{D})$ in which $\mathcal{D}$ is a collection of elements of $P_k(X)$ (called *blocks*) such that every t-subset of X appears in exactly λ blocks. If $\mathcal{D}$ has no repeated blocks, then $\mathbf{D}$ is called *simple.* Here we are concerned only with simple designs. Note that $(X, P_k(X))$ is a t-$(v, k, \binom{v-t}{k-t})$ design which is called the *complete* design. We do not show here, but it is very easy to see that any t-(v, k, λ) design with $t \ge v - k$ is the complete design. Two t-(v, k, λ) designs $(X_1, \mathcal{D}_1)$ and $(X_2, \mathcal{D}_2)$ are said to be *isomorphic* if there is a bijection $\sigma : X_1 \longrightarrow X_2$ such that $\sigma(\mathcal{D}_1) = \mathcal{D}_2$. An isomorphism from $\mathbf{D}$ into itself is called an *automorphism.* An *automorphism group* of $\mathbf{D}$ is a group whose elements are automorphisms of $\mathbf{D}$.

A simple counting argument shows that a t-(v, k, λ) design is also an i-(v, k, λ_i) design for $0 \le i \le t$, where $\lambda_i = \lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$. Hence, a set of necessary conditions for the existence of a t-(v, k, λ) design is

$$\lambda\binom{v-i}{t-i} \equiv 0 \pmod{\binom{k-i}{t-i}}, \qquad 0 \le i \le t. \tag{2.1}$$

Using $\binom{v-i}{t-i}\binom{v-t}{k-t} = \binom{v-i}{k-i}\binom{k-i}{t-i}$, one can easily see that the conditions (2.1) are equivalent to

$$\lambda\binom{v-i}{k-i} \equiv 0 \pmod{\binom{v-t}{k-t}}, \qquad 0 \le i \le t. \tag{2.2}$$

The minimum value of λ satisfying (2.1) is denoted by $\lambda_{\min}$ and any other feasible λ is clearly an integral multiple of $\lambda_{\min}$. By (2.2), we also obtain that

$$\lambda_{\min} = \frac{\binom{v-t}{k-t}}{\gcd\left(\binom{v-i}{k-i} \mid 0 \le i \le t\right)}. \tag{2.3}$$

The λ of the complete design is denoted by $\lambda_{\max}$.

Some more notation. Let

$$\begin{aligned}
\mathcal{D}^d(x) &= \{B \setminus \{x\} \mid x \in B \in \mathcal{D}\},\\
\mathcal{D}^r(x) &= \{B \mid x \notin B \in \mathcal{D}\},\\
\mathcal{D}^c(x) &= \{X \setminus B \mid B \in \mathcal{D}\},\\
\mathcal{D}^s &= \{B \mid B \notin \mathcal{D}\}.
\end{aligned}$$

Then $\mathbf{D}^d(x) = (X \setminus \{x\}, \mathcal{D}^d(x))$ and $\mathbf{D}^r(x) = (X \setminus \{x\}, \mathcal{D}^r(x))$ are $(t-1)$-$(v-1, k-1, \lambda)$ and $(t-1)$-$(v-1, k, \lambda_{t-1} - \lambda)$ designs, respectively, and are called *derived* and *residual* designs of $\mathbf{D}$ with respect to x. By the inclusion-exclusion principle, it is also seen that for $t \leq v-k$, $\mathbf{D}^c = (X, \mathcal{D}^c)$ is a t-$(v, v-k, \lambda^c)$ design, where $\lambda^c = \sum_{i=0}^{t}(-1)^t\binom{t}{i}\lambda_i$ and is called the *complement* of $\mathbf{D}$. The *supplement* of $\mathbf{D}$, $\mathbf{D}^s = (X, \mathcal{D}^s)$, is a t-$(v, k, \lambda_{\max} - \lambda)$ design.

Let $N \geq 1$. A *large set* of t-(v, k, λ) designs of size N, denoted by LS$[N](t, k, v)$, is a set $\mathbf{L}$ of N disjoint t-(v, k, λ) designs $\mathbf{D}_i = (X, \mathcal{D}_i)$ such that $\{\mathcal{D}_i |\ i \leq i \leq N\}$ is a partition of $P_k(X)$. Note that we have $N = \binom{v-t}{k-t}/\lambda$. Sometimes LS$[N](t, k, v)$ is denoted by LS$_\lambda(t, k, v)$ to show λ. If λ is one, it can be omitted. Two LS$[N](t, k, v)$, say $\{(X_1, \mathcal{D}_i)\}$ and $\{(X_2, \mathcal{E}_i)\}$, are called *isomorphic* if there is a a bijection $\sigma : X_1 \longrightarrow X_2$ such that for each i, $\sigma(\mathcal{D}_i) = \mathcal{E}_j$ for some j.

By (2.2), we observe that a set of necessary conditions for the existence of an LS$[N](t, k, v)$ is

$$N \Big| \binom{v-i}{k-i}, \qquad 0 \leq i \leq t. \tag{2.4}$$

The *derived, residual* and *complementary* large sets of $\mathbf{L} = \{\mathbf{D}_i\}$ are defined as $\mathbf{L}^d(x) = \{\mathbf{D}_i^d(x)\}$, $\mathbf{L}^r(x) = \{\mathbf{D}_i^r(x)\}$ and $\mathbf{L}^c = \{\mathbf{D}_i^c\}$ (when $t \leq v-k$) which are LS$[N](t-1, k-1, v-1)$, LS$[N](t-1, k, v-1)$ and LS$[N](t, v-k, v)$, respectively. Note that we can obtain more large sets from a given large set as the following theorem suggests.

Theorem 2.1 [3, 23] *If there exists an LS$[N](t, k, v)$, then there exist LS$[N](t-i, k-j, v-l)$ for all* $0 \leq j \leq l \leq i \leq t$.

Proof We prove the statement by induction on t. From the derived and residual large sets LS$[N](t-1, k-1, v-1)$ and LS$[N](t-1, k, v-1)$ and by the induction hypothesis we obtain LS$[N](t-i, k-j, v-l)$ for $l \geq 1$ and $0 \leq j \leq l \leq i \leq t$. On the other hand LS$[N](t, k, v)$ is at the same time LS$[N](i, k, v)$ for $0 \leq i \leq t$. This completes the proof. □

Notation Let N, t, and k be given integers such that $N > 0$ and $0 \leq t \leq k$. The set of all v for which an LS$[N](t, k, v)$ exists is denoted by $A[N](t, k)$. The set of all v which satisfy the necessary conditions (2.4) is denoted by $B[N](t, k)$. Any quadruple $(N; t, k, v)$ satisfying (2.4) is called an *admissible set of parameters*. Throughout these notes, when we speak of quadruples such as $(N; t, k, v)$, we implicity suppose that $N > 0$ and $0 \leq t \leq k \leq v$.

Example The block sets of two designs of the unique LS$[2](2, 3, 6)$ are as follows.

$$\mathcal{D}_1 = \{123, 124, 135, 146, 156, 236, 245, 256, 345, 346\},$$

$$\mathcal{D}_2 = \{125, 126, 134, 136, 145, 234, 235, 246, 356, 456\},$$

where 123 stands for $\{1, 2, 3\}$, etc.

Example The necessary conditions (2.4) are not always sufficient. A hundred and fifty years ago, Cayley showed that it is possible to have two disjoint 2-(7,3,1) designs and no more [10]. So there are no $\mathrm{LS}(2,3,7)$ and $\mathrm{LS}(3,4,8)$.

Example The previous example showed that there are no large sets of projective planes of order 2. But there is a large set of projective planes of order 3, i. e. $\mathrm{LS}[55](2,4,13)$ [13]. This is the only known example of $\mathrm{LS}(t,k,v)$ with $t \geq 2$ and $k \geq 4$.

Example There exist exactly two nonisomorphic $\mathrm{LS}[7](2,3,9)$ [28] while we know that there is a unique 2-(9,3,1) design.

Example There exist exactly 3,013,287 nonisomorphic $\mathrm{LS}[2](2,3,10)$ [18].

3 Review of the known large sets

In this section we give a brief account of known results on the existence of large sets of t-designs found by various methods. The results obtained by the approach of partitionable sets which is the main subject of this note will be presented in the final sections. Some parts of this section has been taken from [29].

In 1975, Baranyai showed that there exists an $\mathrm{LS}(1,k,v)$ if and only if $k|v$. The proofs related to this result employ the integrality theorem on flows in transportation networks. Two proofs can be found in [8, 52]. Hartman has extended the result for all values of k and v as stated in the following theorem.

Theorem 3.1 **[7, 20]** *$A[N](1,k) = B[N](1,k)$ for all positive integers N and k.*

Another celebrated theorem was obtained by Lu and Teirlink who showed that $\mathrm{LS}(2,3,v)$ exists if and only if $v > 7$ and $v \equiv 1,3 \pmod 6$. This result was obtained after a lot of works done by many researchers. The whole story about triple systems is given in the following theorem.

Theorem 3.2 **[33, 34, 39, 40, 46, 50]** *$A[N](2,3) = B[N](2,3) \setminus \{7\}$ for all positive integers N.*

In 1987, Teirlink proved the following theorem which was greatly acknowledged at the time since it did offer a proof of existence of t-designs for all values of t.

Theorem 3.3 [48] *For all positive integers N and t, there is an integer v such that an $LS[N](t, t+1, v)$ exists.*

Note As Theorems 3.1 and 3.2 show all admissible $\mathrm{LS}(1, k, v)$ and all admissible $\mathrm{LS}(2, 3, v)$ except for $v = 7$ exist. Beyond these cases the only known $\mathrm{LS}(t, k, v)$ is an $\mathrm{LS}(2, 4, 13)$ constructed in [13]. Etzion and Hartman have constructed $v - 5$ disjoint 3-$(v, 4, 1)$ designs for $v = 5.2^n$. This leaves only two more to go for an $\mathrm{LS}(3, 4, v)$ [19].

Some other miscellaneous results on the existence of large sets are as follows.

(i) An $\mathrm{LS}_{\lambda_{\min}}(3, 4, v)$ exists if $v \equiv 0 \pmod 3$ [49].

(ii) An $\mathrm{LS}_{\lambda_{\min}}(4, 5, 20v + 4)$ exists if $\gcd(v, 30) = 1$ [47].

(iii) An $\mathrm{LS}_{60}(4, 5, 60v + 4)$ exists if $\gcd(v, 60) = 1, 2$ [47].

Small cases of large sets play an important role in the constructions of large sets in general. They are initial points in recursive methods to produce infinite families of large sets. In [26], all parameter sets on less than or equal 12 points have been settled. In [12], a table on the existence of large sets with at most 18 points is presented. We will try to update it in the final section. At this point, let us review briefly the main approach used for constructing most of small cases of large sets. This approach was formulated for the first time by Kramer and Mesner [27]. The idea is simply that if there exist t-(v, k, λ) designs, then probably some of them have nontrivial automorphism groups. Therefore, we can reverse the procedure and try some suitable groups as automorphism groups of desired designs. In search for t-(v, k, λ) designs and $\mathrm{LS}_\lambda(t, k, v)$, we first choose a suitable permutation group G on a v-set X. Now if there is a t-(v, k, λ) design with automorphism group G, then its block set must be a union of orbits of G on $P_k(X)$. Different combinations of orbits are examined to see that if they provide for the block sets of t-(v, k, λ) designs. If the set of all orbits can be partitioned into subsets each of them being the block set of a t-(v, k, λ) design, then we find an $\mathrm{LS}_\lambda(t, k, v)$. This approach can be used both computationally and theoretically. Using computer and sometimes hand checking, many small designs and large sets have been constructed by the method. The results can be found in the literature. A reference list includes [11, 12, 14, 25, 26, 27, 30, 31, 32, 38]. The only remarkable theoretic works done so far are related to the groups $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$. Here, we do not have the intention to present those results. The reader is referred to [9, 15, 21, 22, 32, 37]

4 A theorem of Alltop

Alltop [6] has proved a theorem on extending t-designs. We state a similar result for large sets.

Theorem 4.1 *Let t be even and N be a positive integer or let t be odd and $N = 2$. If there exists an LS$[N](t, k, 2k+1)$, then there exists an LS$[N](t+1, k+1, 2k+2)$.*

Proof Let X be a $(2k+1)$-set and $x \notin X$. Suppose that $\{(X, \mathcal{D}_i) | \ 1 \le i \le N\}$ is an LS$[N](t, k, 2k+1)$. For $1 \le i \le N$, define

$$\mathcal{E}_i = \{B \cup \{x\} | \ B \in \mathcal{D}_i\},$$

$$\mathcal{F}_i = \{X \setminus B | \ B \in \mathcal{D}_i\}.$$

Clearly, $\mathcal{E}_i$ and $\mathcal{F}_i$ partition $P_{k+1}(X \cup \{x\})$. Let $n(T, \mathcal{D})$ denote the number of occurrences of T in $\mathcal{D}$.

Let t be even. Let $\mathcal{S}_i = \mathcal{E}_i \cup \mathcal{F}_i$. We show that $\{(X \cup \{x\}, \mathcal{S}_i) \mid 1 \le i \le N\}$ is an LS$[N](t+1, k+1, 2k+2)$. Let $T \in P_t(X)$. Then it is clear that $n(T \cup \{x\}, \mathcal{S}_i)$ is independent of i. Now let $T \in P_{t+1}(X)$. Then by the inclusion and exclusion principle, we have

$$\begin{aligned} n(T, \mathcal{S}_i) &= n(T, \mathcal{D}_i) + n(T, \mathcal{D}_i^c) \\ &= n(T, \mathcal{D}_i) + \sum_{R \subset T} (-1)^{|R|} n(R, \mathcal{D}_i) \\ &= n(T, \mathcal{D}_i) - n(T, \mathcal{D}_i) + \sum_{R \subset T, R \ne T} (-1)^{|R|} n(R, \mathcal{D}_i) \\ &= \sum_{R \subset T, R \ne T} (-1)^{|R|} n(R, \mathcal{D}_i), \end{aligned}$$

which is independent of i.

Now let t be odd and $N = 2$. Let $\mathcal{S}_1 = \mathcal{E}_1 \cup \mathcal{F}_2$ and $\mathcal{S}_2 = \mathcal{E}_2 \cup \mathcal{F}_1$. We show that $\{(X \cup \{x\}, \mathcal{S}_i) \mid 1 \le i \le N\}$ is an LS$[N](t+1, k+1, 2k+2)$. If $T \in P_t(X)$, then $n(T \cup \{x\}, \mathcal{S}_i)$ is independent of i. Let $T \in P_{t+1}(X)$. Then

$$\begin{aligned} n(T, \mathcal{E}_1 \cup \mathcal{F}_2) &= n(T, \mathcal{D}_1) + n(T, \mathcal{D}_2^c) \\ &= n(T, \mathcal{D}_1) + \sum_{R \subset T} (-1)^{|R|} n(R, \mathcal{D}_2) \\ &= n(T, \mathcal{D}_1) + n(T, \mathcal{D}_2) + \sum_{R \subset T, R \ne T} (-1)^{|R|} n(R, \mathcal{D}_2) \\ &= n(T, \mathcal{D}_2) + n(T, \mathcal{D}_1) + \sum_{R \subset T, R \ne T} (-1)^{|R|} n(R, \mathcal{D}_1) \\ &= n(T, \mathcal{D}_2) + \sum_{R \subset T} (-1)^{|R|} n(R, \mathcal{D}_1) \\ &= n(T, \mathcal{D}_2) + n(T, \mathcal{D}_1^c) \\ &= n(T, \mathcal{E}_2 \cup \mathcal{F}_1), \end{aligned}$$

Therefore, we are done. □

Corollary 4.1 *An* $LS[2](2,k,2k)$ *exists if and only if* k *is not a power of* 2.

Proof $\binom{2k-1}{k-1}$ and $\binom{2k-2}{k-2}$ are even if and only if k is not a power of 2 (see for example Theorem 5.1). Therefore, by Theorem 3.1, an LS[2]$(1,k-1,2k-1)$ exists if and only if k is not a power of 2. Now the assertion follows from Theorem 4.1. □

5 The necessary conditions

In this section, the necessary conditions for the existence of LS[N](t,k,v) as given in (2.4) are dealt with. It is possible to give an alternative description of (2.4) when N is a prime power. If N is not a prime power, then we can factorize it into prime powers and apply our results to its prime power factors. Hereafter, we let p^α be a prime power where p is prime.

Let m and n be positive integers. We denote the quotient and remainder of division m by n by $[m/n]$ and (m/n), respectively. It is well known that the largest integral value of e such that $p^e|m!$ is equal to $\sum_{i\geq 1}[m/p^i]$. Let $m \geq n$ and denote the largest integral value of f such that $p^f\,|\binom{m}{n}$ by $(m,n)_p$. Then

$$(m,n)_p = \sum_{i\geq 1} \left[\frac{m}{p^i}\right] - \left[\frac{n}{p^i}\right] - \left[\frac{m-n}{p^i}\right]. \tag{5.1}$$

Note that one can evaluate the expression $[m/p^i]-[n/p^i]-[(m-n)/p^i]$ in the following way:

$$\left[\frac{m}{p^i}\right] - \left[\frac{n}{p^i}\right] - \left[\frac{m-n}{p^i}\right] = \begin{cases} 1 & \text{if} \left(\frac{m}{p^i}\right) < \left(\frac{n}{p^i}\right), \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

We now state the main theorem.

Theorem 5.1 *The quadruple* $(p^\alpha;t,k,v)$ *is admissible if and only if there exist distinct positive integers* ℓ_i $(1\leq i\leq \alpha)$ *such that* $t \leq (v/p^{\ell_i}) < (k/p^{\ell_i})$.

Proof Note that by (5.1), the assertion holds for $t=0$. So let $t>0$. First assume that $v \in \mathrm{B}[p^\alpha](t,k)$. For $0\leq j\leq t$ we have

$$\begin{aligned}(v-j,k-j)_p &= \sum_{r\geq 1} \left[\frac{v-j}{p^r}\right] - \left[\frac{k-j}{p^r}\right] - \left[\frac{v-k}{p^r}\right] \\ &\geq \alpha. \end{aligned} \tag{5.3}$$

Let ℓ_0 be the largest positive integer such that $(v/p^{\ell_0}) \geq t$, but $(v/p^{\ell_0-1}) < t$. Let $j_0 = (v/p^{\ell_0-1})+1$, if $(v/p^{\ell_0-1}) < (k/p^{\ell_0-1})$ and $j_0 = (k/p^{\ell_0-1})$, otherwise. Then,

$$j_0 \leq \left(\frac{k}{p^r}\right), \qquad \text{for } r \geq \ell_0 - 1. \tag{5.4}$$

By (5.2), we have

$$\sum_{r=1}^{\ell_0-1} \left[\frac{v-j_0}{p^r}\right] - \left[\frac{k-j_0}{p^r}\right] - \left[\frac{v-k}{p^r}\right] = 0. \tag{5.5}$$

Now by (5.2)-(5.5), there exist distinct positive integers $\ell_i \geq \ell_0$ for $1 \leq i \leq \alpha$ such that $((v-j_0)/p^{\ell_i}) < ((k-j_0)/p^{\ell_i})$ or $t \leq (v/p^{\ell_i}) < (k/p^{\ell_i})$.

Now suppose that there exist distinct positive integers ℓ_i $(1 \leq i \leq \alpha)$ such that $t \leq (v/p^{\ell_i}) < (k/p^{\ell_i})$. Therefore, $((v-j)/p^{\ell_i}) < ((k-j)/p^{\ell_i})$ for all $0 \leq j \leq t$ and consequently $(v-j, k-j)_p \geq \alpha$ which in turn implies that $v \in \mathrm{B}[p^\alpha](t,k)$. □

Example By Theorem 5.1, LS[55](2, 4, 13) is admissible. Since we have $2 \leq (13/5) < (4/5)$ and $2 \leq (13/11) < (4/11)$.

Example What is the largest value of t for which LS[13]$(t, 9, 18)$ is admissible? By Theorem 5.1, we must have $t \leq (18/13^\alpha) < (9/13^\alpha)$ and hence $\alpha = 1$ and $t_{\max} = 5$.

Using Theorem 5.1, we can easily determine all the admissible sets of parameters for $N = p$:

$$(p; t, k, v) = (p; t, mp^z + r, np^z + s), \tag{5.6}$$

where $0 \leq t \leq s < r < p^z$ and $0 \leq m < n$. We can also assume that z is the smallest or the greatest number with the properties above to be assured of the uniqueness of the representation (5.6). By Theorem 5.1, we are also able to identify $\mathrm{B}[N](t, t+1)$ completely.

Theorem 5.2 *Let $\prod_{i=1}^s p_i^{\alpha_i}$ be the prime power factorization of N. For $1 \leq i \leq s$, suppose that $p_i^{s_i-1} \leq t+1 < p_i^{s_i}$. Then*

$$B[N](t, t+1) = \left\{ v \middle|\ v \equiv t \mod \prod_{i=1}^s p_i^{\alpha_i+s_i-1} \right\}.$$

Proof By Theorem 5.1, $v \in B[p_i^{\alpha_i}](t, t+1)$ if and if only $v = n_i p_i^{\alpha_i+s_i-1} + t$ for some n_i. Therefore, $v \in B[N](t, t+1)$ if and if only $v = n\prod_{i=1}^s p_i^{\alpha_i+s_i-1} + t$ for some n. □

The following result is due to Teirlink and it can be obtained from Theorem 5.2.

Theorem 5.3 *For $k = t + 1$, we have*

$$\lambda_{min} = gcd(v - t, lcm(1, \dots, t + 1)).$$

Proof Let $\prod_{i=1}^{s} p_i^{\alpha_i}$ be the prime power factorization of $v - t$ and let $p_i^{s_i - 1} \le t + 1 < p_i^{s_i}$ for $1 \le i \le s$. If $v \in \mathrm{B}[N](t, t + 1)$, then N is at most equal to $\prod_{i=1}^{s} p_i^{\alpha_i - s_i + 1}$. Therefore, $\lambda_{\min} = \lambda_{\max}/N = \prod_{i=1}^{s} p_i^{s_i - 1}$. This proves the assertion. □

We bring this section to an end by presenting another useful application of Theorem 5.1.

Theorem 5.4 *Let $0 \le t < k$. Then the minimal element of $B[p^\alpha](t, k)$ is equal to*

$$v_{min} = ([k/p^{\ell+\alpha-1}] + 1)p^{\ell+\alpha-1} + t$$

in which ℓ is the smallest positive integer such that $(k/p^\ell) > t$.

Proof Let $\ell_1 = \ell, \ell_2 = \ell + 1, \dots,$ and $\ell_\alpha = \ell + \alpha - 1$. It is easy to check that $v = ([k/p^{\ell_\alpha}] + 1)p^{\ell_\alpha} + t \in B[p^\alpha](t, k)$. Now suppose that $v' \in B[p^\alpha](t, k)$. By Theorem 5.1, there are distinct positive integers ℓ'_i, $1 \le i \le \alpha$, such that $t \le (v'/p^{\ell'_i}) < (k/p^{\ell'_i})$. Clearly $\ell'_i \ge \ell_i$ for all i and so we have

$$\begin{aligned} v' &= \left[\frac{v'}{p^{\ell'_\alpha}}\right] p^{\ell'_\alpha} + \left(\frac{v'}{p^{\ell'_\alpha}}\right) \\ &\ge \left(\left[\frac{k}{p^{\ell'_\alpha}}\right] + 1\right) p^{\ell'_\alpha} + t \\ &\ge \left(\left[\frac{k}{p^{\ell_\alpha}}\right] + 1\right) p^{\ell_\alpha} + t \\ &= v. \end{aligned}$$

Therefore, $v = v_{\min}$ and the proof is complete. □

6 The approach of partitionable sets

A powerful approach for the construction of large sets is obtained from the notion of (N, t)-partitionable sets which was first introduced in [5]. This idea is indeed a generalization of the notion of large sets, where we consider t-balanced partition of a subset $\mathcal{B}$ of $P_k(X)$ instead of the whole set $P_k(X)$. Let $\mathcal{B}_1, \mathcal{B}_2 \subseteq P_k(X)$. We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are *t-equivalent* if every t-subset of X appears in the same number of blocks of $\mathcal{B}_1$ and $\mathcal{B}_2$. If there exists a partition of $\mathcal{B} \subseteq P_k(X)$ into N mutually t-equivalent subsets, then $\mathcal{B}$ is called an *(N, t)-partitionable* set. In the literature of design theory, *$(2, t)$-partitionable* sets are very well known objects called *trades.*

So one can consider (N,t)-*partitionable* sets as a generalization of trades. Let X_1 and X_2 be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1, 2$. Then we define

$$\mathcal{B}_1 * \mathcal{B}_2 = \{B_1 \cup B_2 |\ B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2\}.$$

There are two important lemmas concerning (N,t)-*partitionable* sets. The first one is trivial while the other one is a very unexpected.

Lemma 6.1 (i) *t-equivalence implies i-equivalence for all $0 \le i \le t$.*
(ii) *The union of disjoint (N,t)-partitionable sets is again an (N,t)-partitionable set.*

Lemma 6.2 *Let X_1 and X_2 be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1, 2$. Suppose that $\mathcal{B}_1$ is (N, t_1)-partitionable. Then*

(i) *$\mathcal{B}_1 * \mathcal{B}_2$ is (N, t_1)-partitionable.*
(ii) *If $\mathcal{B}_2$ is (N, t_2)-partitionable, then $\mathcal{B}_1 * \mathcal{B}_2$ is $(N, t_1 + t_2 + 1)$-partitionable.*

Proof Let $n(T, \mathcal{B})$ denote the number of occurrences of T in $\mathcal{B}$. Suppose that $\{\mathcal{R}_i |\ 1 \le i \le N\}$ is a partition of $\mathcal{B}_1$ into mutually t_1-equivalent subsets.

(i) For $1 \le i \le N$, define $\mathcal{S}_i = \mathcal{R}_i * \mathcal{B}_2$. We show that $\{\mathcal{S}_i |\ 1 \le i \le N\}$ is a partition of $\mathcal{B}_1 * \mathcal{B}_2$ into mutually t_1-equivalent subsets. Clearly, $\mathcal{B}_1 * \mathcal{B}_2 = \bigcup_{i=1}^{N} \mathcal{S}_i$ and $\mathcal{S}_i$ are mutually disjoint. For any t_1-subset T of $X_1 \cup X_2$, we have

$$n(T, \mathcal{S}_i) = n(T \cap X_1, \mathcal{R}_i) n(T \cap X_2, \mathcal{B}_2).$$

By Lemma 6.1, $n(T \cap X_1, \mathcal{R}_i)$ is independent from i and so is $n(T, \mathcal{S}_i)$.

(ii) Suppose that $\{\mathcal{H}_i |\ 1 \le i \le N\}$ is a partition of $\mathcal{B}_2$ into mutually t_2-equivalent subsets. Let $A = (a_{ij})$ be a Latin square of order N. For $1 \le i \le N$, define $\mathcal{S}_i = \bigcup_{j=1}^{N} \mathcal{R}_j * \mathcal{H}_{a_{ij}}$. We show that $\{\mathcal{S}_i |\ 1 \le i \le N\}$ is a partition of $\mathcal{B}_1 * \mathcal{B}_2$ into mutually $(t_1 + t_2 + 1)$-equivalent subsets. It is clear that $\mathcal{S}_i$ partition $\mathcal{B}_1 * \mathcal{B}_2$. Let $T \in P_{t_1+t_2+1}(X_1 \cup X_2)$ and for $1 \le i \le 2$ define $T_i = T \cap X_i$ and $r_i = |T_i|$. It is easy to see that

$$n(T, \mathcal{S}_i) = \sum_{j=1}^{N} n(T_1, \mathcal{R}_j) n(T_2, \mathcal{H}_{a_{ij}}).$$

Now we consider the following two cases:

case (a): $r_1 \le t_1$. By Lemma 6.1, $n(T_1, \mathcal{R}_j)$ is independent from j and therefore,

$$\begin{aligned} n(T, \mathcal{S}_i) &= n(T_1, \mathcal{R}_1) \sum_{j=1}^{N} n(T_2, \mathcal{H}_{a_{ij}}) \\ &= n(T_1, \mathcal{R}_1) \sum_{j=1}^{N} n(T_2, \mathcal{H}_j). \end{aligned}$$

So $n(T, \mathcal{S}_i)$ is independent from i and we are done.

case (b): $r_1 > t_1$. In this case, $r_2 \le t_2$ and a similar argument to (a) can be applied. □

The importance of Lemma 6.2 is seen at the first glance. In the theory of t-designs, extension theorems which increase the value of t are very rare (one example is Theorem 4.1). If Lemma 6.2 is employed in a clever way, then very useful extension theorems can be found. We can now state our method for constructing large sets based on Lemmas 6.1 and 6.2. Suppose that we are looking for an LS$[N](t, k, v)$ on a v-set X. We try to partition $P_k(X)$ in a such a way that each part of the partition is an (N, t)-partitionable set. If this done, then by Lemma 6.1, $P_k(X)$ will be an (N, t)-partitionable set which means that we have obtained an LS$[N](t, k, v)$. Each part $\mathcal{B}$ in the partition is usually of the form $P_{k_1}(X_1) * P_{k_2}(X_2)$ where X_1 and X_2 are disjoint subsets of X and $k = k_1 + k_2$. If there exist LS$[N](t_1, k_1, v_1)$ and LS$[N](t_2, k_2, v_2)$ and $t = t_1 + t_2 + 1$, then by Lemma 6.2, $\mathcal{B}$ is (N, t)-partitionable. The approach is understood better with the following simple examples.

Example. Construction of an LS$[2](2, 3, 10)$ from an LS$[2](2, 3, 6)$. Let $X = \{1, 2, \ldots, 10\}$ and consider the following partitioning of $P_3(X)$:

$$\begin{aligned} \mathcal{B}_1 &= P_3(\{1, \ldots, 6\}), \\ \mathcal{B}_2 &= P_2(\{1, \ldots, 5\}) * P_1(\{7, \ldots, 10\}), \\ \mathcal{B}_3 &= P_1(\{1, \ldots, 4\}) * P_2(\{6, \ldots, 10\}), \\ \mathcal{B}_4 &= P_3(\{5, \ldots, 10\}). \end{aligned}$$

$\mathcal{B}_1$ and $\mathcal{B}_4$ are (2,2)-partitionable by the assumption. By Theorem 2.1, there exist LS$[2](1, 2, 5)$ and LS$[2](0, 1, 4)$. Therefore, $\mathcal{B}_2$ and $\mathcal{B}_3$ are (2,2)-partitionable sets by Lemma 6.2. Now Lemma 6.1 shows that $P_3(X)$ is (2,2)-partitionable set, i. e. an LS$[2](2, 3, 10)$ is constructed.

Example. A similar argument for constructing an LS$[3](2, 12, 29)$ from LS$[3](2, 3, 11)$, LS$[3](2, 7, 15)$, LS$[3](2, 8, 16)$ and LS$[3](2, 12, 20)$. Note that there is no other known construction method for LS$[3](2, 12, 29)$. Let $X = \{1, 2, \ldots, 29\}$ and consider the following partitioning of $P_{12}(X)$:

$$\mathcal{B}_i = P_{12-i}(\{1, \ldots, 20 - i\}) * P_i(\{22 - i, \ldots, 29\}), \quad 0 \le i \le 12.$$

$\mathcal{B}_0$ and $\mathcal{B}_{12}$ are (3,2)-partitionable sets since there exists an LS$[3](2, 12, 20)$. By Theorem 2.1, there exist LS$[3](1, 2, 10)$, LS$[3](0, 1, 9)$, LS$[3](1, 6, 14)$, LS$[3](0, 10, 18)$ and LS$[3](1, 11, 19)$. Using Lemma 6.2, it is an easy task to see that all the remaining $\mathcal{B}_i$ are also (3,2)-partitionable sets. For example, consider $\mathcal{B}_2$ and $\mathcal{B}_3$. $P_{10}(\{1, \ldots, 18\})$ is (3,0)-partitionable since there is an LS[3]$(0, 10, 18)$. $P_2(\{20, \ldots, 29\})$ is (3,1)-partitionable since LS[3]$(1, 2, 10)$ exists. Now Lemma 6.2 shows that $\mathcal{B}_2$ is (3,2)-partitionable. Because of the existence of an LS[3]$(2, 3, 11)$, $P_3(\{19, \ldots, 29\})$ is a (3,2)-partitionable set and so is $\mathcal{B}_3$ by Lemma 6.2. The remaining $\mathcal{B}_i$ are dealt with in similar ways. Hence, by Lemma 6.1, it results that $P_{12}(X)$ is (3,2)-partitionable and so an LS[3]$(2, 12, 29)$ is constructed.

The general form of the partitioning given in the examples above is as follows.

Lemma 6.3 *Let $X = \{1, 2, \ldots, u+v\}$ and also for $1 \leq j \leq u+v$, let $X_j = \{1, 2, \ldots, j\}$ and $Y_j = X \setminus X_j$. For $0 \leq i \leq k$, define*

$$\mathcal{B}_i = P_{k-i}(X_{u-i}) * P_i(Y_{u-i+1}).$$

Then $\mathcal{B}_i$ provide a partitioning of $P_k(X)$.

Proof Let $0 \leq j < i \leq k$ and $A \in \mathcal{B}_i$. Then $|A \cap X_{u-i}| = k - i$ and

$$\begin{aligned} |A \cap X_{u-j}| &\leq |A \cap X_{u-i}| + |X_{u-j} \setminus X_{u-i}| - 1 \\ &= k - j - 1. \end{aligned}$$

Therefore, $A \notin \mathcal{B}_j$. It yields that all $\mathcal{B}_i$ are mutually disjoint.

Now let $A \in P_k(X)$. Let $0 \leq i \leq k$ be the smallest integer such that $|A \cap X_{u-i}| \geq k - i$. Then $|A \cap X_{u-i+1}| \leq k - i + 1$ and therefore,

$$\begin{aligned} k - i &\leq |A \cap X_{u-i}| \\ &\leq |A \cap X_{u-i+1}| \\ &\leq k - i. \end{aligned}$$

So $|A \cap X_{u-i}| = |A \cap X_{u-i+1}| = k - i$ and $A \in \mathcal{B}_i$. □

A more complicated generalization of Lemma 6.3 is given in the following lemma.

Lemma 6.4 *Let a, b, s, k, v_1 and v_2 be nonnegative integers such that $s < k \leq min\{v_1, v_2\}$ and $s = k - 1 - a - b$. Let $X = \{1, 2, \ldots, v_1 + v_2 - s\}$ and also for $1 \leq j \leq v_1 + v_2 - s$, let $X_j = \{1, 2, \ldots, j\}$ and $Y_j = X \setminus X_j$. Consider the following subsets of $P_k(X)$:*

$$\begin{aligned} \mathcal{A}_i &= P_{k-i}(X_{v_1}) * P_i(Y_{v_1}), \quad 0 \leq i \leq a, \\ \mathcal{B}_j &= P_{k-a-j}(X_{v_1-j}) * P_{a+j}(Y_{v_1-j+1}), \quad 1 \leq j \leq s, \\ \mathcal{C}_l &= P_l(X_{v_1-s}) * P_{k-l}(Y_{v_1-s}), \quad 0 \leq l \leq b. \end{aligned}$$

Then $\mathcal{A}_i, \mathcal{B}_j$ and $\mathcal{C}_l$ partition $P_k(X)$.

Proof By Lemma 6.3, a partitioning of $P_k(X)$ is given by

$$\mathcal{B}_h = P_{k-h}(X_{v_1+a-h}) * P_h(Y_{v_1+a-h+1}), \quad 0 \le h \le k.$$

Consider $\mathcal{B}_h, 0 \le h \le a$. These sets constitute exactly those elements B of $P_k(X)$ which $|B \cap X_{v_1}| \ge k-a$. So we can replace the sets $\mathcal{B}_h$ $(0 \le h \le a)$ with $\mathcal{A}_i = P_{k-i}(X_{v_1}) * P_i(Y_{v_1})$ $(0 \le i \le a)$ in the partition. Similarly, consider $\mathcal{B}_h,\ k-b \le h \le k$. These sets constitute exactly those elements B of $P_k(X)$ which $|B \cap X_{v_1-s}| \le b$. Therefore, if we replace the sets $\mathcal{B}_h$ $(k-b \le h \le k)$ with $\mathcal{C}_l = P_l(X_{v_1-s}) * P_{k-l}(Y_{v_1-s}),\ (0 \le l \le b)$ in the partition, then we are done. □

Another useful partitioning is given in the following lemma.

Lemma 6.5 *Let $X = \{1, 2, \dots, v\}$ and also for $1 \le j \le v$, let $X_j = \{1, 2, \dots, j\}$ and $Y_j = X \setminus X_j$. For $a \le i \le v-b-1$, define*

$$\mathcal{B}_i = P_a(X_i) * \{\{i+1\}\} * P_b(Y_{i+1}).$$

Then $\mathcal{B}_i$ provide a partitioning of $P_{a+b+1}(X)$.

Proof We sort $(a+b+1)$-subsets of X in lexicographical order. Now look at the entries in the position $a+1$ of the subsets. All subsets with element $i+1$ in this position are next to each other and constitute a set $\mathcal{B}_i$ which is of the form $P_a(X_i) * \{\{i+1\}\} * P_b(Y_{i+1})$. □

We now use the approach to prove a simple recursive method which has been known for a long time at least for t-designs.

Lemma 6.6 *If there exist an $LS[N](t,k,v)$ and an $LS[N](t,k+1,v)$, then there exists an $LS[N](t,k+1,v+1)$.*

Proof Let X be a v-set and $x \notin X$. Consider the following partitioning of $P_{k+1}(X \cup \{x\})$:

$$\mathcal{B}_0 = P_{k+1}(X),$$
$$\mathcal{B}_1 = \{\{x\}\} * P_k(X).$$

By the assumption $\mathcal{B}_0$ is (N,t)-partitionable. Also $P_k(X)$ is an (N,t)-partitionable set by the assumption and therefore by Lemma 6.2, $\mathcal{B}_1$ is (N,t)-partitionable. Now the assertion follows from Lemma 6.1. □

7 General recursive constructions

In this section we present some recursive constructions for large sets of any size which are obtained by the approach of (N, t)-partitionable sets. Large sets of prime sizes will be tackled in the next section. It is worth to note that except for Theorem 4.1, all known recursive constructions for large sets were found through this approach. The first theorem is a result of Lemma 6.6 and an induction argument.

Theorem 7.1 *If there exist* $LS[N](t, k + i, v)$ *for all* $0 \leq i \leq l$*, then there exist* $LS[N](t, k + i, v + j)$ *for all* $0 \leq j \leq i \leq l$.

Proof We use induction on l. For $l = 0$, there is nothing to be proved. Hence, let $l > 0$. By the induction hypothesis, from $\mathrm{LS}[N](t, k + i, v)$ $(0 \leq i \leq l - 1)$ and $\mathrm{LS}[N](t, k + 1 + i, v)$ $(0 \leq i \leq l - 1)$, we obtain $\mathrm{LS}[N](t, k + i, v + j)$ for all $0 \leq j \leq i \leq l$ and $j \neq l$. It remains to establish the existence of an $\mathrm{LS}[N](t, k+l, v+l)$ which is found from $\mathrm{LS}[N](t, k+l-1, v+l-1)$ and $\mathrm{LS}[N](t, k + l, v + l - 1)$ using Lemma 6.6. □

Theorem 7.2 **[2]** *Let* a, b, c, d, t, s, k, v_1 *and* v_2 *be nonnegative integers such that* $t \leq s < k \leq \min\{v_1, v_2\}$ *and* $s = k - 1 - a - b = t + c + d$. *Let* $v_1 \in \cap_{i=k-a}^{k} A[N](t, i)$, $v_2 \in \cap_{i=k-b}^{k} A[N](t, i)$, $v_1 - l \in A[N](t, k - a - l)$ *for* $1 \leq l \leq c$ *and* $v_2 - l \in A[N](t, k - b - l)$ *for* $1 \leq l \leq d$. *Then* $v_1 + v_2 - s \in A[N](t, k)$.

Proof Let X, X_j, Y_j, $\mathcal{A}_i, \mathcal{B}_j$ and $\mathcal{C}_l$ be as defined in Lemma 6.4. We show that $\mathcal{A}_i, \mathcal{B}_j$ and $\mathcal{C}_l$ are (N, t)-partitionable sets. Let $0 \leq i \leq a$ and $0 \leq l \leq b$. By the assumption, $P_{k-i}(X_{v_1})$ and $P_{k-l}(Y_{v_1-s})$ are (N, t)-partitionable sets and so are $\mathcal{A}_i$ and $\mathcal{C}_l$ by Lemma 6.2. Let $1 \leq j \leq s$. If $1 \leq j \leq c$, then by the assumption, $P_{k-a-j}(X_{v_1-j})$ is (N, t)-partitionable and so is $\mathcal{B}_j$ by Lemma 6.2. If $s - d < j \leq s$, then by the assumption, $P_{a+j}(Y_{v_1-j+1})$ is (N, t)-partitionable and so is $\mathcal{B}_j$ by Lemma 6.2. Now let $c < j \leq s - d$. Then, by Theorem 2.1, $P_{k-a-j}(X_{v_1-j})$ and $P_{a+j}(Y_{v_1-j+1})$ are $(N, t - j + c)$-partitionable and $(N, j - c - 1)$-partitionable, respectively. Therefore, by Lemma 6.2, $\mathcal{B}_j$ is (N, t)-partitionable. □

Corollary 7.1 *If* $LS[N](t, i, v)$ *exist for* $t + 1 \leq i \leq k$ *and an* $LS[N](t, k, u)$ *also exists, then* $LS[N](t, k, u + l(v - t))$ *exist for all* $l \geq 1$.

Proof It suffices to prove the assertion for $l = 1$. The statement then will follow by induction. In Theorem 7.2, put $a = k - t - 1, b = c = d = 0, v_1 = v$ and $v_2 = u$. □

Corollary 7.2 *If $LS[N](t, i, v+i)$ exist for $t+1 \le i \le k$ and an $LS[N](t, k, u)$ also exists, then $LS[N](t, k, u + l(v+1))$ exist for all $l \ge 1$.*

Proof In Theorem 7.2, put $a = b = d = 0, c = k - t - 1, v_1 = v + k$ and $v_2 = u$. This proves the assertion for $l = 1$. Now use induction. □

Corollary 7.3 *If an $LS[N](t, t+1, v+t)$ exists, then $LS[N](t, t+1, lv+t)$ exist for all $l \ge 1$.*

Proof This is an immediate result of Corollary 7.1 for $k = t + 1$. □

8 Two general theorems for large sets of prime sizes

The approach of (N, t)-partitionable sets has been mainly used to obtain recursive constructions for large sets of prime sizes. In this section, we use the approach to find two general recursive theorems for large sets of prime sizes. These theorems are due to Ajoodani-Namini and provide an alternative proof of Teirlink's result on the existence of t-designs for all t. Ajoodani-Namini's method has two merits: first it is simpler than Teirlink's, and secondly it provides designs with parameters which are much smaller than the parameters of those of Teirlink.

Let k, v and l be positive integers such that $pk \le l < p(k+1)$ and let $X = \{1, 2, \ldots, pv\}$. For $1 \le i \le v$, define $A_i = \{(i-1)p + 1, \ldots, pi\}$. Consider the ordered set $\mathcal{Y} = \{A_1, A_2, \ldots, A_v\}$ with order $A_1 < A_2 < \cdots < A_v$ and define ϕ from the power set of $\mathcal{Y}$ to the power set of A as

$$\phi(\mathcal{B}) = \cup_{A_i \in \mathcal{B}} A_i, \quad \mathcal{B} \subseteq \mathcal{Y}.$$

From the definition we immediately obtain the following.

Lemma 8.1 *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are i-equivalent subsets of $P_j(\mathcal{Y})$, then $\phi(\mathcal{B}_1)$ and $\phi(\mathcal{B}_1)$ are i-equivalent subsets of $P_{pj}(X)$.*

Now let

$$\mathcal{P}_l = \left\{ \mathcal{F}_m(\mathcal{T}, a_1, \ldots, a_r) \middle| \ \mathcal{T} = \{C_1 < \cdots < C_r\} \subseteq \mathcal{Y}, \ 0 < a_i < p, \ l = \sum_{i=1}^{r} a_i + pm \right\}, \quad (8.1)$$

where

$$\mathcal{F}_m(\mathcal{T}, a_1, \ldots, a_r) = \phi(P_m(\mathcal{Y} \setminus \mathcal{T})) * P_{a_1}(C_1) * \cdots * P_{a_r}(C_r). \quad (8.2)$$

The following lemma is easily observed by the definitions.

Lemma 8.2 *If $l \neq pk$, then $\mathcal{P}_l$ is a partition of $P_l(X)$. Otherwise, $\mathcal{P}_l$ is a partition of $P_l(X) \setminus \phi(P_k(\mathcal{Y}))$.*

Lemma 8.3 *$\mathcal{F}_m(\mathcal{T}, a_1, \ldots, a_r)$ as defined in* (8.1) *and* (8.2) *is $(p, r-1)$-partitionable. Moreover, if there exists an LS$[p](s, m, v-r)$, then $\mathcal{F}_m(\mathcal{T}, a_1, \ldots, a_r)$ is $(p, s+r)$-partitionable.*

Proof Notice that or $0 < a < p$, $p|\binom{p}{a}$. Therefore, $P_{a_i}(C_i)$ is $(p, 0)$-partitionable. By Lemma 6.2 and induction, one can easily see that $\mathcal{F}_m(\mathcal{T}, a_1, \ldots, a_r)$ is $(p, r-1)$-partitionable. Now suppose that there exists an LS$[p](s, m, v-r)$. Since $P_m(\mathcal{Y} \setminus \mathcal{T})$ is a (p, s)-partitionable set, by Lemma 8.1, $\phi(P_m(\mathcal{Y} \setminus \mathcal{T}))$ is also (p, s)-partitionable. Hence, by Lemma 6.2, $\mathcal{F}_m(\mathcal{T}, a_1, \ldots, a_r)$ is $(p, s+r)$-partitionable. □

Lemma 8.4 *If there exists an LS$[p](t, k, v)$, then there exists an LS$[p](t, l, pv)$.*

Proof Consider $\mathcal{F}_m(\mathcal{T}, a_1, \ldots, a_r) \in \mathcal{P}_l$. If $r > t$, then by Lemma 8.3, $\mathcal{F}_m(\mathcal{T}, a_1, \ldots, a_r)$ is (p, t)-partitionable. Now let $r \leq t$. We have $l - mp = \sum_{i=1}^{r} a_i < pr$, so $k - r < m \leq k$. From LS$[p](t, k, v)$ we obtain LS$[p](t-r, m, v-r)$ using Theorem 2.1. Hence, by Lemma 8.3, $\mathcal{F}_m(\mathcal{T}, a_1, \ldots, a_r)$ is (p, t)-partitionable. On the other hand, $P_k(\mathcal{Y})$ is a (p, t)-partitionable set and so is $\phi(P_k(\mathcal{Y}))$. Therefore, by Lemmas 8.2 and 6.1, $P_l(X)$ is (p, t)-partitionable. □

Theorem 8.1 **[3]** *If there exists an LS$[p](t, k, v-1)$, then there exist LS$[p](t+1, pk+i, pv+j)$ for all $0 \leq j < i \leq p-1$.*

Proof Let $pk < l < p(k+1)$. Consider $\mathcal{F}_m(\mathcal{T}, a_1, \ldots, a_r) \in \mathcal{P}_l$. If $r > t+1$, then by Lemma 8.3, $\mathcal{F}_m(\mathcal{T}, a_1, \ldots, a_r)$ is $(p, t+1)$-partitionable. Now let $r \leq t+1$. We have $l - mp = \sum_{i=1}^{r} a_i < pr$, so $k-r+1 \leq m < k$. From LS$[p](t, k, v-1)$ we obtain LS$[p](t-r+1, m, v-r)$ using Theorem 2.1. Hence, by Lemma 8.3, $\mathcal{F}_m(\mathcal{T}, a_1, \ldots, a_r)$ is $(p, t+1)$-partitionable. Now by Lemmas 8.2 and 6.1, $P_l(X)$ is $(p, t+1)$-partitionable. Therefore, we have LS$[p](t+1, l, pv-1)$ for all $pk < l < p(k+1)$. Finally, using Theorem 7.1, one obtains LS$[p](t+1, pk+i, pv+j)$ for all $0 \leq j < i \leq p-1$. □

Theorem 8.2 **[3, 43]** *If there exists an LS$[p](t, k, v-1)$, then there exist LS$[p](t, pk+i, pv+j)$ exist for all $-p \leq j < i \leq p-1$.*

Proof By Lemma 8.4, there exist LS$[p](t, pk+i, pv-p)$ for all $0 \leq i < p$. From LS$[p](t-1, k-1, v-2)$, we also obtain LS$[p](t, pk+i, pv-p)$ for all $-p < i < 0$ using Theorem 8.1. Therefore, there exist LS$[p](t, pk+i, pv-p)$ for all $-p < i < p$. Now Lemma 7.1 implies the existence of LS$[p](t, pk+i, pv+j)$ exist for all $-p \leq j < i \leq p-1$. □

Theorems 8.1 and 8.2 can be used to find numerous number of infinite families of large sets. Note that these theorems are unique in design theory in the sense that they impose no further conditions on the parameters.

We conclude this section by presenting some applications of Theorems 8.1 and 8.2.

Theorem 8.3 *Let $t \geq 6$ and $m \geq 2$. Then there exists an LS[2]$(t, 2^{t-3} - 1, m2^{t-3} - 2)$. Especially, there exists a t-design for any t.*

Proof Since LS[2]$(6, 7, 8m - 2)$ exist for all values of $m \geq 2$ [5], the assertion follows from induction and Theorem 8.1. □

Theorem 8.4 *Let $t \geq 0$ and let a_i and b_i $(0 \leq i \leq t)$ be integers such that $1 \leq b_i \leq a_i \leq p - 1$ for $0 \leq i < t$ and $p | \binom{b_t - 1}{a_t}$. Then there exists an LS[p]$(t, \sum_{i=0}^{t} a_i p^i, \sum_{i=0}^{t} b_i p^i - 1)$.*

Proof We use induction on t. If $t = 0$, then there is an LS[p]$(0, a_0, b_0 - 1)$ since $p | \binom{b_0 - 1}{a_0}$. Now let $t > 0$. By the induction hypothesis, there is an LS[p]$(t - 1, \sum_{i=0}^{t-1} a_{i+1} p^i, \sum_{i=0}^{t-1} b_{i+1} p^i - 1)$. Hence, by Theorem 8.1, an LS[p]$(t, \sum_{i=0}^{t} a_i p^i, \sum_{i=0}^{t} b_i p^i - 1)$ exists. □

Theorem 8.2 is generalized in the following way.

Theorem 8.5 *Let a_i and b_i $(0 \leq i \leq n)$ be integers such that $-p < b_i \leq a_i < p$ for $0 \leq i < t$. If there exists an LS[p]$(t, a_n, b_n - 1)$, then there exists an LS[p]$(t, \sum_{i=0}^{n} a_i p^i, \sum_{i=0}^{n} b_i p^i - 1)$.*

Proof We use induction on n. If $n = 0$, then there is nothing to be proved. So let $n > 0$. By the induction hypothesis, there is an LS[p]$(t, \sum_{i=0}^{n-1} a_{i+1} p^i, \sum_{i=0}^{n-1} b_{i+1} p^i - 1)$. Hence, by Theorem 8.2, an LS[p]$(t, \sum_{i=0}^{n} a_i p^i, \sum_{i=0}^{n} b_i p^i - 1)$ exists. □

9 Large sets of prime sizes

In the previous section we gave two general extension theorems for large sets of prime sizes. In this section, we present some recursive theorems which are more specific and need more assumptions.

Theorem 9.1 [44] *Let t, k, v and f be positive integers such that $v > k > p^f$ and $t \leq (v/p^f) < (k/p^f)$. Suppose that for every $u < v$ the following holds:*
(i) *If $u \geq p^f - 1$ and $t \leq (u/p^f) < p^f - 1$, then $u \in A[p](t, p^f - 1)$,*

(ii) *If* $u \geq k - p^f$ *and* $(u/p^f) = (v/p^f)$, *then* $u \in A[p](t, k - p^f)$.

Then $v \in A[p](t, k)$.

Proof Let $X = \{1, \ldots, v\}$ and let $X_j = \{1, \ldots, j\}$ and $Y_j = X \setminus X_j$ for $j = 1, \ldots, v$. Assume that

$$\mathcal{B}_h = P_{p^f - 1}(X_h) * \{\{h+1\}\} P_{k-p^f}(Y_{h+1}), \qquad p^f - 1 \leq h \leq v - k + p^f - 1.$$

By Lemma 6.5, the sets $\mathcal{B}_h$ partition $P_k(X)$. By Lemma 6.1, it suffices to show that each $\mathcal{B}_h$ is (p, t)-partitionable.

First suppose that $(h/p^f) = p^f - 1$. Then $((v-1-h)/p^f) = (v/p^f)$ and hence $P_{k-p^f}(Y_{h+1})$ is (p, t)-partitionable by the assumption (ii) which in turn concludes that $\mathcal{B}_h$ is (p, t)-partitionable by Lemma 6.2. If $t \leq (h/p^f) < p^f - 1$, then $P_{p^f-1}(X_h)$ is (p, t)-partitionable by the assumption (i) and so is $\mathcal{B}_h$ by Lemma 6.2. Now let $(h/p^f) = r < t$. Then $P_{p^f-1}(X_{h+t-r})$ is (p, t)-partitionable by the assumption (i). It yields that $P_{p^f-1}(X_h)$ is (p, r)-partitionable by Theorem 2.1. We also have $((v-h+r)/p^f) = (v/p^f)$. Therefore, $P_{k-p^f}(Y_{h-r})$ is (p, t)-partitionable by the assumption (ii). By Theorem 2.1, we obtain that $P_{k-p^f}(Y_{h+1})$ is $(p, t-r-1)$-partitionable. Therefore, by Lemma 6.2, $\mathcal{B}_h$ is a (p, t)-partitionable set. □

We use Theorem 9.1 to obtain the following result.

Theorem 9.2 **[23]** *Let* t, k, v *and* f *be positive integers such that* $t \leq (v/p^f) < (k/p^f)$. *Suppose that* $p^f + t \in A[p](t, i)$ *for* $t + 1 \leq i \leq min(k, (p^f + t)/2)$. *Then* $v \in A[p](t, k)$.

Proof Let $t + 1 \leq j \leq \min(k, p^f - 1)$. We show that $w \in \mathrm{A}[p](t, j)$ for all w such that $t \leq (w/p^f) < j$. If $(p^f + t)/2 \leq j \leq \min(k, p^f - 1)$, then $p^f + t - j < (p^f + t)/2$ and so $p^f + t \in \mathrm{A}[p](t, p^f + t - j)$ by the assumption. Hence, for all $t + 1 \leq j \leq \min(k, p^f - 1)$, we have $p^f + t \in \mathrm{A}[p](t, j)$. Using Corollary 7.1 we obtain that $lp^f + t \in \mathrm{A}[p](t, j)$ for $l \geq 1$. Now Theorem 7.1 shows that if $t \leq (w/p^f) < j$, then $w \in \mathrm{A}[p](t, j)$.

If $k > p^f$, then the assertion follows from Theorem 9.1. □

It is possible to generalize Theorem 9.2 in the following way.

Theorem 9.3 **[44]** *Let* t, k, v, f *and* h *be positive integers such that* $f \leq h$, $t < p^f - 1$ *and* $tp^{h-f} \leq (v/p^h) < (k/p^h)$. *Suppose that* $p^f + t \in A[p](t, i)$ *for* $t + 1 \leq i \leq min(k, (p^f + t)/2)$. *Then* $v \in A[p](t, k)$.

Proof The proof is by induction on h. If $h = f$, then we are done by Theorem 9.2. Therefore, assume that $f < h$. Let $k = r_0 p^h + r_1 p + r_2$ and $v = s_0 p^h + s_1 p + s_2$, where $tp^{h-f} \leq s_1 p + s_2 < r_1 p + r_2 < p^h$ and $0 \leq s_2, r_2 < p$. Clearly, $tp^{h-1-f} \leq s_1 \leq r_1 < p^{h-1}$. Now we define

$$(v', k') = \begin{cases} (s_0 p^{h-1} + s_1, r_0 p^{h-1} + r_1) & \text{if} \quad s_1 < r_1, \\ (s_0 p^{h-1} + s_1, r_0 p^{h-1} + r_1 + 1) & \text{if} \quad s_1 = r_1 < p^{h-1} - 1, \\ (s_0 p^{h-1} + s_1 - 1, r_0 p^{h-1} + r_1) & \text{if} \quad s_1 = r_1 = p^{h-1} - 1. \end{cases}$$

Then $tp^{h-1-f} \leq (v'/p^{h-1}) < (k'/p^{h-1})$ and by the induction hypothesis we have $v' \in \mathrm{A}[p](t, k')$. Now Theorem 8.2 shows that $v \in \mathrm{A}[p](t, k)$. □

Theorem 9.4 **[44]** *Let t, k, f and n be positive integers such that $f \leq n$, $t \leq p^{f-1}/2$ and $p^{n-1} \leq k < p^n$. Suppose that $p^f + t \in A[p](t, i)$ for $t + 1 \leq i \leq min(k, (p^f + t)/2)$. Then the following holds:*

(i) *If $v \in A[p](t, k)$, then $v + p^n \in A[p](t, k)$,*
(ii) *If $t \leq (v/p^n) < k$ and $v > 2p^n$, then $v \in A[p](t, k)$.*

Proof By Theorem 9.3, $p^n + i - 1 \in \mathrm{A}[p](t, i)$ for $t + 1 \leq i \leq k$. Therefore, by Corollary 7.2, the assertion (i) holds. We use an induction argument on n to prove (ii). By (i) and Theorem 9.3, it suffices to show that $2p^n + j \in \mathrm{A}[p](t, k)$ for $t \leq j < tp^{n-f}$. We make use of Theorem 7.2 to prove it. Let $a = k - p^{n-1}$, $b = p^{n-1} - 1 + t - 2tp^{n-f}$, $c = d = tp^{n-f} - t$ and $s = 2tp^{n-f} - t$. Then $s = k - 1 - a - b = t + c + d$. Also let $v_1 = p^n + tp^{n-f}$ and $v_2 = p^n + tp^{n-f} + j - t$, where $t \leq j < tp^{n-f}$. Since $t \leq p^{f-1}/2$, we have $b \geq 0$. By Theorem 9.3, we have $v_1 \in \cap_{i=p^{n-1}}^{k} \mathrm{A}[p](t, i)$ and $v_2 \in \cap_{i=k-b}^{k} \mathrm{A}[p](t, i)$. By the induction hypothesis, $v_1 - l \in \mathrm{A}[p](t, p^{n-1} - l)$ for $1 \leq l \leq c$ and $v_2 - l \in \mathrm{A}[p](t, k - b - l)$ for $1 \leq l \leq d$. Therefore, by Theorem 7.2, $v_1 + v_2 - s = 2p^n + j \in \mathrm{A}[p](t, k)$. □

10 Root cases of large sets of prime sizes

Theorem 9.1 shows that many large sets of prime sizes can be constructed from smaller large sets. Theorem 9.2 demonstrates that for given t and k there are a finite number of certain large sets which suffice to produce large sets for every possible value of v. We call these large sets *root cases*. The root cases of large sets of size 2 have already been determined by Ajoodani-Namini [1]. He has also constructed them for $t = 2$ and arbitrary k. There are similar results for large sets of any prime size. The proofs of Theorems 10.1 and 10.2 below are similar and hence we only present the proof of the latter case.

Theorem 10.1 **[1]** *Let t, k and s be positive integers such that $2^s - 1 \leq t < 2^{s+1} - 1$ and $t \leq k$. Suppose that for every j and n such that $0 \leq j \leq [t/2]$ and $t + 1 \leq 2^n + j \leq k$, there exists an*

$LS[2](t, 2^n + j, 2^{n+1} + t)$. Then $A[2](t_1, k_1) = B[2](t_1, k_1)$ for all $2^s - 1 \leq t_1 \leq t$ and $k_1 \leq k$.

Theorem 10.2 **[23]** *Let p be an odd prime and let t, k and s be nonnegative integers such that $p^s - 1 \leq t < p^{s+1} - 1$ and $t \leq k$. Suppose that the following conditions hold:*
(i) *There exists an $LS[p](t, k', p^{s+1} + t)$ for every $t + 1 \leq k' \leq min(k, (p^{s+1} + t)/2)$,*
(ii) *There exists an $LS[p](t, ip^n + j, p^{n+1} + t)$ for every i, j and n such that $0 \leq j \leq t, 1 \leq i \leq (p-1)/2, ip^n + j \leq k$ and $n > s$.*

Then $A[p](t_1, k_1) = B[p](t_1, k_1)$ for all $p^s - 1 \leq t_1 \leq t$ and $k_1 \leq k$.

Proof We use induction on t_1+k_1. First let $t_1 = p^s-1$ and $k_1 = p^s$. From $\mathrm{LS}[p](t, t+1, p^{s+1}+t)$ and Theorem 2.1 we obtain $\mathrm{LS}[p](t_1, k_1, p^{s+1} + t_1)$. Therefore we are done by Theorem 5.2 and Theorem 9.2.

Now suppose that $2p^s - 1 < t_1 + k_1 \leq t + k$ and $t_1 \leq t$. Suppose that ℓ_1 is the smallest positive integer such that $(k_1/p^{\ell_1}) > t_1$. Assume that we have shown that

$$p^{\ell_1} + t_1 \in \mathrm{A}[p](t_1, k'), \text{ for all } t_1 + 1 \leq k' \leq \min(k_1, (p^{\ell_1} + t_1)/2). \tag{10.1}$$

Let $v \in \mathrm{B}[p](t_1, k_1)$. By Theorem 5.1, there exists $r \geq \ell_1$ such that $t_1 \leq (v/p^r) < (k_1/p^r)$. We have

$$[v/p^r]p^r + t_1 \in \mathrm{A}[p](t_1, [k_1/p^r]\, p^r + j),$$

for all $(k_1/p^r) - (v/p^r) + t_1 \leq j \leq (k_1/p^r)$. Since if $j < (k_1/p^r)$, then we are done by the induction hypothesis. If $j = (k_1/p^r)$, then we are done by (10.1) and Theorem 9.2. Hence, by Theorem 7.1, $v = [v/p^r]p^r + (v/p^r) \in \mathrm{A}[p](t_1, k_1)$.

Now we prove (10.1). If $k_1 > (p^{\ell_1} + t_1)/2$, then we are done by the induction hypothesis. So let

$$k_1 \leq (p^{\ell_1} + t_1)/2. \tag{10.2}$$

By the induction hypothesis, it is sufficient to establish the existence of an $\mathrm{LS}[p](t_1, k_1, p^{\ell_1} + t_1)$. From (10.2), we have $[k_1/p^{\ell_1}] = 0$. Therefore, $\ell_1 \geq s+1$. If $\ell_1 = s+1$, then by (i), we can obtain $\mathrm{LS}[p](t_1, k_1, p^{s+1} + t_1)$ from $\mathrm{LS}[p](t, \max(t+1, k_1), p^{s+1} + t)$ using Theorem 2.1. So suppose that $\ell_1 > s + 1$. Let $[k_1/p^{\ell_1 - 1}] = i$ and $(k_1/p^{\ell_1-1}) = j$. Clearly $j \leq t_1 \leq t$. By (10.2), we also obtain that $i \leq (p-1)/2$. Now $\mathrm{LS}[p](t, ip^{\ell_1-1} + j, p^{\ell_1} + t)$, which exists by (ii), may be employed to obtain $\mathrm{LS}[p](t_1, ip^{\ell_1-1} + j, p^{\ell_1} + t_1)$ via Theorem 2.1. □

11 More results on large sets of sizes two and three

In the previous sections we presented recursive constructions for large sets of prime sizes. It is possible to find more comprehensive results for large sets of sizes two and three. We have to wait until Section 13 to find out what existence results are obtained by the following theorems.

Theorem 11.1 **[2]** *Let t, k, f and n be positive integers such that $f < n$, $t \leq 2^{f-2}$ and $2^{n-1} \leq k < 2^n$. Suppose that $A[2](t,i) = B[2](t,i)$ for $t < i < 2^f$. Then*

(i) *$B[2](t,k) \setminus A[2](t,k) \subset \{2^n + j \mid t \leq j < t2^{n-f}\}$,*
(ii) *If $2^{n-1} + t2^{n-f} \leq k < 2^n$, then $A[2](t,k) = B[2](t,k)$.*

Proof The proof in by induction on n. Assume that $v \in \mathrm{B}[2](t,k)$ and $v \neq 2^n + j$, $t \leq j < t2^{n-f}$. Let $w = (v/2^n)$. If $w < k$, then by Theorem 9.3 (if $v \leq 2^{n+1}$) and Theorem 9.4 (if $v > 2^{n+1}$), $v \in \mathrm{A}[2](t,k)$. So let $k \leq w < 2^n$. By Theorem 9.4, it suffices to show that $w \in \mathrm{A}[2](t,k)$, or equivalently $w \in \mathrm{A}[2](t, w-k)$. If $w - k < 2^{n-2}$, then we are done by the induction hypothesis. If $2^{n-2} \leq w - k < 2^{n-1}$, then $w \geq 2^{n-1} + 2^{n-2} \geq 2^{n-1} + t2^{n-1-f}$. Therefore, $w \in \mathrm{A}[2](t, w-k)$ by the induction hypothesis.

Now let $2^{n-1} + t2^{n-f} \leq k < 2^n$ and $v = 2^n + j$, $t \leq j < t2^{n-f}$. Then $v - k < 2^n + t2^{n-f} - 2^{n-1} - t2^{n-f} = 2^{n-1}$. Hence, by the induction hypothesis, $v \in \mathrm{A}[2](t, v-k)$ which in turn yields that $v \in \mathrm{A}[2](t,k)$. □

Theorem 11.2 **[44]** *Let t, k, f and n be positive integers such that $f < n$, $t \leq 3^{f-2}$ and $3^{n-1} \leq k < 3^n$. Suppose that $A[3](t,i) = B[3](t,i)$ for $t < i < 3^f$. Then*

(i) *$B[3](t,k) \setminus A[3](t,k) \subset \{3^n + j \mid t \leq j < t3^{n-f}\}$,*
(ii) *If $2.3^{n-1} + t3^{n-f} \leq k < 3^n$, then $A[3](t,k) = B[3](t,k)$.*

Proof The proof is mostly similar to that of Theorem 11.1. We use an induction argument on k. Assume that $v \in \mathrm{B}[3](t,k)$ and $v \neq 3^n + j$, $t \leq j < t3^{n-f}$. Let $w = (v/3^n)$. If $w < k$, then by Theorems 9.3 and 9.4, $v \in \mathrm{A}[3](t,k)$. So let $k \leq w < 3^n$. By Theorem 9.4, it suffices to show that $w \in \mathrm{A}[3](t,k)$, or equivalently $w \in \mathrm{A}[3](t, w-k)$. If $w - k < 3^{n-2}$ or $3^{n-1} \leq w - k < k$, then we are done by the induction hypothesis. If $3^{n-2} \leq w - k < 3^{n-1}$, then $w \geq 3^{n-1} + 3^{n-2} \geq 3^{n-1} + t3^{n-1-f}$. Therefore, $w \in \mathrm{A}[3](t, w-k)$ by the induction hypothesis. Now suppose that $w - k > k$. Then we have $w = 2.3^{n-1} + i$ and $k = 3^{n-1} + j$, where $0 \leq j < i < 3^{n-1}$. By Theorem 5.1, there is $h < n-1$ such that $t \leq (w/3^h) < (k/3^h)$. Such an h is not necessarily unique and we can assume that $(k/3^h) \geq 3^{h-1}$. We now have $2.3^{n-2} + t3^{n-1-f} \leq k - 3^h < k$. Therefore, by the induction hypothesis and Theorem 9.1, it yields that $w \in \mathrm{A}[3](t,k)$.

Now let $2.3^{n-1} + t3^{n-f} \leq k < 3^n$ and $v = 3^n + j$, $t \leq j < t3^{n-f}$. Then $v - k < 3^n + t3^{n-f} - 2.3^{n-1} - t3^{n-f} = 3^{n-1}$. Hence, by the induction hypothesis, $v \in \mathrm{A}[3](t, v-k)$ which concludes that $v \in \mathrm{A}[3](t,k)$. □

12 Another classes of root cases for large sets of sizes 2 and 3

In Section 10, we showed that one can construct all possible large sets of sizes two and three from the root cases $\mathrm{LS}[2](t, 2^n + j_1, 2^{n+1} + t)$ and $\mathrm{LS}[3](t, 3^n + j_2, 3^{n+1} + t)$, respectively, where j_1, j_2, and n are nonnegative integers such that $j_1 \leq t/2$ and $j_2 \leq t$. It is quite interesting that we can introduce different classes of root cases which are not related to t and say the story for all t. These classes are identified in the following theorems. They have similar proofs and we only present the proof of Theorem 12.2.

Theorem 12.1 *If there exists an* $LS[2](2^n - 2, 2^n - 1, 2^{n+1} - 2)$ *for every positive integer* n*, then* $A[2](t, k) = B[2](t, k)$ *for any* t *and* k*.*

Theorem 12.2 *If there exists an* $LS[3](3^n - 2, 3^n - 1, 2.3^n - 2)$ *for every positive integer* n*, then* $A[3](t, k) = B[3](t, k)$ *for any* t *and* k*.*

Proof For a given t, assume that $3^s - 1 \leq t < 3^{s+1} - 1$. By Theorem 10.2, it is sufficient to show the existence of the following families of large sets:

(i) $\mathrm{LS}[3](t, 3^s + j_0, 3^{s+1} + t)$, where $t - 3^s + 1 \leq j_0 \leq (3^s + t)/2$,

(ii) $\mathrm{LS}[3](t, 3^n + j_0, 3^{n+1} + t)$, where $n > s$ and $0 \leq j_0 \leq t$.

Large sets in the class (i) can be obtained from $\mathrm{LS}[3](3^{s+1} - 2, 3^{s+1} - 1, 2.3^{s+1} - 2)$ using Theorem 2.1. Simply, suppose that $i = 3^{s+1} - 2 - t, j = 2.3^s - 1 - j_0$ and $l = 3^{s+1} - 2 - t$. Similarly, the class (ii) can be constructed from $\mathrm{LS}[3](3^{n+1} - 2, 3^{n+1} - 1, 2.3^{n+1} - 2)$ by taking $i = 3^{n+1} - 2 - t, j = 2.3^n - 1 - j_0$ and $l = 3^{n+1} - 2 - t$ in Theorem 2.1. □

Finally, we note that by Theorem 4.1, large sets $\mathrm{LS}[2](2^n - 2, 2^n - 1, 2^{n+1} - 2)$ and $\mathrm{LS}[3](3^n - 2, 3^n - 1, 2.3^n - 2)$ can be considered as the extensions of $\mathrm{LS}[2](2^n - 3, 2^n - 2, 2^{n+1} - 3)$ and $\mathrm{LS}[3](3^n - 3, 3^n - 2, 2.3^n - 3)$, respectively. Therefore, it is possible to consider these latter classes as root cases which have to be constructed.

13 Existence results

We use the results of the previous sections to find some existence results on large sets of prime sizes. In 1987, Hartman [20] conjectured that the necessary conditions (2.4) are sufficient for the existence of large sets of size 2. Later Khosrovshahi extended this conjecture to large sets of sizes 3 and 4 [4]. These conjectures have not yet been settled and their proofs seem to be far from reach. Note that Theorems 11.1 and 11.2 indicate that for given t if these conjectures are true for some small values of k, then they will be true for infinitely many values of k. We use them to find some existence results in Theorems 13.3 and 13.6 below. By now, the best result is the following theorem due to Ajoodani-Namini.

Theorem 13.1 **[1]** *$A[2](2,k) = B[2](2,k)$ for all $k \geq 2$.*

By Theorem 10.1, to establish this result, one should construct two families of large sets LS[2]$(2, 2^n+1, 2^{n+1}+2)$ and LS[2]$(2, 2^n, 2^{n+1}+2)$. The first family exists according to Corollary 4.1. Ajoodani-Namini has also constructed the second family by the use of $(2,2)$-partitionable sets. His construction is long and complicated enough to be rewritten here. The interested reader is referred to [1] or [2]. We note that Ajoodani-Namini has also shown that Hartman's conjecture is true asymptotically for $k = t+1$ [2]. He makes use of partitionable sets and also Teirlink's methods in his proof.

Theorem 13.2 **[1, 5, 20, 31]** *If $3 \leq t \leq 5$ and $k \leq 15$ or, $t = 6$ and $k = 7, 8, 9$, then $A[2](t,k) = B[2](t,k)$.*

Proof First suppose that $3 \leq t \leq 6$ and $k \leq 9$. By Theorem 10.1, we need the following large sets:

(i) LS[2]$(6, 7, 14)$, (ii) LS[2]$(6, 8, 22)$, (iii) LS[2]$(6, 9, 22)$.

These large sets exist, the first one by [30] and the last two by [31].

To complete the proof we also need LS[2]$(5, 10, 21)$ which is known to exist by [31]. □

Theorem 13.3 **[2]** *If $2^{n-1} + 3.2^{n-4} \leq k < 2^n$ for a positive integer $n > 4$, then $A[2](3,k) = B[2](3,k)$.*

Proof By Theorem 13.2, A[2]$(3,i)$ = B[2]$(3,i)$ for $i < 16$ and so the assertion holds by Theorem 11.1. □

Theorem 13.4 **[23]** *If $k \leq 80$, then $A[3](2,k) = B[3](2,k)$.*

Proof By Theorem 10.2, we have to establish the existence of the following large sets:

(i) LS[3]$(2, 9, 29)$, (ii) LS[3]$(2, 10, 29)$, (iii) LS[3]$(2, 11, 29)$,
(iv) LS[3]$(2, 27, 83)$, (v) LS[3]$(2, 28, 83)$, (vi) LS[3]$(2, 29, 83)$.

There exist LS[3]$(2,9,29)$ and LS[3]$(3,11,30)$ by [32]. By the last one we obtain LS[3]$(2,10,29)$ and LS[3]$(2,11,29)$ via Theorem 2.1. Large sets (iv)-(vi) are constructed through Theorem 6 in [32]. □

Theorem 13.5 **[43]** *If $t \leq 4$ and $k \leq 8$, then $A[3](t,k) = B[3](t,k)$.*

Proof Large sets LS[3]$(4,5,13)$ and LS[3]$(4,6,13)$ exist by [25] and [32], respectively. Therefore, by Theorem 10.2, we are done. □

Theorem 13.6 **[44]** *If $2.3^{n-1} + 2.3^{n-4} \leq k < 3^n$ for a positive integer $n > 4$, then $A[3](2,k) = B[3](2,k)$.*

Proof By Theorem 13.4, A[3]$(2,i)$ = B[3]$(2,i)$ for $i < 81$. Therefore, the assertion follows from Theorem 11.2. □

Theorem 13.7 **[32]** *If $k \leq 5$, then $A[5](2,k) = B[5](2,k) \setminus \{7\}$.*

Proof If $k = 3$, then we are done by Theorem 3.2. So first let $k = 4$. By Theorem 5.1, we have

$$\mathrm{B}[5](2,4) = \{5l + i |\ l \geq 1, \quad i = 2,3\}.$$

It is well known that LS[5]$(2,4,7)$ does not exist. There exist LS[5]$(2,4,8)$ [41], LS[5]$(2,4,12)$ [26], LS[5]$(2,4,13)$ [13] and LS[5]$(2,4,17)$ [49]. Therefore, by Corollary 7.1, we are able to construct LS[5]$(2,4,5l+i)$ for all $l \geq 2$ and $i = 2,3$.

For $k = 5$, we use Theorem 9.2. By Theorem 5.1, we have

$$\mathrm{B}[5](2,5) = \{25l + i |\ l \geq 1, \quad i = 2,3,4\}.$$

It suffices to have LS[5]$(2,3,27)$ and LS[5]$(2,4,27)$ which exist by the paragraphs above and LS[5]$(2,5,27)$ which exists by [32]. □

Theorem 13.8 **[32]** *If $k \leq 5$, then $A[5](3,k) = B[5](3,k) \setminus \{8\}$.*

Proof By Theorem 5.1, we have

$$\mathrm{B}[5](3,4) = \{5l + 3 |\ l \geq 1\}.$$

LS[5]$(3,4,8)$ does not exist because of the non-existence of LS[5]$(2,3,7)$. Since LS[5]$(3,4,13)$ [26] and LS[5]$(3,4,18)$ [49] exist, we obtain LS[5]$(3,4,5l+3)$ for $l \geq 2$ by Corollary 7.1.

For $k = 5$, we use Theorem 9.2. By Theorem 5.1, we have

$$\mathrm{B}[5](3,5) = \{25l + i|\ l \geq 1,\quad i = 3,4\}.$$

It suffices to have LS[5](3, 4, 28) which exist by the paragraph above and LS[5](3, 5, 28) which exists by [32]. □

Theorem 13.9 **[32]** *If $k \leq 6$, then $A[7](2,k) = B[7](2,k)$.*

Proof By Theorem 10.2, we need LS[7](2, 3, 9) and LS[7](2, 4, 9) which exist by [24] and [26], respectively. □

Theorem 13.10 **[32]** *If $k \leq 10$, then $A[11](2,k) = B[11](2,k)$.*

Proof By Theorem 10.2, we need large sets LS[11](2, 3, 13), LS[11](2, 4, 13), LS[11](2, 5, 13) and LS[11](2, 6, 13) which exist by [33], [13] and the last two by [12], respectively. □

Theorem 13.11 **[32]** *If $k \leq 5$, then $A[29](2,k) = B[29](2,k)$.*

Proof By Theorem 10.2, we need LS[29](2, 3, 31), LS[29](2, 4, 31) and LS[29](2, 5, 31) which exist by [33] and the last two by [32], respectively. □

We summarize the results in Table I.

Table I ($*$ means all admissible values)

N	t	k	v	Ref.
$*$	1	$*$	$*$	[7, 20]
$*$	2	3	$\neq 7$	[33, 34, 46]
2	2	$*$	$*$	[1]
2	≤ 5	≤ 16	$*$	[1, 5, 20, 31]
2	6	7, 8, 9	$*$	[1, 31]
3	2	≤ 80	$*$	[23]
3	≤ 4	≤ 8	$*$	[43]
5	2	≤ 5	$\neq 7$	[32]
5	3	≤ 5	$\neq 8$	[32]
7	2	≤ 6	$*$	[32]
11	2	≤ 10	$*$	[32]
29	2	≤ 5	$*$	[32]

14 Open problems

As the previous sections suggest there are many unsolved problems on large sets of t-designs. We list some open problems here for further researches.

Problem 1 Construct an LS[3]$(5, 6, 14)$. There are five 5-$(14, 6, 3)$ designs known [36], but the existence of LS[3]$(5, 6, 14)$ is in doubt. In the case of nonexistence, it will be a counterexample for Khosrovshahi's conjecture on large sets of size 3.

Problem 2 Is it possible to find an LS[2]$(6, 7, 14)$ through partitionable sets? All known examples of this large set have been found by prescribing some groups as automorphism group of designs.

Problem 3 Construct LS[3]$(2, 3^n + j, 3^{n+1} + 2)$ for $j = 0, 1, 2$ and for any $n > 3$. If these exist, then we will have $A[3](2, k) = B[3](2, k)$ for all $k \geq 2$.

Problem 4 Prove or disprove the existence of LS[2]$(2^n - 2, 2^n - 1, 2^{n+1} - 2)$ for $n > 4$. If these large sets exist, then Hartman's conjecture will be true.

Problem 5 Prove or disprove the existence of LS[3]$(3^n - 2, 3^n - 1, 2.3^n - 2)$ for $n > 1$. If these large sets exist, then Khosrovshahi's conjecture on large sets of size 3 will be true.

Problem 6 Determine root cases for large sets of any sizes. In particular, determine root cases for large sets of prime power sizes.

Problem 7 Are there general theorems similar to Theorems 8.1 and 8.2 for large sets of prime power sizes.

15 A table of small large sets

In [26], all parameter sets on less than or equal 12 points have been settled. In [12], a table on the existence of large sets with at most 18 points is presented. We here present an updated version of it. We exclude large sets of 1-designs from the table since according to Theorem 3.1, their existence problem has completely been answered.

Table II Large sets for $2 \le t < k \le v/2$ and $v \le 18$.

large set	Reference	large set	Reference
LS[2](2, 3, 6)	Table I	LS[3](2, 5, 12)	Table I
LS[2](2, 3, 10)	Table I	LS[3](2, 5, 13)	Table I
LS[2](2, 3, 14)	Table I	LS[3](2, 6, 12)	Table I
LS[2](2, 3, 18)	Table I	LS[3](2, 6, 13)	Table I
LS[2](2, 4, 10)	Table I	LS[3](2, 6, 14)	Table I
LS[2](2, 4, 11)	Table I	LS[3](2, 7, 14)	Table I
LS[2](2, 4, 18)	Table I	LS[3](2, 7, 15)	Table I
LS[2](2, 5, 10)	Table I	LS[3](2, 8, 16)	Table I
LS[2](2, 5, 11)	Table I	LS[3](3, 4, 12)	Table I
LS[2](2, 5, 12)	Table I	LS[3](3, 5, 12)	Table I
LS[2](2, 5, 18)	Table I	LS[3](3, 5, 13)	Table I
LS[2](2, 6, 12)	Table I	LS[3](3, 6, 12)	Table I
LS[2](2, 6, 13)	Table I	LS[3](3, 6, 13)	Table I
LS[2](2, 6, 18)	Table I	LS[3](3, 6, 14)	Table I
LS[2](2, 7, 14)	Table I	LS[3](3, 7, 14)	Table I
LS[2](2, 7, 18)	Table I	LS[3](3, 7, 15)	Table I
LS[2](2, 8, 18)	Table I	LS[3](3, 8, 16)	Table I
LS[2](2, 9, 18)	Table I	LS[3](4, 5, 13)	Table I
LS[2](3, 4, 11)	Table I	LS[3](4, 6, 13)	Table I
LS[2](3, 5, 11)	Table I	LS[3](4, 6, 14)	Table I
LS[2](3, 5, 12)	Table I	LS[3](4, 7, 14)	Table I
LS[2](3, 6, 12)	Table I	LS[3](4, 7, 15)	Table I
LS[2](3, 6, 13)	Table I	LS[3](4, 8, 16)	Table I
LS[2](3, 7, 14)	Table I	LS[3](5, 6, 14)	
LS[2](4, 5, 12)	Table I	LS[3](5, 7, 14)	ext. of LS[3](4, 6, 13)
LS[2](4, 6, 12)	Table I	LS[3](5, 7, 15)	
LS[2](4, 6, 13)	Table I	LS[3](5, 8, 16)	ext. of LS[3](4, 7, 15)
LS[2](4, 7, 14)	Table I	LS[3](6, 7, 15)	
LS[2](5, 6, 13)	Table I	LS[3](6, 8, 16)	
LS[2](5, 7, 14)	Table I	LS[3](7, 8, 16)	
LS[2](6, 7, 14)	Table I	LS[4](2, 3, 10)	Table I
LS[3](2, 3, 11)	Table I	LS[4](2, 3, 18)	Table I
LS[3](2, 4, 11)	Table I	LS[4](2, 4, 18)	
LS[3](2, 4, 12)	Table I	LS[4](2, 5, 18)	
LS[3](2, 5, 11)	Table I	LS[4](2, 6, 18)	

Table II Continued.

large set	Reference	large set	Reference
LS[4](2, 7, 14)		LS[7](2, 5, 10)	Table I
LS[4](2, 7, 18)		LS[7](2, 5, 11)	Table I
LS[5](2, 3, 7)	no [10]	LS[7](2, 5, 16)	Table I
LS[5](2, 3, 12)	Table I	LS[7](2, 5, 17)	Table I
LS[5](2, 3, 17)	Table I	LS[7](2, 5, 18)	Table I
LS[5](2, 4, 8)	Table I	LS[7](2, 6, 12)	Table I
LS[5](2, 4, 12)	Table I	LS[7](2, 6, 16)	Table I
LS[5](2, 4, 13)	Table I	LS[7](2, 6, 17)	Table I
LS[5](2, 4, 17)	Table I	LS[7](2, 6, 18)	Table I
LS[5](2, 4, 18)	Table I	LS[7](3, 4, 10)	no [28]
LS[5](2, 8, 17)	Laue	LS[7](3, 4, 17)	[12]
LS[5](2, 9, 18)		LS[7](3, 5, 10)	ext. of LS[7](2, 4, 9)
LS[5](3, 4, 8)	no LS[5](2, 3, 7)	LS[7](3, 5, 11)	[17]
LS[5](3, 4, 13)	Table I	LS[7](3, 5, 17)	Laue
LS[5](3, 4, 18)	Table I	LS[7](3, 5, 18)	Laue
LS[5](3, 9, 18)	ext. of LS[5](2, 8, 17)	LS[7](3, 6, 12)	LS[42](3, 6, 12)
LS[6](2, 4, 11)	[11]	LS[7](3, 6, 17)	Laue
LS[6](2, 5, 11)	LS[42](2, 5, 11)	LS[7](3, 6, 18)	Laue
LS[6](2, 5, 12)	LS[6](2, 5, 12)	LS[7](4, 5, 11)	no LS[7](3, 4, 10)
LS[6](2, 6, 12)	LS[42](2, 6, 12)	LS[7](4, 5, 18)	
LS[6](2, 6, 13)	[12]	LS[7](4, 6, 12)	
LS[6](2, 7, 14)		LS[7](4, 6, 18)	
LS[6](3, 5, 12)	[26]	LS[7](5, 6, 12)	no LS[7](3, 4, 10)
LS[6](3, 6, 12)	LS[42](3, 6, 12)	LS[8](2, 3, 18)	
LS[6](3, 6, 13)	[12]	LS[8](2, 7, 18)	
LS[6](3, 7, 14)		LS[10](2, 4, 18)	
LS[6](4, 6, 13)		LS[10](2, 9, 18)	
LS[6](4, 7, 14)		LS[11](2, 3, 13)	Table I
LS[6](5, 7, 14)		LS[11](2, 4, 13)	LS[55](2, 4, 13)
LS[7](2, 3, 9)	Table I	LS[11](2, 4, 14)	[25]
LS[7](2, 3, 16)	Table I	LS[11](2, 5, 13)	[12]
LS[7](2, 4, 9)	Table I	LS[11](2, 5, 14)	[12]
LS[7](2, 4, 10)	Table I	LS[11](2, 5, 15)	[12]
LS[7](2, 4, 16)	Table I	LS[11](2, 6, 13)	[12]
LS[7](2, 4, 17)	Table I	LS[11](2, 6, 14)	Table I

Table II Continued.

large set	Reference	large set	Reference
LS[11](2, 6, 15)	Table I	LS[11](5, 7, 16)	
LS[11](2, 6, 16)	Table I	LS[11](5, 7, 17)	
LS[11](2, 7, 14)	Table I	LS[11](5, 8, 16)	
LS[11](2, 7, 15)	Table I	LS[11](5, 8, 17)	
LS[11](2, 7, 16)	Table I	LS[11](5, 8, 18)	
LS[11](2, 7, 17)	Table I	LS[11](5, 9, 18)	
LS[11](2, 8, 16)	Table I	LS[11](6, 7, 17)	no LS[11](4, 5, 15)
LS[11](2, 8, 17)	Table I	LS[11](6, 8, 17)	
LS[11](2, 8, 18)	Table I	LS[11](6, 8, 18)	
LS[11](2, 9, 18)	Table I	LS[11](6, 9, 18)	
LS[11](3, 4, 14)		LS[11](7, 8, 18)	no LS[11](4, 5, 15)
LS[11](3, 5, 14)		LS[11](7, 9, 18)	
LS[11](3, 5, 15)		LS[12](2, 7, 14)	
LS[11](3, 6, 14)		LS[13](2, 3, 15)	Table I
LS[11](3, 6, 15)		LS[13](2, 4, 15)	Laue
LS[11](3, 6, 16)		LS[13](2, 4, 16)	Magliveras
LS[11](3, 7, 14)	ext. of LS[11](2, 6, 13)	LS[13](2, 5, 15)	
LS[11](3, 7, 15)		LS[13](2, 5, 16)	
LS[11](3, 7, 16)		LS[13](2, 5, 17)	
LS[11](3, 7, 17)		LS[13](2, 6, 15)	
LS[11](3, 8, 16)	ext. of LS[11](2, 7, 15)	LS[13](2, 6, 16)	
LS[11](3, 8, 17)		LS[13](2, 6, 17)	
LS[11](3, 8, 18)		LS[13](2, 6, 18)	
LS[11](3, 9, 18)	ext. of LS[11](2, 8, 17)	LS[13](2, 7, 15)	
LS[11](4, 5, 15)	no [35]	LS[13](2, 7, 16)	
LS[11](4, 6, 15)		LS[13](2, 7, 17)	
LS[11](4, 6, 16)		LS[13](2, 7, 18)	
LS[11](4, 7, 15)		LS[13](2, 8, 16)	
LS[11](4, 7, 16)		LS[13](2, 8, 17)	
LS[11](4, 7, 17)		LS[13](2, 8, 18)	
LS[11](4, 8, 16)		LS[13](2, 9, 18)	
LS[11](4, 8, 17)		LS[13](3, 4, 16)	
LS[11](4, 8, 18)		LS[13](3, 5, 16)	
LS[11](4, 9, 18)		LS[13](3, 5, 17)	
LS[11](5, 6, 16)	no LS[11](4, 5, 15)	LS[13](3, 6, 16)	

Table II Continued.

large set	Reference	large set	Reference
LS[13](3, 6, 17)		LS[22](2, 6, 13)	
LS[13](3, 6, 18)		LS[22](2, 7, 14)	
LS[13](3, 7, 16)		LS[22](2, 8, 18)	
LS[13](3, 7, 17)		LS[22](2, 9, 18)	
LS[13](3, 7, 18)		LS[22](3, 7, 14)	
LS[13](3, 8, 16)		LS[26](2, 6, 18)	
LS[13](3, 8, 17)		LS[26](2, 7, 18)	
LS[13](3, 8, 18)		LS[26](2, 8, 18)	
LS[13](3, 9, 18)		LS[26](2, 9, 18)	
LS[13](4, 5, 17)		LS[28](2, 5, 18)	
LS[13](4, 6, 17)		LS[28](2, 6, 18)	
LS[13](4, 6, 18)		LS[33](2, 5, 13)	
LS[13](4, 7, 17)		LS[33](2, 6, 13)	
LS[13](4, 7, 18)		LS[33](2, 6, 14)	
LS[13](4, 8, 17)		LS[33](2, 7, 14)	
LS[13](4, 8, 18)		LS[33](2, 7, 15)	
LS[13](4, 9, 18)		LS[33](2, 8, 16)	
LS[13](5, 6, 18)		LS[33](3, 6, 14)	
LS[13](5, 7, 18)		LS[33](3, 7, 14)	
LS[13](5, 8, 18)		LS[33](3, 7, 15)	
LS[13](5, 9, 18)		LS[33](3, 8, 16)	
LS[14](2, 4, 10)	[26]	LS[33](4, 7, 15)	
LS[14](2, 5, 10)	[26]	LS[33](4, 8, 16)	
LS[14](2, 5, 11)	LS[42](2, 5, 11)	LS[33](5, 8, 16)	
LS[14](2, 5, 18)		LS[35](2, 4, 17)	
LS[14](2, 6, 12)	LS[42](2, 6, 12)	LS[39](2, 7, 15)	
LS[14](2, 6, 18)		LS[39](2, 8, 16)	
LS[14](3, 5, 11)	no [16]	LS[39](3, 8, 16)	
LS[14](3, 6, 12)		LS[42](2, 5, 11)	[26]
LS[14](4, 6, 12)	no LS[14](3, 5, 11)	LS[42](2, 6, 12)	LS[42](3, 6, 12)
LS[15](2, 4, 12)	[26]	LS[42](3, 6, 12)	ext. of LS[42](2, 5, 11)
LS[20](2, 4, 18)		LS[44](2, 7, 14)	
LS[21](2, 5, 11)	LS[42](2, 5, 11)	LS[52](2, 6, 18)	
LS[21](2, 6, 12)	LS[42](2, 6, 12)	LS[52](2, 7, 18)	
LS[21](3, 6, 12)	LS[42](3, 6, 12)	LS[55](2, 4, 13)	[13]

Table II Continued.

large set	Reference	large set	Reference
LS[55](2, 8, 17)		LS[143](2, 8, 16)	
LS[55](2, 9, 18)		LS[143](2, 8, 17)	
LS[55](3, 9, 18)		LS[143](2, 8, 18)	
LS[65](2, 8, 17)		LS[143](2, 9, 18)	
LS[65](2, 9, 18)		LS[143](3, 6, 16)	
LS[65](3, 9, 18)		LS[143](3, 7, 16)	
LS[66](2, 6, 13)		LS[143](3, 7, 17)	
LS[66](2, 7, 14)		LS[143](3, 8, 16)	
LS[66](3, 7, 14)		LS[143](3, 8, 17)	
LS[77](2, 6, 16)		LS[143](3, 8, 18)	
LS[91](2, 4, 16)		LS[143](3, 9, 18)	
LS[91](2, 5, 16)		LS[143](4, 7, 17)	
LS[91](2, 5, 17)		LS[143](4, 8, 17)	
LS[91](2, 6, 16)		LS[143](4, 8, 18)	
LS[91](2, 6, 17)		LS[143](4, 9, 18)	
LS[91](2, 6, 18)		LS[143](5, 8, 18)	
LS[91](3, 5, 17)		LS[143](5, 9, 18)	
LS[91](3, 6, 17)		LS[182](2, 6, 18)	
LS[91](3, 6, 18)		LS[286](2, 8, 18)	
LS[91](4, 6, 18)		LS[286](2, 9, 18)	
LS[104](2, 7, 18)		LS[364](2, 6, 18)	
LS[110](2, 9, 18)		LS[429](2, 7, 15)	
LS[130](2, 9, 18)		LS[429](2, 8, 16)	
LS[132](2, 7, 14)		LS[429](3, 8, 16)	
LS[143](2, 5, 15)		LS[715](2, 8, 17)	
LS[143](2, 6, 15)		LS[715](2, 9, 18)	
LS[143](2, 6, 16)		LS[715](3, 9, 18)	
LS[143](2, 7, 15)		LS[1001](2, 6, 16)	
LS[143](2, 7, 16)		LS[1430](2, 9, 18)	
LS[143](2, 7, 17)			

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