SOME NEW 4-DESIGNS

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Abstract. The existence question for the family of 4-$(15, 5, \lambda)$ designs has long been answered for all values of λ except $\lambda = 2$. Here, we resolve this last undecided case and prove that 4-$(15, 5, 2)$ designs are constructible.

Introduction

A t-(v, k, λ) *design* $D = (V, \mathcal{B})$ is a family $\mathcal{B}$ of k-subsets, called *blocks*, of a v-set V of *points*, such that every t-subset of V is contained in exactly λ blocks of $\mathcal{B}$. If $\mathcal{B}$ has no repeated blocks, then the design D is called *simple*. Here, simplicity is always taken for granted. We further assume that $V = \{1, 2, \ldots, v\}$. The set of all k-subsets of V will be denoted here by V_k. (V, V_k) is called the *complete* design. Elementary counting arguments show that a t-(v, k, λ) design is also an i-(v, k, λ_i) design, for all i, $0 \leq i \leq t$, where $\lambda_i = \lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$.

An *isomorphism* between $(V, \mathcal{B})$ and $(V, \mathcal{B}')$ is a one-one mapping on the elements of V such that the blocks of $\mathcal{B}$ are mapped onto the blocks of $\mathcal{B}'$. If no such mapping exists, then the designs are said to be *non-isomorphic*. The set of all *automorphisms* of a design (that is, isomorphisms from a design to itself) forms a group which acts in a natural way as a permutation group on the points of the design and consequently on its blocks and is called the *full automorphism group* of design. The full automorphism group of D is denoted by $Aut(D)$. A design is called *rigid* if its full automorphism group is the trivial group. We also recall the notion of the *normalizer* of a group G in a bigger group H as the sub-group of H consisting of all elements $g \in H$ such that $g^{-1}Gg = G$.

Let $(V, \mathcal{B})$ be a t-(v, k, λ) design and consider the set $W \subset V$ with $|W| = w < t$. Let $V' = V \backslash W$ and $\mathcal{B}' = \{B \backslash W : B \in \mathcal{B}, W \subseteq B\}$. Then $(V', \mathcal{B}')$ is a $(t-w)$-$(v-w, k-w, \lambda)$ design called the *derived* design with respect to W.

In 1976, Kramer and Mesner [6] observed that finding a t-design with a given automorphism group can be reduced to solving a matrix problem of the form $A\mathbf{u} = b$, where A is an $m \times n$ positive integer matrix built from the required automorphism group, b is a particular m dimensional integer vector, and $\mathbf{u} \in \{0, 1\}^n$.

In order to effectively use this observation, one needs to obtain an efficient procedure for solving the matrix equation. In this respect, backtracking plays a prominent role [2, 7]. Backtracking constructs a series of feasible solutions by extending

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and collapsing partial feasible solutions one step at a time in an orderly fashion, until a complete feasible solution is constructed. A search tree is usually pruned by rejecting partial solutions which do not have the correct structure or are isomorphic to the ones generated before.

The existence question for the family of 4-$(15, 5, \lambda)$ has been answered for all values of $1 \leq \lambda \leq 11$ except $\lambda = 2$ [1, 5, 9]. In [3, 4], some incidence matrices are exploited to enforce pruning of the search space in backtracking algorithms. In Section 1, an outline of the algorithm is provided. In this paper, we employ, with slight modifications, the same approach to tackle the existence question for the family of 4-$(15, 5, 2)$ designs and completely classify all 4-$(15, 5, 2)$ designs admitting an automorphism of order 7 and 13. We also consider possible extensions of these designs to a 5-$(16, 6, 2)$ design.

1. Constructing t-designs with a prescribed automorphism group

Given a t-(v, k, λ) design $(V, \mathcal{B})$, let $V_t = \{x_1, \ldots, x_m\}$ and $V_k = \{y_1, \ldots, y_n\}$, where $m = \binom{v}{t}$ and $n = \binom{v}{k}$. Construct the $m \times n$ $(0, 1)$ Kramer-Mesner matrix $A^v_{tk} = (a_{ij})$, where $a_{ij} = 1$ if and only if $x_i \subseteq y_j$. Now, it is clear that a t-(v, k, λ) design exists if and only if there is a solution $\mathbf{u} \in \{0, 1\}^n$ to the equation $A^v_{tk}\mathbf{u} = \lambda J_m$, where J_m is the m-dimensional all-one column vector. For most designs of interest, however, A^v_{tk} is prohibitively large so that to reduce the size, one assumes the action of a group G on the set V. Let τ_t and τ_k be the number of orbits under the induced action of G on V_t and V_k, respectively. Denote by $A^v_{tk}(G) = (a_{ij})$ a $\tau_t \times \tau_k$ matrix, where a_{ij} is the number of k-subsets in the j-th orbit of V_k containing a representative t-subset in the i-th orbit of V_t. Clearly $A^v_{tk}(G)$ can be obtained from A^v_{tk} by adding the columns in each orbit of V_k and keeping one representative row from each orbit of V_t. A t-design with G as a sub-group of its full automorphism group exists if and only if there is a vector $\mathbf{u} \in \{0, 1\}^{\tau_k}$ satisfying the equation $A^v_{tk}(G)\mathbf{u} = \lambda J_{\tau_t}$. Note that $\mathbf{u}$ is indeed the vector representation of $(V, \mathcal{B})$, i. e., $\mathbf{u}$ is a column vector whose rows are indexed by the elements of the orbits of V_k such that $\mathbf{u}_i = 1$ if and only if $\mathcal{B}$ contains the i-th orbit of V_k.

Let $S \subseteq V$ such that $|S| = s$. Let ψ_s be the number of orbits of V_s under the action of G. Define a $\psi_s \times \tau_k$ matrix $M^{vs}_{tk}(G)$ whose (i, j)-th entry is $\sum_{K \in \kappa_j} \binom{|S \cap K| - 1}{t}$, where S is any representative in the i-th orbit of V_s and κ_j is the j-th orbit of V_k under G. In [4], it is shown that $M^{vs}_{tk}(G)\mathbf{u} = b^{vs}_{tk} J_{\psi_s}$, where $b^{vs}_{tk} = \sum_{i=0}^{t} (-1)^{i+t} \binom{s}{i} \lambda_i$. Note that these equations are not independent from the original equations. However, they are crucial in backtracking algorithms to find new designs.

Therefore, to construct a t-(v, k, λ) design $(V, \mathcal{B})$ admitting G as its automorphism group, we can employ a backtracking algorithm to find a $\mathbf{u} \in \{0, 1\}^{\tau_k}$ satisfying the equations $A^v_{tk}(G)\mathbf{u} = \lambda J_{\tau_t}$ and $M^{vs}_{tk}(G)\mathbf{u} = b^{vs}_{tk} J_{\psi_s}$.

2. Using The derived designs To prune the search space

In the previous section, group actions were employed to reduce the size of the problem. In this section, we explain how the derived designs can be used to prune the search space.

Lemma 1. *Let $D_1 = (V_1, \mathcal{B}_1)$ be a $(t-1)$-$(v-1, k-1, \lambda)$ design on the point set $V_1 = \{2, \ldots, v\}$ with a non-trivial automorphism π_1. Let $V = \{1, \ldots, v\}$ and fix σ_1 as an element of the normalizer of $< \pi_1 >$ in S_{V_1}, i.e., $\sigma_1 \in N(< \pi_1 >)$. Define $\pi \in S_V$ by $\pi = (1)\pi_1$ and let $D = (V, \mathcal{B})$ be a t-(v, k, λ) design with the following properties:*
a)D admits π as an automorphism.
b) The derived design of D with respect to the point 1 is D_1.
Then there exists a design D' isomorphic to D with properties (i) and (ii):
i)D' admits π as an automorphism.
ii)The derived design of D' with respect to the point 1 is $\sigma_1(D_1)$.

Proof. Define $\sigma \in S_V$ by $\sigma = (1)\sigma_1$. We prove that $D' = \sigma(D)$ satisfies the properties (i) and (ii). From the definition of σ and π we have: $\sigma \in N(< \pi >)$ and $\sigma^{-1}\pi\sigma(D) = D$, which means $\pi \in Aut(D')$. Since σ fixes 1, the derived design of $\sigma(D)$ with respect to the point 1 is $\sigma_1(D_1)$, and this completes the proof. □

Now suppose that we want to find all t-(v, k, λ) designs D with a given automorphism π of prime order p with $\pi(1) = 1$. We can do as follows: Let D_1 and π_1 be as in Lemma 1. We first solve the equations mentioned in the previous section to produce all candidates for D_1. According to Lemma 1, for each design D_1 and each $\sigma_1 \in N(< \pi_1 >)$, there exists an isomorphic copy $\sigma_1(D_1)$ among solutions which can be discarded. We call this process *normalizer isomorphism test*. We can now obtain possible extensions of the remaining derived designs which admit π as an automorphism. Clearly, there might still be isomorphic copies of solutions which have to be rejected. We do this process in three phases: First we use the normalizer isomorphism test to reject isomorphisms under normalizers. Second we determine the order of the full automorphism group of each of the remaining designs. In this step any two designs D and D' with $|Aut(D)| = |Aut(D')| = p$ are clearly non-isomorphic. Thus in the third step we concentrate on designs D with more than p automorphisms and extract non-isomorphic designs among them. For the second and third steps of this process, we proceed as in [8].

3. 4-(15,5,2) designs

In this section we find all 4-$(15, 5, 2)$ designs with an automorphism of prime order p, $7 \leq p \leq 13$. Let D be such a design on the point set $V = \{1, \ldots, 15\}$ with π as an automorphism. We say an automorphism is of type p^r if it consists exactly of r cycles of length p. Clearly, there exists no 4-$(15, 5, 2)$ design with an automorphism of type 11^1 or 7^1. Hence, we consider the types 13^1 and 7^2.

For the type 13^1, let $\pi = (3\ 4\ \cdots\ 15)$. We employ the algorithm described in the previous sections to get all 19 extensions of the derived designs (with respect to the point sets $\{1, 2\}$ and $\{1\}$). Since all of these solutions have exactly 13 automorphisms, they are non-isomorphic:

Theorem 1. *There exist exactly* 19 *non-isomorphic* 4-$(15, 5, 2)$ *designs admitting an automorphism of order* 13. *These designs have all exactly* 13 *automorphisms.*

Now let $\pi = \sigma_1\sigma_2$, where $\sigma_1 = (2\ 3\ \cdots\ 8)$, and $\sigma_2 = (9\ 10\ \cdots\ 15)$. Here, we get a large number of derived designs and hence we prefer to employ the properties of the normalizers of $< \pi >$ so that the normalizer isomorphism test can be applied to reject partially-completed isomorphic solutions.

Let $- = \{\Gamma_i\}$ be the set of orbits of the blocks of D_1 under the action of $< \pi_1 >$ (as defined in Lemma 1) and let $-^j = \{\Gamma_i :$ there exists a block $B \in \Gamma_i$ such that $|B \cap point(\sigma_1)| = j$ or $|B \cap point(\sigma_2)| = j\}$, $2 \leq j \leq 4$.

Clearly $-^j$ is a well defined subset of $-$ and for any $\sigma \in N(G_1)$, $\sigma(-^j) = -^j$. Let $\mathbf{u}_1$ be the vector representation of D_1 and permute the rows of $\mathbf{u}_1$ so that the orbits of $-^4$, $-^3$, and $-^2$ appear one group after another. We solve the equations $A^{14}_{3,4}(G_1)\mathbf{u}_1 = 2J$ and $M^{14,7}_{3,4}(G_1)\mathbf{u}_1 = 0$ (where the columns of matrices are permuted accordingly) to obtain all partial solutions of length $|-^4|$, $|-^4| + |-^3|$, $|-|$, respectively. We can employ the normalizers to reduce the number of solutions at each step. Finally, **9048** 3-$(14, 4, 2)$ designs are produced. The equations $A^{15}_{4,5}(G)D = 2J$ and $M^{15,7}_{4,5}(G)D = 14J$ are then solved to determine possible extensions to 4-$(15, 5, 2)$ designs. Applying necessary isomorphism tests described in Section 2, one obtains:

Theorem 2. *There exist exactly* 575 *non-isomorphic* 4-$(15, 5, 2)$ *designs admitting an automorphism of order* 7. *The number of automorphisms of* 557 *designs is* 7 *and* 18 *designs have* 14 *automorphisms.*

Examples of designs with $7, 13$ and 14 automorphisms are given in the Appendix.

Solving the equations $A^{16}_{5,6}(G)D' = 2J$ and $M^{16,8}_{5,6}(G)D' = 0$ shows that none of the above designs extends to a 5-$(16, 6, 2)$ design D' . Therefore:

Theorem 3. *There are no* 5-$(16, 6, 2)$ *designs with an automorphism of order* $7, 11,$ *and* 13.

Appendix

The orbit representations of some new 4-(15,5,2) designs are given below. The point set is $V = \{1, \ldots, 9, A, \ldots, F\}$. Design $\#i$ $(1 \leq i \leq 3)$ has G_i as the full automorphism group where G_i are as follows:

$$G_1 =< \sigma_1 >,\ G_2 =< \sigma_2 >,\ G_3 =< \sigma_2, \sigma_3 >,$$
$$\sigma_1 = (3\ \cdots\ F),$$
$$\sigma_2 = (2\ \cdots\ 8)(9\ \cdots\ F),$$
$$\sigma_3 = (2\ F)(3\ 9)(4\ A)(5\ B)(6\ C)(7\ D)(8\ E).$$

Note that $|G_1| = 13$, $|G_2| = 7$, and $|G_3| = 14$.

Design #1.

12345	12359	1236A	1236B	1345A	13469	1346C	13478	1347B	1348D
134AE	134BE	13579	1358A	2345C	23468	2346E	2347C	2347E	2348A
2349B	2349D	234AD	2357A	34567	3457B	3458B	3458E	3459A	3459D
345CD	3467E	3468D	3469B	346AC	346BD	3479C	347AD	348AB	348CE
349CE	3579D								

Design #2.

12345	12356	1239A	1239E	123AE	123BD	123BF	123CD	123CF	12469
1246A	1249F	124AC	124BD	124BE	124CD	1259A	1259D	125AB	125BF
125CE	125EF	129CE	12ADF	19ABC	19ACD	2345B	2346D	2346E	23479
2347D	2349F	234AB	234CE	234CF	23569	2357E	2357F	2359F	235AC
235AD	235BE	2369B	236AC	236AE	236BF	237AB	237AF	237BC	237CE
239BC	239CD	239DE	23ADF	23DEF	2469C	246AF	249AB	249AD	249DE
24AEF	24BCE	24BCF	24BDF	259AE	259BC	259CF	25ABF	25BCD	25DEF
29ABE	29ACF	29BDF	29BEF	2ABCD	2ABDE	2ACDE	2ACEF		

Design #3.

12345	12356	1239B	1239D	123AD	123AE	123BF	123CE	123CF	12469
1246B	124AC	124AD	2345F	2346A	2346F	2347B	2347C	2349A	2349C
234BD	234DE	2356C	2357A	2357D	2359B	235AF	235BC	235DE	2369D
236BE	236CE	236DF	2379C	237AF	237BE	237EF	246BD	246CF	

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