Median eigenvalues of bipartite graphs

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Abstract

For a graph G of order n and with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$, the HL-index $R(G)$ is defined as $R(G) = \max\left\{|\lambda_{\lfloor (n+1)/2\rfloor}|, |\lambda_{\lceil (n+1)/2\rceil}|\right\}$. We show that for every connected bipartite graph G with maximum degree $\Delta \geqslant 3$, $R(G) \leqslant \sqrt{\Delta - 2}$ unless G is the the incidence graph of a projective plane of order $\Delta - 1$. We also present an approach through graph covering to construct infinite families of bipartite graphs with large HL-index.

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1 Introduction

Unless explicitly stated, we assume that all graphs in this paper are simple, i.e. multiple edges and loops are not allowed. The *adjacency matrix* of

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G, denoted by $A(G) = (a_{uv})_{u,v \in V(G)}$, is a $(0,1)$-matrix whose rows and columns are indexed by the vertices of G such that $a_{uv} = 1$ if and only if u is adjacent to v. We use $\deg_G(v)$ to denote the degree of vertex v in G. The set of all neighbours of v is denoted by $N_G(v)$ and we write $N_G[v] = N_G(v) \cup \{v\}$. The smallest and largest degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of G. Then $\lambda_{\lfloor (n+1)/2 \rfloor}$ and $\lambda_{\lceil (n+1)/2 \rceil}$ are called the *median* eigenvalue(s) of G. These eigenvalues play an important role in mathematical chemistry since they are related to the HOMO-LUMO separation, see, e.g. [7] and [3, 4]. Following [9], we define the *HL-index* $R(G)$ of the graph G as

$$R(G) = \max \left\{ |\lambda_{\lfloor (n+1)/2 \rfloor}|, |\lambda_{\lceil (n+1)/2 \rceil}| \right\}.$$

If G is a bipartite graph, then $R(G)$ is equal to $\lambda_{n/2}$ if n is even and 0, otherwise. In this paper, we show that for every connected bipartite graph G with maximum degree Δ, $R(G) \leqslant \sqrt{\Delta - 2}$ unless G is the the incidence graph of a projective plane of order $\Delta - 1$, in which case it is equal to $\sqrt{\Delta - 1}$. This extends the result of one of the authors [13] who proved the same for subcubic graphs.

On the other hand, we present an approach through graph covering to construct infinite families of connected graphs with large HL-index. Graph coverings and analysis of their eigenvalues were instrumental in a recent breakthrough in spectral graph theory by Marcus, Spielman, and Srivastava who used graph coverings to construct infinite families of Ramanujan graphs of arbitrary degrees [11] (and for solving the Kadison-Singer Conjecture [12]). In our paper, we find another application of a different character. As opposed to double covers used in [11], we use k-fold covering graphs with cyclic permutation representation and show that the behavior of median eigenvalues can be controlled in certain instances. The main ingredient is a generalization of a result of Bilu and Linial [1] that eigenvalues of double covers over a graph G are the union of the eigenvalues of G and the eigenvalues of certain cover matrix A^- that is obtained from the adjacency matrix by replacing some of its entries by -1. In our case, we use a family A^λ of such matrices, where instead of -1 we use certain powers of a parameter $\lambda \in [-1, 1]$. This result seems to be of independent interest.

2 Bounds for bipartite graphs

In this section we obtain upper bounds on the HL-index of bipartite graphs in terms of maximum and minimum degrees of graphs. We consider regular graphs first.

Theorem 1. *Let G be a connected bipartite k-regular graph, where $k \geq 3$. If $R(G) > \sqrt{k-2}$, then $R(G) = \sqrt{k-1}$ and G is the incidence graph of a projective plane of order $k-1$.*

Proof. Let $|V(G)| = 2n$. The adjacency matrix of G can be written as

$$A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where B is a square matrix of order n. The matrix $E = BB^T - kI$ is a symmetric matrix of order n. Assuming that $R(G) > \sqrt{k-2}$, every eigenvalue λ of G satisfies $\lambda^2 > k-2$ and hence all eigenvalues of E are greater than -2. Hence, $E + 2I$ is a positive definite matrix. All diagonal entries of this matrix are equal to 2. Positive definiteness in turn implies that all off diagonal entries are 0 or 1. It follows that E is the adjacency matrix of a graph H with the least eigenvalue greater than -2. We see that H is regular since $E\,\mathbf{j} = (BB^T - kI)\,\mathbf{j} = (k^2-k)\,\mathbf{j}$. The connectedness of G also yields that H is connected. By Corollary 2.3.22 of [2], a connected regular graph with least eigenvalue greater than -2 is either a complete graph or an odd cycle. If H is an odd cycle, then it is 2-regular and so from $k^2 - k = 2$, we have $k = 2$, a contradiction. Hence H is a complete graph. It is easy to see that this implies that G is the incidence graph of a projective plane of order $k-1$. □

For the next theorem, we need the following result [2, Theorem 2.3.20].

Theorem 2 ([2])**.** *If G is a connected graph with the least eigenvalue greater than -2, then one of the following holds:*

(i) *G is the line graph of a multigraph K, where K is obtained from a tree by adding one edge in parallel to a pendant edge;*
(ii) *G is the line graph of a graph K, where K is a tree or is obtained from a tree by adding one edge giving a nonbipartite unicyclic graph;*
(iii) *G is one of the 573 exceptional graphs on at most 8 vertices.*

We can now prove an analogue to Theorem 1 for non-regular graphs.

Theorem 3. *Let G be a connected bipartite nonregular graph with maximum degree $\Delta \geqslant 3$. Then $R(G) \leqslant \sqrt{\Delta - 2}$.*

Proof. Let $d = \Delta - 1$. Suppose, for a contradiction, that $R(G) > \sqrt{d-1}$. Let $\{U, W\}$ be the bipartition of $V(G)$. Then U and W have the same size, say m, since otherwise $R(G)$ would be zero. We proceed in the same way as in the proof of Theorem 1. The adjacency matrix of G can be written in the form

$$A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where the rows of B are indexed by the elements of U and the columns by W. The matrix $E = BB^T - (d-1)I$ is a symmetric matrix of order m. Since $R(G) > \sqrt{d-1}$, we have $\lambda_m(E) > 0$. Hence E is a positive definite matrix whose diagonal entries are the integers $\deg_G(u) - (d-1) \leqslant 2, u \in U$. Since E is positive definite, these are all equal to 1 or 2 and hence the degrees of vertices in U are either Δ or $\Delta - 1$. Moreover, this in turn implies that all off-diagonal entries of E are either 0 or 1. Since the off-diagonal entries in E count the number of walks of length 2 between vertices in U, the last conclusion in particular implies that G has no 4-cycles. Let D be the diagonal matrix whose diagonal is the same as the main diagonal of E. Let H be the graph on U with the adjacency matrix $A(H) = E - D$. Then the least eigenvalue of H is greater than -2 and $A(H) + D$ is positive definite. The connectedness of G yields that H is connected.

Suppose that $v_1, v_2 \in U$ are distinct vertices of degree d in G. Let P be a shortest path in H connecting v_1 to v_2. The vertices v_1, v_2 and the path P can be chosen so that all internal vertices on P are of degree $d+1$ in G. Then $A(P) + \mathrm{diag}(1, 2, \ldots, 2, 1)$, which is a principal submatrix of $A(H) + D$, has eigenvalue 0 with the eigenvector $(1, -1, 1, -1, \ldots)^T$, a contradiction. This shows that U contains at most one vertex which has degree d in G. Note that the same argument can be applied to W. As G is not regular, we conclude that it has precisely two vertices of degree d, one in U and one in W. Therefore, we may assume that $D = \mathrm{diag}(2, \ldots, 2, 1)$.

Let us now consider degrees of vertices in H. Since G has no 4-cycles, we have for every $u \in U$:

$$\deg_H(u) = \sum_{uv \in E(G)} (\deg_G(v) - 1). \tag{1}$$

Since U and W each has precisely one vertex whose degree in G is d, (1) implies the following: If $\deg_G(u) = d$, then $\deg_H(u) \in \{d^2, d^2 - 1\}$; if $\deg_G(u) = d + 1$, then $\deg_H(u) \in \{d^2 + d, d^2 + d - 1\}$ with the smaller value only when u is adjacent to the vertex of degree d in W. Thus, H has a unique vertex of degree d^2 or d^2-1, at most d vertices of degree d^2+d-1, and all other vertices are of degree d^2+d.

Let $v \in U$ be the vertex with $\deg_G(v) = d$. We claim that the neighbourhood of v in H is a complete graph. This is since

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

is not positive definite and so it cannot be a principal submatrix of $A(H)+D$. We also claim that $N_H[v_1] \cap N_H[v_2] \subseteq N_H[v]$ for arbitrary distinct vertices $v_1, v_2 \in N_H(v)$. If this were not the case, the matrix

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

which is not positive definite, would be a principal submatrix of $A(H)+D$, a contradiction. Therefore, H has at least $1+(d^2-1)+(d^2-1)((d^2+d-1)-(d^2-1)) = d^3+d^2-d$ or $1+d^2+d^2((d^2+d-1)-d^2) = d^3+1$ vertices. In any case, we have $|V(H)| \geqslant 9$ and so H is the case (i) or (ii) of Theorem 2.

Let H be the line graph of a multigraph K as in Theorem 2, where $|V(K)| = m$ or $m+1$ and $|E(K)| = m$. Note that K is not a cycle, since then H would be regular in that case. So, from $\sum_{v \in V(K)} \deg_K(v) = 2m$ and $|V(K)| \geqslant m$, we have a vertex u such that $\deg_K(u) = 1$. Let u' be the unique neighbour of u in K. The degree of the vertex in H corresponding to the edge uu' is at least d^2-1. Thus, $\deg_K(u)+\deg_K(u')-2 \geqslant d^2-1$ and so $\deg_K(u') \geqslant d^2$. Let r be the number of vertices of degree one in K. The sum of degrees in K is at least $r+d^2+2(|V(K)|-r-1) \leqslant 2m$ which gives $r \geqslant d^2-2 \geqslant 2$. Since H has only one vertex of degree d^2-1 or d^2, we may take u to be a vertex of degree 1 in K such that $\deg_K(u)+\deg_K(u')-2 \geqslant d^2+d-1$ and now a similar argument as above gives $r \geqslant d^2+d-2 > d+1$. Now again since H has at most $d+1$ vertices of degree d^2-1, d^2 or d^2+d-1, we may take u such that $\deg_K(u)+\deg_K(u')-2 = d^2+d$. It follows that H has the complete graph of order d^2+d+1 as a subgraph which in turn implies that H is in fact the complete graph of order d^2+d+1 since H is connected with maximum degree d^2+d. And this contradicts the fact that H is nonregular. $\square$

Theorems 1 and 3 can be combined into our main result.

Theorem 4. *Let G be a connected bipartite graph with maximum degree $\Delta \geqslant 3$. Then $R(G) \leqslant \sqrt{\Delta - 2}$ unless G is the the incidence graph of a projective plane of order $\Delta - 1$, in which case $R(G) = \sqrt{\Delta - 1}$.*

There are connected graphs that are not incidence graphs of a projective planes and attain the bound $\sqrt{\Delta - 2}$ of Theorem 4. The incidence graph of a biplane ($(v, k, 2)$ symmetric design) has degree k and HL-index $\sqrt{k - 2}$. Only 17 biplanes are known and the question of existence of infinitely many biplanes is an old open problem in design theory [8]. An infinite family of cubic graphs with HL-index equal to 1 is constructed in [6].

Finally, we give an upper bound for the HL-index of bipartite graphs in term of the minimum degree.

Theorem 5. *Let G be a bipartite graph with minimum degree δ. Then $R(G) \leqslant \sqrt{\delta}$. Moreover, if $\delta \geq 2$ or $\delta = 1$ and G has a component with minimum degree 1 that is not isomorphic to K_2, then $R(G) < \sqrt{\delta}$.*

Proof. We may assume that G is connected and of even order $n = 2m$. Let v be a vertex of degree δ and $H = G - v$. Since G is connected, $\lambda_1(G) > \lambda_1(H)$. By interlacing, $\lambda_i(G) \geqslant \lambda_i(H)$ for $1 < i \leq m$. We also have $\lambda_m(H) = 0$ since H is bipartite and has an odd number of vertices.

The sum of the squares of the eigenvalues of G is the trace of A^2, which is equal to $2|E(G)|$. By considering only half of the eigenvalues and using the fact that eigenvalues of a bipartite graph are symmetric about zero, we have:

$$
\begin{aligned}
\lambda_m^2(G) &= |E(G)| - \sum_{i=1}^{m-1} \lambda_i^2(G) \\
&= \delta + \sum_{i=1}^{m-1} \lambda_i^2(H) - \sum_{i=1}^{m-1} \lambda_i^2(G) \\
&\leqslant \delta.
\end{aligned}
$$

If $m \geqslant 2$, then for $i = 1$, we have $\lambda_1^2(H) - \lambda_1^2(G) < 0$, so the last inequality is strict. This proves the assertion of the theorem. $\square$

3 Covering graphs and their eigenvalues

If $\hat{G}$ is a covering graph of G, then all eigenvalues of G are included in the spectrum of $\hat{G}$. The essence of this section is to show how to control the newly arising eigenvalues in the covering graph.

We will denote the eigenvalues of a symmetric $n \times n$ matrix M by $\lambda_1(M) \geqslant \cdots \geqslant \lambda_n(M)$. Also, if $\mathbf{x}$ is an eigenvector of M, then we denote the corresponding eigenvalue by $\lambda_{\mathbf{x}}(M)$. For a positive integer t, let I_t and $\mathbf{0}_t$ denote the $t \times t$ identity matrix and the $t \times t$ all-zero matrix, respectively. A *permutation matrix* C of *size* t and *order* m is a $t \times t$ $(0,1)$-matrix that has exactly one entry 1 in each row and each column and m is the smallest positive integer such that $C^m = I_t$.

Let us replace each edge of a multigraph G by two oppositely oriented directed edges joining the same pair of vertices and let $\overrightarrow{E}(G)$ denote the resulting set of directed edges. We denote by $(e,u,v) \in \overrightarrow{E}(G)$ the directed edge in $\overrightarrow{E}(G)$ corresponding to an edge $e = uv$ that is oriented from u to v. Let S_t denote the symmetric group of all permutations of size t. We shall consider a representation of S_t as the set of all permutation matrices of size t. A function $\phi : \overrightarrow{E}(G) \to S_t$ is a *permutation voltage assignment* for G if $\phi(e,u,v) = \phi(e,v,u)^{-1}$ for every $e \in E(G)$. A *t-lift* of G *associated to* ϕ and denoted by $G(\phi)$, is a multigraph with the adjacency matrix obtained from the adjacency matrix of G by replacing any (u,v)-entry of $A(G)$ by the $t \times t$ matrix $\sum_{(e,u,v) \in \overrightarrow{E}(G)} \phi(e,u,v)$. Note that if G is bipartite, then so is $G(\phi)$. We say that $G(\phi)$ is an *Abelian lift* if all matrices in the image of ϕ commute with each other.

Bilu and Linial [1] found an expression for the spectrum of 2-lifts. They proved that the spectrum of a 2-lift $G(\phi)$ consists of the spectrum of G together with the spectrum of the matrix A^- which is obtained from the adjacency matrix of G by replacing each (u,v)-entry by -1 whenever the voltage $\phi(e,u,v)$ is not the identity. Note that 2-lifts are always Abelian since the permutation matrices of size 2 commute with each other. Below we extend the result of [1] to arbitrary Abelian t-lifts. Since permutation matrices are diagonalizable and any commuting family of diagonalizable $t \times t$ matrices has a common basis of eigenvectors, we observe that any commuting set of permutation matrices of the same size has a common basis of eigenvectors.

In the proofs we will use the following result, see [10, Theorem 1].

Theorem 6 ([10])**.** *Let t and n be positive integers and for $i, j \in \{1, \ldots, n\}$, let B_{ij} be $t \times t$ matrices over a commutative ring that commute pairwise. Then*

$$\det \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{bmatrix} = \det\Bigg(\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) B_{1\sigma(1)} \cdots B_{n\sigma(n)} \Bigg),$$

where S_n is the set of all permutations of $\{1,\ldots,n\}$.

Theorem 7. *Let G be a multigraph and ϕ be a permutation voltage assignment for an Abelian t-lift $G(\phi)$ of G. Let $\mathcal{B}$ be a common basis of eigenvectors of the permutation matrices in the image of ϕ. For every $\mathbf{x}\in\mathcal{B}$, let $A_{\mathbf{x}}$ be the matrix obtained from the adjacency matrix of G by replacing any (u,v)-entry of $A(G)$ by $\sum_{(e,u,v)\in\overrightarrow{E}(G)}\lambda_{\mathbf{x}}(\phi(e,u,v))$. Then the spectrum of $G(\phi)$ is the multiset union of the spectra of the matrices $A_{\mathbf{x}}$ $(\mathbf{x}\in\mathcal{B})$.*

Proof. The adjacency matrix of a t-lift can be written in the block form, with the blocks being indexed by $V(G)$, where the (u,v)-block D_{uv} is equal to the permutation matrix $\phi(e,u,v)$ if u,v are joined by a single edge e, or to $\sum_{(e,u,v)\in\overrightarrow{E}(G)}\phi(e,u,v)$ if there are multiple edges, or $\mathbf{0}_t$ if u,v are not adjacent in G. Thus, assuming $V(G)=\{1,\ldots,n\}$, we can write

$$\lambda I - A(G(\phi)) = \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{bmatrix},$$

where the diagonal blocks are λI_t, while the off-diagonal blocks are $B_{uv} = -D_{uv}$. All block matrices B_{uv} commute with each other and all their products and sums also commute and have $\mathcal{B}$ as a common basis of eigenvectors. By Theorem 6, we have

$$\begin{aligned}\det(\lambda I - A(G(\phi))) &= \det\Big(\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)B_{1\sigma(1)}\cdots B_{n\sigma(n)}\Big)\\ &= \prod_{\mathbf{x}\in\mathcal{B}}\lambda_{\mathbf{x}}\Big(\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)B_{1\sigma(1)}\cdots B_{n\sigma(n)}\Big)\\ &= \prod_{\mathbf{x}\in\mathcal{B}}\Big(\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\lambda_{\mathbf{x}}(B_{1\sigma(1)})\cdots\lambda_{\mathbf{x}}(B_{n\sigma(n)})\Big)\\ &= \prod_{\mathbf{x}\in\mathcal{B}}\det(\lambda I - A_{\mathbf{x}}).\end{aligned}$$

This equality gives the conclusion of the theorem. □

Let us keep the notation of Theorem 7 and its proof. There are two things to be observed. Since every permutation matrix D_{uv} satisfies $D_{uv}^{-1} = D_{uv}^{T}$, we have that the matrices $A_{\mathbf{x}}$ $(\mathbf{x}\in\mathcal{B})$ are Hermitian. The (u,v)-entry of $A_{\mathbf{x}}$ is the eigenvalue of D_{uv} corresponding to the eigenvector $\mathbf{x}$. Thus the characteristic polynomial $\varphi(A_{\mathbf{x}}, \lambda) = \det(\lambda I - A_{\mathbf{x}})$ is a polynomial in λ, whose coefficients are polynomials in the $\mathbf{x}$-eigenvalues of the permutation matrices in the image of ϕ.

4 Median eigenvalues of covering graphs

In this section we present an approach through graph covering to construct infinite families of graphs with large HL-index. The main tool is Theorem 7, which will be invoked in a very special situation.

Let us select a set $F \subseteq \overrightarrow{E}(G)$ of oriented edges such that whenever $(e, u, v) \in F$, the opposite edge (e, v, u) is not in F. For every positive integer t, let C_t be a cyclic permutation of size and order t. Now, let us consider an infinite family of Abelian lifts $\phi_1, \phi_2, \phi_3, \ldots$ such that ϕ_t is an Abelian t-lift over the graph G, whose voltages are given by the following rule:

$$\phi_t(e, u, v) = \begin{cases} C_t, & (e, u, v) \in F; \\ C_t^{-1}, & (e, v, u) \in F; \\ I_t, & \text{otherwise.} \end{cases} \tag{2}$$

In this way, we obtain an infinite family of graphs $G(\phi_t)$. By Theorem 7, we can express the characteristic polynomial of $A(G(\phi_t))$ as a product of the characteristic polynomials of matrices $A_{\mathbf{x}}$. For each $\mathbf{x} \in \mathcal{B}$, the characteristic polynomial of $A_{\mathbf{x}}$ depends only on λ and on the eigenvalue $\alpha = \lambda_{\mathbf{x}}(C_t)$ and on $\lambda_{\mathbf{x}}(C_t^{-1}) = \alpha^{-1} = \overline{\alpha}$. The dependence on α can be expressed in terms of the real parameter $\nu = \alpha + \overline{\alpha}$. For cyclic permutations, every such ν is an eigenvalue of the t-cycle, which is of the form $\nu = 2\cos(2\pi j/t)$ for some $j \in \{0, 1, \ldots, t-1\}$. Thus, there is a polynomial $\Phi(\lambda, \nu)$ such that

$$\det(\lambda I - A(G(\phi_t)) = \prod_{0 \leqslant j < t} \Phi(\lambda, 2\cos(2\pi j/t)). \tag{3}$$

Note that $\Phi(\lambda, \nu)$ is independent of t and only depends on the underlying graph G and the values for ν lie in the interval $[-2, 2]$ for every t. All eigenvalues of $G(\phi_t)$ correspond to the zero-set of the polynomial $\Phi(\lambda, \nu)$ with $\nu \in [-2, 2]$. When t gets large, the appropriate values of ν become dense in the interval $[-2, 2]$. This shows that if G is bipartite, then $R(G(\phi_t))$ converges to some value when t goes to infinity. This is better seen in a special case which is given in the following theorem.

Theorem 8. *Let G be a bipartite graph and let E_0 be a set of edges all incident with some fixed vertex v_0. Let $F \subset \overrightarrow{E}(G)$ be the set of directed edges*

$\{(e, u, v_0) \mid uv_0 \in E_0\}$. *For each positive integer* t*, fix a cyclic permutation matrix* C_t *of size and order* t *and define a permutation voltage assignment* ϕ_t *by* (2)*. Then*

$$R(G(\phi_{2t})) = R(G(\phi_2))$$

for every $t \geq 1$*, whereas the values* $R(G(\phi_{2t+1}))$ *are non-increasing as a function of* t *and*

$$\lim_{t\to\infty} R(G(\phi_{2t+1})) = R(G(\phi_2)).$$

Proof. By Theorem 7, there is a polynomial $\Phi(\lambda, \nu)$ such that (3) holds and every eigenvalue of any $G(\phi_t)$ lies among the values λ for which there is a $\nu \in [-2, 2]$ such that $\Phi(\lambda, \nu) = 0$. It is easy to see that our choice of F implies that $\Phi(\lambda, \nu)$ is linear in ν, so it can be expressed in the form

$$\Phi(\lambda, \nu) = p(\lambda) - \nu q(\lambda).$$

If $R(G(\phi_2))$ is zero, then $R(G(\phi_{2t})) = 0$ for every $t \geq 1$, since by Theorem 7, the spectrum of $G(\phi_2)$ is contained in the spectrum of $G(\phi_{2t})$. Hence, we may assume that $R(G(\phi_2)) \neq 0$. Note that $R(G) \geq R(G(\phi_2))$, so we also have $R(G) \neq 0$.

Let us first assume that $q(0) \neq 0$. Let $\Phi(0, \nu_0) = 0$. Then $\nu_0 = p(0)/q(0)$. We have $\Phi(0, 2) = p(0) - 2q(0)$ and $\Phi(0, -2) = p(0) + 2q(0)$ which results in

$$\nu_0 = p(0)/q(0) = \frac{2(\Phi(0, -2) + \Phi(0, 2))}{\Phi(0, -2) - \Phi(0, 2)}.$$

On the other hand, Eq.(3) gives that $\Phi(0, 2) = \det(-A(G))$ and $\Phi(0, -2) = \det(-A(G(\phi_2)))/\det(-A(G))$. Since the eigenvalues of bipartite graphs G and $G(\phi_2)$ are symmetric about zero, this implies that the above determinants, and thus also $\Phi(0, 2)$ and $\Phi(0, -2)$, have the same sign. It follows that $|\nu_0| > 2$. Since $\Phi(\lambda, \nu)$ is linear in ν, for each λ there exists at most one value ν such that $\Phi(\lambda, \nu) = 0$ (and there is exactly one if $q(\lambda) \neq 0$). Therefore, the continuity of $\Phi(\lambda, \nu)$ and its linearity in ν show that the eigenvalue $R(G(\phi_t))$ is either a root of $\Phi(\lambda, 2)$ or a root of $\Phi(\lambda, -2)$ (if t is even) or a root of $\Phi(\lambda, 2\cos(\pi(t-1)/t))$ (if t is odd). This is independent of t when t is even and is already among the eigenvalues of $G(\phi_2)$. For odd values of t, this shows the behavior as claimed in the theorem.

Suppose next that $q(0) = 0$. Then $p(0) \neq 0$, since otherwise we have $\Phi(0, 2) = 0$ and so $R(G) = 0$, a contradiction. This shows that if $\Phi(\lambda_0, \nu_0) = 0$ and λ_0 goes to zero, then ν_0 goes to infinity. Again the continuity of $\Phi(\lambda, \nu)$ and its linearity in ν show that $R(G(\phi_t))$ is either a root of $\Phi(\lambda, 2)$ or a root of $\Phi(\lambda, -2)$ (if t is even) or a root of $\Phi(\lambda, 2\cos(\pi(t-1)/t))$ (if t is odd). We now complete the proof in the same was as above. $\square$

Theorem 9. *For any integer k for which $k-1$ is a prime power, there exist infinitely many connected bipartite k-regular graphs G with $\sqrt{k-2}-1 < R(G) < \sqrt{k-1}-1$.*

Proof. Let G be the incidence graph of a projective plane of order $q = k-1$. Note that G is bipartite and k-regular. It is well-known (see, e.g., [5]) that G has eigenvalues $\pm k$ and $\pm\sqrt{q}$. Thus, $R(G) = \sqrt{q}$. Let e_0 be any edge of G and $E_0 = \{e_0\}$. For each positive integer t, define the permutation assignment ϕ_t as in Theorem 8.

The adjacency matrix of $G(\phi_2)$ can be written as $A(G(\phi_2)) = A(G) \otimes I_2 + B$, where B is a matrix with only ± 2 as nonzero eigenvalues. Let r be the number of vertices of $G(\phi_2)$. By the Courant-Weyl inequalities $\lambda_{i+j-r}(A+B) \geqslant \lambda_i(A) + \lambda_j(B)$, we have $\lambda_{r/2-1}(A(G(\phi_2))) \geqslant \lambda_{r/2}(A(G) \otimes I_2) + \lambda_{r-1}(B)$ which gives $\lambda_{r/2-1}(G(\phi_2)) \geqslant \sqrt{q}$. In fact, since $G(\phi_2)$ has $\sqrt{q}$ as an eigenvalue with big multiplicity, one observes that $\lambda_{r/2-1}(G(\phi_2)) = \sqrt{q}$ and so $R(G) = \lambda_{r/2}(G(\phi_2)) \leqslant \sqrt{q}$.

Let $e_0 = \{v_0, v_1\}$. Let us consider the partition $\{v_0, v_1\} \cup W \cup W'$ of $V(G)$, where W is the set of all vertices adjacent to v_0 or v_1 and W' is the set of vertices nonadjacent to v_0, v_1. Define the vector $\mathbf{x}$ on $V(G)$ as

$$\mathbf{x}(w) = \begin{cases} 1 & w = v_0 \text{ or } w = v_1, \\ a & w \in W, \\ b & w \in W'. \end{cases}$$

We now consider the vector $\mathbf{y} = (\mathbf{x}, -\mathbf{x})$ on $V(G(\phi_2))$. Since the girth of G is six, it is easy to see that $\mathbf{y}$ is an eigenvector of $G(\phi_2)$ with the corresponding eigenvalue λ if and only if $\lambda = qa - 1$, $a\lambda = qb + 1$, and $b\lambda = qb + a$. Solving these equations in term of λ gives $\lambda^3 + (1-q)\lambda^2 - 3q\lambda + q^2 - q = 0$. The value of the expression on the left side of this equation is 2 and $\sqrt{q} - q$ for $\lambda = \sqrt{q-1} - 1$ and $\lambda = \sqrt{q} - 1$, respectively. Therefore, there is a root between $\sqrt{q-1} - 1$ and $\sqrt{q} - 1$. This implies that $\sqrt{q-1} - 1 < R(G(\phi_2)) < \sqrt{q} - 1$. Finally, since $R(G) = \sqrt{q}$, Theorem 8 implies that $\sqrt{q-1} - 1 < R(G(\phi_{2t})) = R(G(\phi_2)) < \sqrt{q} - 1$ for all t. □

The proof of Theorem 9 can be used to obtain a slightly better bound on $R(G)$. Let $t = \sqrt{k-1}$. The value of $\lambda^3 + (1-q)\lambda^2 - 3q\lambda + q^2 - q$ is

$$(ht)^{-3}((2h^2 - h^3)t^5 + (h^3 - 2h^2)t^4 + (4h^2 - h)t^3 + (3h - h^2)t^2 - 2ht - 1) \quad (4)$$

for $\lambda = t - 1 - (ht)^{-1}$. Note that (4) is positive for $h = 2$ and any t, whereas it is negative for any $h > 2$ if t is large enough. Therefore we find that $\sqrt{k-1} - 1 - \frac{1}{2\sqrt{k-1}} < R(G) < \sqrt{k-1} - 1 - \frac{1}{(2+\epsilon)\sqrt{k-1}}$ for every $\epsilon > 0$ and any large k.

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