A note on order and eigenvalue multiplicity of strongly regular graphs

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Abstract. In this note, we consider a well known upper bound for the order of a strongly regular graph in terms of the multiplicity of a non-principal eigenvalue of its adjacency matrix.

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A *strongly regular graph* with parameters (n, k, λ, μ), denoted $srg(n, k, \lambda, \mu)$, is a regular graph of order n and valency k such that (i) it is not complete or edgeless, (ii) every two adjacent vertices have λ common neighbors, and (iii) every two non-adjacent vertices have μ common neighbors. Strongly regular graphs form an important class of graphs which lie somewhere between highly structured graphs and apparently random graphs. Obviously, complete multipartite graphs with equal part sizes and their complements are trivial examples of strongly regular graphs. In this note, to exclude these examples, we assume that a strongly regular graph and its complement are connected; in other words, we assume, equivalently, that $0 < \mu < k < n - 1$.

The *adjacency matrix* of a graph G, denoted by $\mathcal{A}_G$, has its rows and columns indexed by the vertex set of G and its (u, v)-entry is 1 if the vertices u and v are adjacent and 0 otherwise. The zeros of the characteristic polynomial of $\mathcal{A}_G$ are called the *eigenvalues* of G. The statement that G is an $srg(n, k, \lambda, \mu)$ is equivalent to

$$\mathcal{A}_G J_n = kJ_n \quad \text{and} \quad \mathcal{A}_G^2 + (\mu - \lambda)\mathcal{A}_G + (\mu - k)I_n = \mu J_n,$$

where I_n and J_n are the $n \times n$ identity matrix and the $n \times n$ all one matrix, respectively. It is not hard to see that the eigenvalues of an $srg(n, k, \lambda, \mu)$ are

$$\begin{cases} k, \text{ with the multiplicity } 1; \\ r = \dfrac{\lambda - \mu + \sqrt{\Delta}}{2}, \text{ with the multiplicity } f = \dfrac{n-1}{2} - \dfrac{2k + (n-1)(\lambda - \mu)}{2\sqrt{\Delta}}; \\ s = \dfrac{\lambda - \mu - \sqrt{\Delta}}{2}, \text{ with the multiplicity } g = \dfrac{n-1}{2} + \dfrac{2k + (n-1)(\lambda - \mu)}{2\sqrt{\Delta}}, \end{cases}$$

where $\Delta = (\lambda - \mu)^2 + 4(k - \mu)$. It is well known that the second largest eigenvalue of a graph G is non-positive if and only if the non-isolated vertices of G form a complete multipartite graph. Also, it is a known fact that the smallest eigenvalue of a graph G is at least -1 if and only if G is a vertex disjoint union of some complete graphs. Therefore, for any $srg(n, k, \lambda, \mu)$, we have $r > 0$ and $s < -1$.

The important conditions satisfied by the parameters of a strongly regular graph are the Krein condition [4] and the absolute bound [3]:

$$\text{The Krein condition:} \quad \begin{cases} (r+1)(k+r+2rs) \leqslant (k+r)(s+1)^2, & (1) \\ (s+1)(k+s+2sr) \leqslant (k+s)(r+1)^2; & (2) \end{cases}$$

$$\text{The absolute bound:} \quad \begin{cases} n \leqslant f(f+3)/2, & (3) \\ n \leqslant g(g+3)/2. & (4) \end{cases}$$

It was shown in [2] that (3) can be improved to $n \leqslant f(f+1)/2$, unless the equality occurs in (1). A similar statement holds for (2) and (4). It is easy to see that the equality occurs in (1) for a strongly regular graph if and only if the graph is the pentagon or an $srg(n, k, \lambda, \mu)$ with integral eigenvalues in which

$$\begin{cases} n = \dfrac{2(s-r)^2(s^2+2s+2sr+r)}{(s^2+r)(s^2+2s-r)}, \\ k = \dfrac{r(-s^2+2sr+r)}{s^2+2s-r}, \\ \lambda = \dfrac{s(r+1)(s^2+2s+r)}{s^2+2s-r}, \\ \mu = \dfrac{r(s+1)(s^2+r)}{s^2+2s-r}. \end{cases}$$

A strongly regular graph with these parameters or the complement of one is called a *Smith* graph. Since $r > 0$ and $s < -1$, the non-negativity of λ shows that $r > s^2 + 2s$. Moreover, we have

$$f = \frac{(s^2-2sr-r)(s^2+2s+2sr+r)}{(s^2+r)(s^2+2s-r)} \quad \text{and} \quad g = \frac{2r(r+1)(-s^2+2sr+r)}{(s^2+r)(s^2+2s-r)}.$$

By an easy calculation, we find that

$$g - f = \frac{(-s^2+2sr+r)\big((s+r+1)^2+r^2+r-1\big)}{(s^2+r)(s^2+2s-r)} > 0.$$

In this note, we improve the aforementioned result of [2].

Lemma 1. *If the equality occurs in (1), then either* $n \leqslant f(f+1)/2$ *or* $n = f(f+3)/2$*, unless the graph is the Clebsch graph, that is, the unique* $srg(16, 10, 6, 6)$.

Proof. Let the equality occurs in (1) for a strongly regular graph. Using (3), we may suppose that $f(f+1)/2 < n < f(f+3)/2$. Since

$$\frac{f(f+3)}{2} - n = \frac{2r(r+1)(r-s)(s^2+2s+2sr+r)(2s^3+3s^2+r)}{(s^2+r)^2(s^2+2s-r)^2},$$

we have $r < -2s^3 - 3s^2$. Also,

$$n - 1 - \frac{f(f+1)}{2} = \frac{2r(r+1)(s^2 - 2sr - r)(s^3 + 2s^2 + 2s^3r + 3s^2r - sr + r^2)}{(s^2+r)^2(s^2+2s-r)^2},$$

and hence

$$s^3 + 2s^2 + 2s^3r + 3s^2r - sr + r^2 = s^2(s+2) + r(2s^3 + 3s^2 - s + r) \geqslant 0.$$

This implies that $r \geqslant -2s^3 - 3s^2 + s$. Letting $\ell = 2s^3 + 3s^2 + r$, we have

$$\mu = 2s^4 + 3s^3 - s\ell - \ell + \frac{\ell(s+1)^2 - \ell}{2s(s+1)^2 - \ell}.$$

Since $s \leqslant \ell \leqslant -1$ and μ is integral, it is straightforward to see that $s = -2$. From $-2s^3 - 3s^2 + s \leqslant r \leqslant -2s^3 - 3s^2 - 1$ and the integrity of n, we find that $r = 2$ and so the graph is $srg(16, 10, 6, 6)$. □

Note that the equality occurs in (1) for a strongly regular graph if and only if the equality occurs in (2) for its complement. So, by Lemma 1, we obtain the following result.

Theorem 1. *For any strongly regular graph, one of the following holds.*

(i) $n \leqslant \min\left\{\frac{f(f+1)}{2}, \frac{g(g+1)}{2}\right\}$.

(ii) $n = \min\left\{\frac{f(f+3)}{2}, \frac{g(g+3)}{2}\right\}$.

(iii) *The graph or its complement is the Clebsch graph.*

Remark 1. There are only three known examples of strongly regular graphs satisfying $n = f(f+3)/2$, that are, the pentagon, the Schläfli graph, and the McLaughlin graph. There are infinitely many feasible parameters of a strongly regular graph with $n = f(f+1)/2$. It is not hard to check that a strongly regular graph with $n = f(f+1)/2$ and $s = -2$ has the parameters $k = 2f - 2$, $\lambda = f - 1$, and $\mu = 4$. By a result of [1], any such strongly regular graph whenever $f \neq 7$ is a triangular graph, that is, the line graph of the complete graph of order $f + 1$. The characterizing problem of the strongly regular graphs with $n = f(f+1)/2$ which is posed in [2] remains unsolved yet.

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