Classification of 6-(14,7,4) designs with nontrivial automorphism groups*

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Abstract

In this paper, we introduce some intersection matrices for t-designs. Using these matrices together with a modified version of a backtracking algorithm we classify all 6-(14,7,4) and 5-(13,6,4) designs with nontrivial automorphism groups and obtain 13 and 21 such designs, respectively.

1. Introduction

Let t, k, v, and λ be integers such that $0 \le t \le k \le v$ and $\lambda > 0$. Let V be a v-set and $P_k(V)$ be the set of all k-subsets (called *blocks*) of V. A

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t-(v,k,λ) *design* is a collection $\mathcal{D}$ of blocks of V such that every t-subset of V occurs exactly λ times in $\mathcal{D}$. It can be easily seen that $\mathcal{D}$ is also a i-(v,k,λ_i) design for $i=0,\ldots,t$, where $\lambda_i=\lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$. $\mathcal{D}$ is *simple* if it contains no repeated block. If $\mathcal{D}$ is simple, then $\mathcal{D}_s=P_k(V)\setminus\mathcal{D}$ is obviously a simple t-$(v,k,\binom{v-t}{k-t}-\lambda)$ design which is called the *supplement* of $\mathcal{D}$. Here, we are concerned only with simple designs. Two t-(v,k,λ) designs $\mathcal{D}_1$ and $\mathcal{D}_2$ are called *isomorphic* if there is a permutation σ on V such that $\sigma\mathcal{D}_1=\mathcal{D}_2$ (Note that σ induces a permutation $\bar{\sigma}$ on the blocks. For simplicity, we write $\sigma\mathcal{D}$ instead of $\bar{\sigma}\mathcal{D}$.). An *automorphism* of $\mathcal{D}$ is a permutation σ such that $\sigma\mathcal{D}=\mathcal{D}$. The group generated by some of the automorphisms of $\mathcal{D}$ is called an *automorphism group* of $\mathcal{D}$ and the group of all its automorphisms is called the *full automorphism group* of $\mathcal{D}$, denoted by $Aut(\mathcal{D})$.

Let $W_{tk}(v)$ be a $\binom{v}{t}$ by $\binom{v}{k}$ $(0,1)$-matrix whose rows and columns are indexed by the t-subsets and k-subsets of V, respectively and for a t-subset T and a k-subset K, $W_{tk}(v)(T,K)=1$ if and only if $T\subseteq K$. $W_{tk}(v)$ is called the *incidence* matrix of t-subsets vs. k-subsets of V. We simply write W_{tk} instead of $W_{tk}(v)$ if there is no danger of confusion. If D is the (0,1)-column vector representation of $\mathcal{D}$ (with the blocks ordered in the same order of the indices of columns of W_{tk} and $D(B)=1$ if and only if $B\in\mathcal{D}$), then we have

$$W_{tk}D=\lambda J, \tag{1}$$

where J is the all-one column vector.

The most natural and common algorithm to find D satisfying (1) is with a backtracking algorithm. However, this method is of practical value only when the incidence matrix is not large. A well known scheme in design theory, due to Kramer and Mesner [4], to handle large incidence matrices is prescribing suitable groups as the automorphism groups of designs and applying them to the incidence matrices. The details of this approach is described in the next section.

The family of 6-(14,7,4) designs are the smallest possible 6-designs. Each

6-(14,7,4) design contains exactly half of the $\binom{v}{k}$ blocks. In 1986, Kreher and Radziszowski [5] found the first 6-(14,7,4) designs using a procedure called the lattice basis reduction. They showed that there are exactly two of these designs with the cyclic group of order 13 as their full automorphism group. Recently, Eslami and Khosrovshahi constructed four further of these designs with the full automorphism group of order 3 using the standard basis of trades [2] and most recently Laue *et al.* obtained the unique 6-(14,7,4) design with the automorphism group A_4 by DISCRETA [6].

At the present time, it seems to be a difficult task to find a 6-(14,7,4) design without prescribing an automorphism group. Therefore, we focused on the classification of the designs with nontrivial automorphism groups. Our results show that there exist exactly 13 6-(14,7,4) designs with nontrivial automorphism group. We also show that there are exactly 21 5-(13,6,4) designs with nontrivial automorphism group. Although we have considered only simple designs, but it is not hard to show that there are no 6-(14,7,4) and 5-(13,6,4) designs with repeated blocks.

2. Algorithm

The most common way for finding combinatorial objects is with a backtracking algorithm. For a description of applications of backtracking algorithms in design theory, the reader is referred to [3]. In this section we present an improved version of the backtracking algorithm which enables us to solve the equation

$$W_{tk}X = \lambda J \tag{2}$$

and find $(0,1)$-solutions of it for a relatively large W_{tk}. We also show that there are some suitable matrices which by appending them to the incidence matrix may result in accelerating the backtracking algorithm.

Let $X^T = [x_1 \ \cdots \ x_{\binom{v}{k}}]$ and consider Equation (2). The support of row i of W_{tk}, denoted by $S(i)$, is the set of all columns j such that $W_{tk}(i,j) \neq 0$.

If in a step of the backtracking algorithm, we have $|\{r \in S(i) : x_r = 1\}| = \lambda$ or $|\{r \in S(i) : x_r = 0\}| = \binom{v-t}{k-t} - \lambda$ for a row i, then we can *preset* every non-determined x_s for $s \in S(i)$ to the proper value. Various practical tests confirm the advantage of this improvement.

Let $\mathcal{D}$ be a t-(v, k, λ) design on the v-set V and let D be its column vector representation. For $S \subseteq V, |S| = s$, let $\alpha_i(S)$ be the number of blocks of $\mathcal{D}$ which have exactly i common elements with S. It is easily seen that

$$\sum_{i=0}^{s} \binom{i}{j} \alpha_i(S) = \binom{s}{j} \lambda_j, \quad j = 0, \ldots, t. \tag{3}$$

Multiplying (3) by $(-1)^{j+t}$ and summing up over j and assuming $b_{tk}^s = \sum_{j=0}^{t} (-1)^{j+t} \binom{s}{j} \lambda_j$, we obtain

$$\begin{aligned} b_{tk}^s =& \sum_{j=0}^{t} \sum_{i=0}^{s} (-1)^{j+t} \binom{i}{j} \alpha_i(S) \\ =& (-1)^t \sum_{i=0}^{s} \alpha_i(S) \sum_{j=0}^{t} (-1)^j \binom{i}{j} \\ =& \sum_{i=0}^{s} \binom{i-1}{t} \alpha_i(S). \end{aligned} \tag{4}$$

Let M_{tk}^s be a matrix whose rows and columns are indexed by the s-subsets and k-subsets of V, respectively and is defined by

$$M_{tk}^s(S, K) = \binom{|S \cap K| - 1}{t},$$

for an s-subset S and a k-subset K of V. By (4), we have

$$M_{tk}^s D = b_{tk}^s J. \tag{5}$$

For example, let $\mathcal{D}$ be a 6-(14,7,4) design. Then

$$M_{67}^7(S, K) = \begin{cases} 1 & \text{if } S = K \text{ or } S \cap K = \varnothing, \\ 0 & \text{otherwise} \end{cases}$$

and $M_{67}^{7}D = J$. This means that a block $B \in \mathcal{D}$ if and only if $V \setminus B \notin \mathcal{D}$ (This is just the Alltop's extension theorem [1]. It also means that every 6-(14,7,4) design is simple.). Therefore, every 5-(13,6,4) design can uniquely be extended to a 6-(14,7,4) design. Now let $\mathcal{D}$ be a 5-(13,6,4) design. Then

$$M_{56}^{6}(S,K) = \begin{cases} 1 & \text{if } S = K \ , \\ -1 & \text{if } S \cap K = \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

and $M_{56}^{6}D = -3J$. In our attempt to classify 6-(14,7,4) and 5-(13,6,4) families of designs, we employ these matrices, namely, M_{67}^{7} and M_{56}^{6}.

In general, for most interesting designs, the W_{tk} matrices are prohibitively large. To overcome this difficulty, we apply some groups to shrink down the size of W_{tk}. More precisely, let G be a permutation group on the v-set V and let $\{\Delta_i\}$ and $\{\Gamma_j\}$ be the orbits of t-subsets and k-subsets of V under the action of G, respectively. Suppose that

$$c_{ij}(T) = |\{K \in \Gamma_j : T \subseteq K\}|, \quad T \in \Delta_i.$$

It can easily be observed that $c_{ij} = c_{ij}(T)$ is independent of T chosen from Δ_i. Let $W_{tk}[G]$ be a matrix whose rows and columns indexed by $\{\Delta_i\}_i$ and $\{\Gamma_j\}_j$, respectively and $W_{tk}[G](\Delta_i, \Gamma_j) = c_{ij}$. Then it is easily seen that each $(0,1)$-solution D of the equation

$$W_{tk}[G]D = \lambda J$$

is a t-(v,k,λ) design with G as its automorphism group. Historically, Kramer and Mesner were first to suggest the use of $W_{tk}[G]$ matrix equations to find t-designs [4]. Now, it is natural to define $M_{tk}^{s}[G]$ in the sense of $W_{tk}[G]$. Note that

$$d_{ij}(T) = \sum_{K \in \Gamma_j} \binom{|T \cap K| - 1}{t}, \quad T \in \Delta_i$$

is independent of T chosen from Δ_i (Say d_{ij}). Therefore, if we let $M_{tk}^{s}[G](\Delta_i, \Gamma_j) = d_{ij}$, then by (5), we have

$$M_{tk}^{s}[G]D = b_{tk}^{s}J.$$

3. Possible Automorphisms

As it was mentioned in Section 2, each 5-(13,6,4) design $\mathcal{D}$ is uniquely extendable to a 6-(14,7,4) design $\overline{\mathcal{D}}$. The extension is simply done by the property that a block lies in $\overline{\mathcal{D}}$ if and only if its complement does not lie in $\overline{\mathcal{D}}$. This property show that if $\mathcal{D}$ has an automorphism σ, then $\overline{\mathcal{D}}$ also admits σ with an additional fixed point as an automorphism. Therefore, one can first classify 5-(13,6,4) designs and then find 6-(14,7,4) designs or vice versa. We choose the reverse procedure because it seems better for the sake of computation. Let $\mathcal{D}$ be 6-(14,7,4) design with a nontrivial automorphism. To classify all designs $\mathcal{D}$, it is sufficient to focus only on automorphisms of prime order. Such automorphisms having r cycles of length p (p being prime) are said to be of type p^r. By [2], there exists no $\mathcal{D}$ having an automorphism of order 11. It can easily be shown that $p \neq 5$ and each automorphism of order 7 has no fixed point and every automorphism of order 3 has exactly one fixed point [2]. Every automorphism of order 2 of $\mathcal{D}$ must have at least one fixed point, otherwise one block and its complement lie in $\mathcal{D}$ which is impossible. Therefore, the possible automorphisms of $\mathcal{D}$ are of types $13^1, 7^2, 3^4, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6$.

4. Results of Computations

The automorphism of type 13^1 has been treated in [5]. To find 6-(14,7,4) designs with each automorphism of the remaining possible types, we first classified the derived 2-(10,3,4) designs with the corresponding derived automorphism groups (except for the type 7^2). Then the results were extended to 6-(14,7,4) designs by the improved backtracking algorithm described in Section 3 using the matrices W_{67} and M_{67}^7. A primary isomorphism testing within the solutions was done using the normalizers of the cyclic group generated by each of the automorphisms. Then a secondary testing was carried out to reject isomorphic copies and to compute the full automorphism groups of the nonisomorphic designs. Computations were done by a PC with a 533 MHz Pentium II CPU and 128 MB RAM running a C program. All programs were independently written and run by two of the authors. A few hours computations showed that there exists no 6-(14,7,4) design with any automorphism of types $2^1, 2^2, 2^3, 2^4, 2^5$. It took about one hour to find designs with the automorphism of type 3^4. The toughest case was the case with automorphism of type 2^6, which took almost a week. We also found 6-(14,7,4) designs with the automorphism of type 7^2 in a few seconds.

We summarize the results in the following theorem.

Theorem 1. *There are exactly* 13 $6\text{-}(14,7,4)$ *designs with nontrivial automorphism group. The orders of the full automorphism groups are as follow:*

$\|Aut(\mathcal{D})\|$	2	3	7	12	13
$\#\mathcal{D}$	4	4	2	1	2

The designs with the full automorphism group of even order are isomorphic to their supplements.

The obtained 6-(14,7,4) designs were used to find the derived 5-(13,6,4) designs with respect to the various points where another isomorphism testing and full automorphism group computation were carried out. Theorem 2 presents the results.

Theorem 2. *There are exactly* 21 $5-(13,6,4)$ *designs with nontrivial automorphism group. The orders of the full automorphism groups are as follow:*

$\|Aut(\mathcal{D})\|$	2	3	13
$\#\mathcal{D}$	9	10	2

Among the designs with the full automorphism group of order 2*, there are* 7

designs which are isomorphic to their supplements.

The 6-(14,7,4) designs with the full automorphisms group of orders 2 and 7 are new and are given in the Appendix.

Open question. Find 6-(14,7,4) designs with trivial automorphism group.

References

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Appendix

The orbit representations of the new 6-(14,7,4) designs are presented here. The point set is $V = \{1, 2, \cdots, A, B, \cdots, E\}$. We only give the orbits which contain 1. If 1 does not appear in an orbit, then it lies in the design if and only if its complement does not lie in the design. Design #1 has the full automorphism group $G =< (1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9\ A\ B\ C\ D\ E) >$ and is nonisomorphic to its supplement (not presented here). Designs #2 to #5 have the full automorphism group $G =< (2\ 3)(5\ 6)(7\ 8)(9\ A)(B\ C)(D\ E) >$ and are isomorphic to their supplements. The designs are also available at http://www.ipm.ac.ir/IPM/tayfeh-r.

Design #1.

1234567	1234568	1234569	123456A	123458A	123458C	123459C	123459D	12345AB
12345AD	12345BC	12345BD	12345BE	12345CE	12345DE	123468B	123468C	123469C
123469D	123469E	12346AB	12346AC	12346AE	12346BC	12346BD	12346DE	123489A
123489B	123489D	123489E	12348AD	12348AE	12348BE	12348CD	12348DE	12349AB
12349AE	12349BC	12349BD	1234ACD	1234ACE	1234BCE	1234CDE	123568A	123568B
123568E	123569A	123569B	123569E	12356AC	12356BC	12356BD	12356CD	12356DE
123589A	123589B	123589C	123589E	12358AD	12358BD	12358CE	12358DE	12359AD
12359BE	12359CD	12359CE	1235ABC	1235ABE	1235ACE	1235ADE	1235BCD	123689C
123689D	123689E	12368AB	12368AD	12368AE	12368CD	12368CE	12369AB	12369AD
12369BE	12369CD	1236ABD	1236ACE	1236BCE	1236BDE	1236CDE	12389AC	12389BC
12389BD	1238ABC	1238ABE	1238ACD	1238BCD	1238BCE	1238BDE	1239ABC	1239ABE
1239ACD	1239ACE	1239ADE	1239BDE	1239CDE	123ABCD	123ABDE	124589B	124589D
12458AC	12458AE	12458BC	12458BD	12458BE	12458CE	12459AC	12459AD	12459AE
12459BC	12459BE	12459CD	1245ABD	1245ADE	1245CDE	124689A	124689B	12468AB
12468AD	12468BE	12468CD	12468CE	12468DE	12469AC	12469BC	12469BD	12469CE
12469DE	1246ABE	1246ACD	1246ADE	1246BCD	12489AB	12489AC	12489CD	12489CE
12489DE	1248ABC	1248ABD	1248ACE	1248BCD	1249ABE	1249ACD	1249ADE	1249BCE
1249BDE	124ABCD	124ABCE	124ABDE	124BCDE	12589AB	12589AE	12589BE	12589CD
12589DE	1258ABC	1258ABD	1258ABE	1258ACD	1258CDE	1259ABC	1259ABD	1259ACE
1259BCD	1259BDE	125ABCE	125ACDE	125BCDE	12689AC	12689AD	12689AE	12689BC
12689BD	12689BE	1268ABC	1268ADE	1268BCE	1268BDE	1269ABD	1269ABE	1269ACE
1269BCD	1269CDE	126ABCD	126ABCE	126ACDE	1289ABD	1289ACE	1289ADE	1289BCE
1289CDE	128ABDE	128ACDE	128BCDE	129ABCD	129BCDE	13589AB	13589AC	13589BC
13589CD	13589DE	1358ABD	1358ACE	1358ADE	1358BCE	1358BDE	1359ABD	1359ABE
1359ACD	1359BCE	1359BDE	135ABCD	135ACDE	135BCDE	1389ABD	1389ABE	1389ACE
1389ADE	1389BCD	1389CDE	138ABCE	138ACDE	139BCDE	13ABCDE	1489ABE	1489ACD
1489BCD	1489BCE	1489BDE	148ABCE	148ABDE	148ACDE	149ABCD	149ACDE	189ABCD
189ABCE	189ABDE	19ABCDE						

Design #2.

1234567	1234569	1234578	1234579	123457A	123458B	123458D	123459B	123459C
12345AC	12345AE	12345BD	12345BE	12345CD	12345CE	12345DE	123478B	123479A
123479D	123479E	12347AB	12347AE	12347BC	12347BD	12347CD	12347DE	12349AD
12349BC	12349BE	12349CE	1235679	123567C	123567D	123569A	123569D	12356BC
12356BD	12356BE	1235789	123578C	123578E	123579D	12357AB	12357AD	12357AE
12357BC	12357BD	12357BE	12357CE	123589B	123589C	123589E	12358AB	12358AC
12358AD	12358CD	12358DE	12359AB	12359AD	12359AE	12359BC	12359CE	12359DE
1235ABC	1235ACD	1235BDE	123789A	123789B	123789E	12378BE	12378DE	12379AB
12379AC	12379BD	12379CD	12379CE	1237ACE	1237ADE	1237BCD	1237BCE	1239ABE
1239BCD	1239BDE	1239CDE	123BCDE	124567A	124567B	124567E	124568C	124568D
124568E	124569B	124569C	124569E	12456AB	12456AD	12456BC	12456CD	12456DE
1245789	124578A	124578D	124579D	124579E	12457AC	12457BC	12457BD	12457BE
12457CD	12457CE	124589A	124589B	124589E	12458AC	12458AE	12458BC	12458BD
12458CE	12459AB	12459AC	12459AD	12459CD	12459DE	1245ABD	1245ABE	1245ADE
1245BCE	124678B	124678D	124678E	124679A	124679B	124679C	124679D	12467AD
12467AE	12467BC	12467CD	124689A	124689C	124689E	12468AB	12468AC	12468BC
12468BD	12468DE	12469AD	12469AE	12469BD	12469CE	1246ABE	1246ACD	1246ACE
1246BCE	1246BDE	124789C	124789D	124789E	12478AB	12478AC	12478AD	12478BE
12478CE	12479AB	12479AC	12479BC	12479BE	1247ABD	1247ACE	1247ADE	1247BDE
1247CDE	12489AB	12489BD	12489BE	12489CD	1248ABE	1248ACD	1248ADE	1248BCD
1248CDE	1249ABC	1249ACE	1249ADE	1249BCD	1249BDE	1249CDE	124ABCD	124ABCE
124BCDE	1256789	125678B	125678C	125678E	125679A	125679B	12567AC	12567AD
12567BD	12567CE	12567DE	125689A	125689C	125689D	12568AB	12568AD	12568BE
12568CD	12569AC	12569BD	12569BE	12569CE	12569DE	1256ABC	1256ABE	1256ACE
1256ADE	1256BCD	125789C	12578AB	12578AD	12578AE	12578BC	12578BD	12578DE
12579AB	12579AC	12579AE	12579BC	12579BE	12579CD	12579DE	1257ABE	1257ACD
1257CDE	12589AD	12589AE	12589BD	12589BE	12589CD	1258ABC	1258ACE	1258BCE
1258BDE	1258CDE	1259ABD	1259ACE	1259BCD	1259BCE	125ABCD	125ABDE	125ACDE
125BCDE	126789A	126789B	126789D	12678AC	12678AE	12678CD	12678CE	12679AE
12679CD	12679CE	12679DE	1267ABC	1267ABD	1267ABE	1267BCD	1267BCE	1267BDE
12689AB	12689BC	12689BE	12689DE	1268ABD	1268ACD	1268ACE	1268ADE	1268BCE
1268BDE	1269ABC	1269ABD	1269ABE	1269ACD	1269BCD	1269CDE	126ACDE	126BCDE
12789AD	12789AE	12789BC	12789BD	12789CE	1278ABC	1278ABE	1278BCD	1278BDE
1278CDE	1279ABD	1279ACD	1279ADE	1279BCE	1279BDE	127ABCD	127ABCE	127ACDE
1289ABC	1289ACD	1289ACE	1289ADE	1289BCE	1289CDE	128ABCD	128ABDE	129ABCE
129ABDE	1456789	145678B	145679C	145679D	145679E	14567AC	14567AE	14567BD
14567CD	14569AB	14569AD	14569BD	1456BCD	1456BDE	145789A	145789B	14578AB
14578BE	14578CD	14578CE	14578DE	14579AD	14579AE	14579BC	14579BD	14579CE
1457ABC	1457ABE	1457ACD	1457ADE	1457BDE	14589AC	14589BC	14589CD	14589CE
14589DE	1458ABD	1458ACD	1458ADE	1458BCD	1458BDE	1459ABC	1459ABE	1459BCE
1459BDE	1459CDE	145ABCD	145ABCE	145ACDE	14789AD	14789BC	14789BD	14789DE
1478BCD	1479ABD	1479ABE	1479ACD	1479ACE	1479BCD	1479CDE	147ABCD	147BCDE
149ABDE	156789D	156789E	15678BC	15678DE	15679AB	15679AD	15679BC	15679BE
15679CD	1567ABC	1567ABD	1567ACE	1567BCE	1567BDE	1567CDE	1569ABE	1569ADE
1569BCE	1569CDE	15789AB	15789AC	15789BE	15789CD	15789CE	15789DE	1578ABD
1578ACD	1578ACE	1578BCD	1578BCE	1579ABD	1579ACE	1579ADE	1579BCD	157ABCE
157ABDE	157BCDE	1589ABC	1589ABD	1589ABE	1589ADE	1589BDE	158ABCE	158ACDE
159ABCD	159ACDE	159BCDE	1789ABC	1789ABE	1789BDE	1789CDE	179ABCE	179BCDE
17ABCDE	19ABCDE							

Design #3.

1234567	1234569	1234578	1234579	123457A	123458B	123458D	123459C	123459E
12345AC	12345AE	12345BC	12345BD	12345BE	12345CD	12345DE	123478B	123479A
123479B	123479D	12347AD	12347AE	12347BD	12347BE	12347CD	12347CE	12349AB
12349BC	12349CE	12349DE	1235679	123567D	123567E	123569A	123569B	12356BC
12356BD	12356BE	123578B	123578C	123578D	123579B	123579C	12357AB	12357AD
12357AE	12357BC	12357CE	12357DE	123589A	123589B	123589C	123589E	12358AC
12358AD	12358BE	12358CE	12359AD	12359AE	12359BD	12359CD	12359DE	1235ABC
1235ABD	1235ABE	1235CDE	123789A	123789C	123789D	123789E	12378DE	12379AC
12379BE	12379CD	12379DE	1237ABC	1237ACE	1237BCD	1237BCE	1237BDE	1239ABD
1239ABE	1239BCE	1239BDE	123BCDE	124567A	124567B	124567C	124568B	124568C
124568E	124569B	124569C	124569E	12456AD	12456AE	12456BD	12456CD	12456DE
1245789	124578A	124578E	124579D	124579E	12457AB	12457BC	12457BD	12457CD
12457CE	12457DE	124589A	124589C	124589D	12458AB	12458AD	12458BC	12458CE
12458DE	12459AB	12459AC	12459AD	12459BD	12459BE	1245ABE	1245ACD	1245ACE
1245BCE	1246789	124678D	124678E	124679A	124679C	124679D	12467AB	12467AC
12467BD	12467BE	12467DE	124689B	124689E	12468AC	12468AD	12468AE	12468BC
12468BD	12468CD	12469AC	12469AD	12469AE	12469BD	12469CE	1246ABC	1246ABE
1246BCE	1246CDE	124789C	124789D	12478AB	12478AC	12478AD	12478BC	12478BE
12478DE	12479AB	12479AE	12479BC	12479BE	12479CE	1247ACD	1247ACE	1247ADE
1247BCD	12489AB	12489AE	12489BC	12489BE	12489CD	1248ABD	1248ACE	1248BDE
1248CDE	1249ACD	1249ADE	1249BCD	1249BDE	1249CDE	124ABCD	124ABCE	124ABDE
124BCDE	1256789	125678A	125678B	125678D	125679C	125679D	12567AC	12567AE
12567BC	12567BE	12567DE	125689A	125689C	125689D	12568AB	12568BE	12568CD
12568CE	12569AB	12569AD	12569BE	12569CE	12569DE	1256ABC	1256ABD	1256ACE
1256ADE	1256BCD	125789B	125789E	12578AC	12578AE	12578BD	12578CD	12578CE
12579AB	12579AC	12579AD	12579AE	12579BE	12579CD	1257ABD	1257ACD	1257BCE
1257BDE	12589AE	12589BC	12589BD	12589DE	1258ABC	1258ABE	1258ACD	1258ADE
1258BCD	1258BDE	1259ABC	1259ACE	1259BCD	1259BCE	1259CDE	125ABDE	125ACDE
125BCDE	126789A	126789B	12678AD	12678AE	12678BC	12678CD	12678CE	12679AE
12679BC	12679BD	12679CE	12679DE	1267ABD	1267ABE	1267ACD	1267BCE	1267CDE
12689AB	12689AC	12689BE	12689CD	12689DE	1268ABD	1268ACE	1268ADE	1268BCE
1268BDE	1269ABC	1269ABE	1269ACD	1269BCD	1269BDE	126ABCD	126ACDE	126BCDE
12789AC	12789AD	12789BD	12789BE	12789CE	1278ABC	1278ABE	1278BCD	1278BDE
1278CDE	1279ABC	1279ABD	1279ADE	1279BCD	1279CDE	127ABCE	127ABDE	127ACDE
1289ABD	1289ACD	1289ACE	1289ADE	1289BCE	1289CDE	128ABCD	128ABCE	129ABCE
129ABDE	145678B	145678D	145679A	145679B	145679D	145679E	14567AD	14567AE
14567CE	14569AB	14569BC	14569CD	1456BCD	1456BDE	145789A	145789B	145789C
14578AC	14578BD	14578BE	14578CD	14579AC	14579AE	14579BC	14579CD	1457ABC
1457ABD	1457ABE	1457ADE	1457BDE	1457CDE	14589BD	14589BE	14589CE	14589DE
1458ABC	1458ABE	1458ACD	1458ADE	1458BCD	1458BCE	1459ABD	1459ACE	1459ADE
1459BCE	1459BDE	1459CDE	145ABCD	145ACDE	14789AD	14789BD	14789CE	14789DE
1478BCD	1479ABC	1479ABD	1479ABE	1479ACD	1479BDE	1479CDE	147ABCE	147BCDE
149ABCD	156789C	156789E	15678BC	15678DE	15679AB	15679AD	15679BD	15679BE
15679CE	1567ABC	1567ACD	1567ACE	1567BCD	1567BDE	1567CDE	1569ABE	1569ADE
1569BCD	1569CDE	15789AB	15789AD	15789BC	15789CD	15789DE	1578ABD	1578ABE
1578ACE	1578ADE	1578BCE	1578CDE	1579ACE	1579ADE	1579BCD	1579BCE	1579BDE
157ABCD	157ABCE	1589ABC	1589ABD	1589ABE	1589ACD	1589ACE	1589CDE	158BCDE
159ABCD	159ABDE	15ABCDE	1789ABC	1789ABE	1789BCE	1789BDE	179ACDE	179BCDE
17ABCDE	19ABCDE							

Design #4.

1234567	1234569	1234578	1234579	123457B	123458A	123458B	123459C	123459D
12345AD	12345AE	12345BC	12345BE	12345CD	12345CE	12345DE	123478D	123479A
123479C	12347AB	12347AC	12347AD	12347BD	12347BE	12347CE	12347DE	12349AD
12349BC	12349BD	12349BE	123567C	123567D	123567E	123569A	123569B	123569D
12356BC	12356BD	1235789	123578C	123578D	123579B	123579E	12357AB	12357AC
12357AD	12357AE	12357BC	12357DE	123589A	123589D	123589E	12358AB	12358AC
12358BE	12358CD	12358CE	12359AB	12359AC	12359BE	12359CD	12359CE	1235ABD
1235ADE	1235BCD	1235BDE	123789A	123789C	123789D	12378BC	12378BD	12379AE
12379BD	12379BE	12379CD	12379DE	1237ABC	1237ACD	1237BCE	1237CDE	1239ABC
1239ABD	1239ADE	1239BCD	1239CDE	123BCDE	124567A	124567B	124567D	1245689
124568C	124568D	124569B	124569E	12456AB	12456AC	12456BE	12456CD	12456CE
12456DE	124578A	124578C	124578E	124579C	124579D	124579E	12457AC	12457AE
12457BD	12457BE	12457CD	124589B	124589C	124589E	12458AD	12458AE	12458BC
12458BD	12458DE	12459AB	12459AC	12459AD	12459AE	12459BD	1245ABC	1245ABD
1245BCE	1245CDE	1246789	124678B	124678E	124679A	124679B	12467AC	12467AE
12467BD	12467CD	12467CE	12467DE	124689A	124689D	12468AD	12468AE	12468BC
12468BD	12468BE	12468CE	12469AC	12469AD	12469BC	12469BE	12469CD	12469CE
1246ABC	1246ABD	1246ADE	124789B	124789D	124789E	12478AB	12478AC	12478AD
12478BC	12478CD	12479AB	12479AD	12479BC	12479CE	12479DE	1247ABE	1247ADE
1247BCD	1247BCE	12489AB	12489AC	12489CD	12489DE	1248ABE	1248ACD	1248ACE
1248BCE	1248BDE	1249ABE	1249ACE	1249BCD	1249BDE	1249CDE	124ABCD	124ABDE
124ACDE	124BCDE	1256789	125678A	125678B	125678E	125679C	125679D	125679E
12567AB	12567AD	12567BC	12567CE	125689A	125689C	12568AB	12568AC	12568BE
12568CD	12568DE	12569AD	12569AE	12569BC	12569BD	12569CE	1256ABE	1256ACD
1256ADE	1256BCD	1256BDE	125789A	125789B	12578AD	12578BD	12578BE	12578CD
12578CE	12579AB	12579AC	12579AD	12579BC	12579DE	1257ABE	1257ACE	1257BCD
1257BDE	1257CDE	12589AE	12589BC	12589BD	12589CD	12589DE	1258ABC	1258ABD
1258ACE	1258ADE	1258BCE	1259ABE	1259ACD	1259BCE	1259BDE	1259CDE	125ABCD
125ABCE	125ACDE	126789C	126789D	12678AC	12678AD	12678BC	12678CD	12678DE
12679AB	12679AE	12679BE	12679CD	12679DE	1267ABC	1267ABD	1267ACE	1267BCE
1267BDE	12689AB	12689AE	12689BD	12689BE	12689CE	1268ABD	1268ABE	1268ACE
1268BCD	1268CDE	1269ABC	1269ACD	1269ADE	1269BCD	1269BDE	126ABCE	126ACDE
126BCDE	12789AC	12789AE	12789BD	12789BE	12789CE	1278ABC	1278ABE	1278ADE
1278BDE	1278CDE	1279ABD	1279ACD	1279ACE	1279BCD	1279BCE	127ABCD	127ABDE
127ACDE	1289ABC	1289ABD	1289ACD	1289ADE	1289BCE	1289CDE	128ABCD	128BCDE
129ABCE	129ABDE	145678B	145678D	145679A	145679C	145679D	145679E	14567AD
14567BC	14567CE	14569AB	14569BD	14569CD	14569DE	1456BCD	145789A	145789B
145789C	145789D	14578AB	14578BE	14578DE	14579AB	14579BC	14579BE	1457ABC
1457ABD	1457ACD	1457ACE	1457ADE	1457BDE	1457CDE	14589AD	14589BE	14589CD
14589CE	1458ABC	1458ABE	1458ACD	1458ACE	1458BCD	1458CDE	1459ACD	1459ACE
1459ADE	1459BCD	1459BCE	1459BDE	145ABDE	145BCDE	14789AD	14789BD	14789CE
1478BCD	1478BDE	1479ABC	1479ACD	1479ACE	1479ADE	1479BDE	1479CDE	147ABCD
147ABCE	149ABCD	156789B	156789D	15678BD	15679AC	15679AE	15679BD	15679BE
1567ABC	1567ABE	1567ACD	1567ADE	1567BCD	1567BDE	1567CDE	1569ABC	1569ABD
1569BCE	1569CDE	15789AC	15789AE	15789CD	15789CE	15789DE	1578ABC	1578ABD
1578ACD	1578ADE	1578BCD	1578BCE	1579ABD	1579ABE	1579ADE	1579BCD	1579BCE
1579CDE	157ABCE	1589ABC	1589ABD	1589ABE	1589ACE	1589BDE	158ABDE	158BCDE
159ABCD	159ACDE	15ABCDE	1789ABC	1789ABE	1789BCD	1789BDE	179ABDE	179BCDE
17ABCDE	19ABCDE							

Design #5.

1234567	1234569	1234578	1234579	123457B	123458B	123458D	123459A	123459C
12345AD	12345AE	12345BC	12345BE	12345CD	12345CE	12345DE	123478D	123479A
123479B	123479D	12347AB	12347AC	12347AD	12347BC	12347CE	12347DE	12349BD
12349BE	12349CD	12349CE	123567B	123567D	123567E	123569B	123569C	123569E
12356BD	1235789	123578A	123578C	123579C	123579E	12357AC	12357AD	12357AE
12357BD	12357BE	12357CD	123589A	123589B	123589E	12358AC	12358AE	12358BC
12358BD	12358DE	12359AB	12359AD	12359BD	12359CD	12359DE	1235ABC	1235ABE
1235BCE	1235CDE	123789C	123789D	12378BD	12378BE	12379AB	12379AE	12379BC
12379CE	12379DE	1237ABE	1237ACD	1237BCD	1237BDE	1239ABC	1239ABD	1239ADE
1239BCE	1239BDE	123BCDE	1245679	124567A	124567B	1245689	124568B	124568E
124569D	12456AC	12456AD	12456BC	12456BE	12456CD	12456CE	12456DE	124578C
124578D	124578E	124579D	124579E	12457AB	12457AC	12457AE	12457BD	12457CD
12457CE	124589B	124589C	124589D	12458AB	12458AC	12458AD	12458AE	12458CE
12459AB	12459AC	12459AE	12459BC	12459BE	12459DE	1245ABD	1245BCD	1245BDE
1246789	124678C	124678D	124679C	124679E	12467AB	12467AD	12467AE	12467BD
12467BE	12467CD	124689A	124689B	12468AB	12468AE	12468BD	12468CD	12468CE
12468DE	12469AC	12469AD	12469BC	12469BD	12469BE	12469CE	1246ABC	1246ACE
1246ADE	124789A	124789B	124789E	12478AB	12478AC	12478AD	12478BC	12478BE
12479AC	12479AE	12479BC	12479BD	12479CD	1247ADE	1247BCE	1247BDE	1247CDE
12489AD	12489CD	12489CE	12489DE	1248ABE	1248ACD	1248BCD	1248BCE	1248BDE
1249ABC	1249ABD	1249ABE	1249ADE	1249CDE	124ABCD	124ABCE	124ACDE	124BCDE
125678A	125678B	125678C	125678D	125679A	125679B	125679C	12567AE	12567CD
12567CE	12567DE	125689A	125689C	125689D	12568AD	12568AE	12568BC	12568BE
12569AB	12569AE	12569BD	12569CE	12569DE	1256ABC	1256ABE	1256ACD	1256BCD
1256BDE	125789A	125789D	125789E	12578AB	12578BC	12578BE	12578DE	12579AC
12579AD	12579BC	12579BD	12579BE	1257ABC	1257ABD	1257ADE	1257BCE	1257CDE
12589AB	12589BE	12589CD	12589CE	1258ABD	1258ACD	1258ACE	1258BCD	1258BDE
1258CDE	1259ACD	1259ACE	1259ADE	1259BCD	1259BCE	125ABCE	125ABDE	125ACDE
126789C	126789E	12678AC	12678AE	12678BD	12678BE	12678DE	12679AB	12679AD
12679BE	12679CD	12679DE	1267ABC	1267ABD	1267ACE	1267BCD	1267BCE	12689AC
12689AE	12689BC	12689BD	12689DE	1268ABC	1268ABD	1268ACD	1268BCE	1268CDE
1269ABD	1269ABE	1269ACD	1269BCE	1269CDE	126ABDE	126ACDE	126BCDE	12789AB
12789AD	12789BC	12789BD	12789CE	1278ACD	1278ACE	1278ADE	1278BCD	1278CDE
1279ABE	1279ACD	1279ACE	1279BDE	1279CDE	127ABCD	127ABCE	127ABDE	1289ABC
1289ABE	1289ACE	1289ADE	1289BCD	1289BDE	128ABCE	128ABDE	129ABCD	129BCDE
1456789	145678D	145679B	145679C	14567AD	14567BC	14567BE	14567CE	14567DE
14569AB	14569AD	14569BD	14569CD	14569CE	145789A	145789B	145789C	14578AE
14578BC	14578BD	14578CD	14579AB	14579AD	14579BE	14579CE	14579DE	1457ABC
1457ABE	1457ACD	1457ADE	1457BCD	14589AC	14589AD	14589AE	14589BD	14589BE
1458ABC	1458ADE	1458BCE	1458BDE	1458CDE	1459ACD	1459BCD	1459BCE	1459CDE
145ABCE	145ABDE	145ACDE	145BCDE	14789BD	14789CD	14789CE	14789DE	1478BDE
1479ABC	1479ABD	1479ACD	1479ACE	1479BCE	1479BDE	147ABCD	147ABCE	149ABDE
156789B	156789E	15678BD	15679AC	15679AE	15679BD	15679CD	15679DE	1567ABC
1567ABD	1567ABE	1567ACD	1567BCE	1567BDE	1569ABC	1569ADE	1569BCE	1569BDE
156BCDE	15789AC	15789AE	15789BC	15789CD	15789DE	1578ABD	1578ABE	1578ACD
1578ACE	1578BCE	1578BDE	1579ABD	1579ABE	1579ACE	1579BCD	1579CDE	157ACDE
157BCDE	1589ABC	1589ABD	1589ADE	1589BDE	1589CDE	158ABCD	158ABCE	159ABCD
159ABCE	159ABDE	1789ABC	1789ABE	1789ADE	1789BCE	178BCDE	179ABDE	179BCDE
17ABCDE	19ABCDE							