Graphs with prescribed star complement for the eigenvalue 1

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October 26, 2008

Abstract

Let G be a graph of order n and let μ be an eigenvalue of multiplicity m. A star complement for μ in G is an induced subgraph of G of order $n-m$ with no eigenvalue μ. In this paper, we study the maximal graphs as well as regular graphs which have $K_{r,s}+tK_1$ as a star complement for eigenvalue 1. It turns out that some well known strongly regular graphs are uniquely determined by such a star complement.

AMS Subject Classification: 05C50.
Keywords: Star complement, maximal graphs, strongly regular graphs, multiple eigenvalue.

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1 Introduction

Let G be a simple graph of order n and the vertex set $V(G)$. Let μ be an eigenvalue of G of multiplicity m. An m-subset X of $V(G)$ is called a *star set* for μ in G if μ is not an eigenvalue of $G \setminus X$. The induced subgraph $H = G \setminus X$ is said to be a *star complement* for μ in G. Star sets exist for any eigenvalue in a graph and they are not necessarily unique. For the background and results on star sets and star complements, one may consult [8, 9, 11, 14].

The following theorem which establishes a relation between a graph and its substructures corresponding to an eigenvalue is the basis of the so called *star complement technique*.

Theorem 1 (The Reconstruction Theorem) *Let G be a graph with adjacency matrix*

$$\begin{pmatrix} A_X & B^t \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by a subset X of vertices. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and $\mu I - A_X = B^t(\mu I - C)^{-1}B$.

This theorem states that the triple (μ,B,C) determines A_X uniquely. In other words, given eigenvalue μ, a star complement H and H-neighborhoods of X, G is uniquely determined. From the theorem, it is seen that for any two vertices u and v of X, we have

$$< \mathbf{b}_u, \mathbf{b}_v >= \mathbf{b}_u^t(\mu I - C)^{-1}\mathbf{b}_v = \begin{cases} \mu & \text{if } u = v, \\ -1 & \text{if } u \sim v, \\ 0 & \text{if } u \nsim v, \end{cases} \qquad (1)$$

where $\mathbf{b}_x$ is the column of B corresponding to a vertex x. It is well known that if $\mu \neq 0, -1$, then the H-neighborhoods of vertices of X are distinct and nonempty.

Let H be a graph of order t with no eigenvalue μ. The star complement technique is a method for determining all graphs G prescribing H as a star complement for eigenvalue μ. It is known that if $\mu \neq -1, 0$, then $|V(G)| \leq \binom{t+1}{2}$ (see [3]) and therefore there are only finitely many such graphs G. Now we briefly review the star complement technique. We use the notation of Theorem 1. Given C (the adjacency matrix of H) with no eigenvalue μ, one is interested in finding the solutions for B (note that by Theorem 1, A_X will then be determined uniquely). Hence, first of all one needs to find $(0,1)$ column vectors of dimension t which are candidates for columns of B. In other words, we need to find all possible extensions $H+u$ of H by adding a new vertex u such that $H+u$ has μ as an eigenvalue. In order to do this, we identify all vectors $\mathbf{b}$ satisfying

$$< \mathbf{b}, \mathbf{b} >= \mu,$$

and let them be the vertices of the *compatibility* graph $\mathcal{G}(H,\mu)$. An edge is inserted between $\mathbf{b}$ and $\mathbf{b}'$ if and only if

$$< \mathbf{b}, \mathbf{b}' >= 0, -1.$$

Now by Theorem 1, any clique in $\mathcal{G}(H,\mu)$ determines the vertices of a star set X and therefore a graph G having H as a star complement for eigenvalue μ. To describe all the graphs with H as a star complement for μ, it suffices to determine the *maximal* graphs, i.e. those graphs for which the corresponding clique in $\mathcal{G}(H,\mu)$ is maximal, since any graph with H as a star complement for μ is an induced subgraph of such a graph.

Two main problems arise in the context of star complement. One of these is the *general problem* which is to find all maximal graphs having a given graph H as a star complement for some eigenvalue. In other words, by the notation of Theorem 1, given C, we want to find all solutions for μ, B, A_X. The other problem is the *restricted problem* which is about the determination of all maximal graphs prescribing a given graph H as a star complement for a given eigenvalue μ. This means that given C and μ, we are interested in finding all solutions for B and A_X. These problems are interesting for some reasons as is described in the following. Sometimes there is only a unique maximal graph and hence that graph is characterized by a means of its star complement. Also the problems usually build unexpected links to other areas of combinatorics such as extremal set theory and t-designs. The general and restricted problems have been dealt with for some special families of graphs such as complete graphs, complete bipartite graphs, stars, paths, cycles and so on. A list of references includes [1, 2, 4, 5, 9, 10, 12, 13, 14, 16, 17].

In this paper, we consider the restricted problem for $H = K_{r,s} + tK_1$ and $\mu = 1$. Our objective is to identify the maximal graphs as well as regular graphs which have H as a star complement for eigenvalue 1. Some special cases of this problem have already been investigated: In [12] (see also [15, 16]), it is shown that the complement of the Clebsch graph (srg$(16,5,0,2)$) is the unique maximal graph which has $K_{1,5}$ as a star complement. For $H = K_{1,9}$, it is known that there are exactly 15 maximal graphs [16] and for $H = K_{1,10}$, there is a unique maximal graph [12]. We also know that the complement of the Schläfli graph is the unique maximal graph which admits $H = K_{2,5}$ as a star complement [12, 13]. Finally, for $H = K_{1,s} + 2K_1$, the maximal graphs are found in [10]. We note that in [12] (see also [13]) some general observations on the general problem for $H = K_{r,s} + tK_1$ and arbitrary μ are given.

The following lemma provides useful information on the location of an eigenvalue of a graph. For a graph G of order n we denote the ith largest eigenvalue of G by $\lambda_i(G)$ and we also let $\lambda_0(G) = \infty$ and $\lambda_{n+1}(G) = -\infty$.

Lemma 1 *Given a graph G of order n with eigenvalue μ of multiplicity $m \geq 1$, let H be a star complement for μ in G. Let $\lambda_{r+1}(H) < \mu < \lambda_r(H)$ for some $0 \leq r \leq n-m$. Then $\lambda_{r+1}(G) = \cdots = \lambda_{r+m}(G) = \mu$.*

Proof. By interlacing, we have the inequalities

$$\lambda_{r+m}(G) \leq \lambda_r(H) \leq \lambda_r(G),$$

$$\lambda_{r+1+m}(G) \leq \lambda_{r+1}(H) \leq \lambda_{r+1}(G),$$

which yield $\lambda_{r+m+1}(G) < \mu < \lambda_r(G)$. Since G has eigenvalue μ of multiplicity m, the assertion follows. □

We introduce some notation which will be used throughout the paper. We assume that $H = K_{r,s} + tK_1$ is a star complement for eigenvalue $\mu = 1$ in G. Note that by Lemma 1, G has 1 as the second largest eigenvalue. With no loss of generality, we suppose that $1 \leq r \leq s$, $(r,s) \neq (1,1)$. Let also $W = \{w_1, w_2, \ldots, w_t\}$ denote the set of isolated vertices in H and let $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$ be the two subsets of vertices of H with all edges of H between U and V. The star set corresponding to H and μ is denoted by X. Let $H(a,b,c)$ be a graph obtained from H by introducing a new vertex and joining it to a vertices of U, b vertices of V and c vertices of W. The (0,1) column vector $\mathbf{b}_u$

denotes the neighborhood of $u \in X$ in H. Let $H+u = H(a,b,c)$ and $H+v = H(\alpha,\beta,\gamma)$. Then it is an easy task to show that

$$(1-rs) < \mathbf{b}_u, \mathbf{b}_v >= (1-rs)\rho + a(\beta+\alpha s) + b(\alpha+\beta r), \tag{2}$$

where ρ is the number of common neighbors of u and v in H (see [12, Eq. (7.4)]).

2 Extension by a vertex

The first step in the star complement technique is to find all possible extensions of a star complement by adding a new vertex. We proceed to determine all possible graphs $H+u$ by adding a vertex u such that $H+u$ has μ as an eigenvalue.

Let $H+u = H(a,b,c)$. Using (1) and (2), we find that

$$1-rs = (a+b+c)(1-rs) + 2ab + a^2 s + b^2 r. \tag{3}$$

Assuming $r = a+x$ and $s = b+y$, (3) is converted to

$$ab(c-3) + (b+c-1)ay + (a+c-1)bx + (a+b+c-1)(xy-1) = 0. \tag{4}$$

We make use of (4) to obtain the solutions of (3). The proofs of the next two lemmas are straightforward.

Lemma 2 *Let $m \geq n \geq 1$ be integers. If $mn \leq m+n$, then $(m,n) = (2,2)$ or $n = 1$.*

Lemma 3 *Let $m \geq n \geq q \geq 1$ be integers. Then the solutions of $mnq = m+n+q+2$ are $(m,n,q) \in \{(2,2,2),(3,3,1),(5,2,1)\}$.*

Lemma 4 *Let $c \geq 3$. Then the solutions of (3) are as follows.*

r	3	2	2	1	1	1	1	1
s	3	5	2	5	3	2	2	2
a	3	2	2	1	1	1	1	0
b	3	5	2	5	3	2	1	2
c	4	4	5	5	6	8	4	3

Proof. First suppose that $xy \neq 0$. Since $c \geq 3$, all sentences in the left hand side of (4) are nonnegative and thus they all must be 0. Consequently, we obtain that $a = b = 0$ and $x = y = 1$, which is not acceptable. So $xy = 0$. Assume that $x \neq 0$ and $y = 0$. Then (4) yields $ab(c-3) + (a+c-1)bx = a+b+c-1$. By Lemma 2, $b = 1, 2$. If $b = 2$, then by Lemma 2, $x = 1, c = 3, a = 0$ and we have the solution $(r, s, a, b, c) = (1, 2, 0, 2, 3)$. If $b = 1$, then $x = 1, c = 4, a = 1$ and the solution $(r, s, a, b, c) = (2, 1, 1, 1, 4)$ is obtained which is not acceptable (since $r \leq s$). The case $x = 0$ and $y \neq 0$ gives the same solutions with the roles of r and s interchanged. Therefore in this case we have the solution $(r, s, a, b, c) = (1, 2, 1, 1, 4)$. Finally, let $x = y = 0$. Then we have $ab(c-3) = a+b+c-1$ and hence by Lemma 3, $(a, b, c) = (2, 2, 5), (1, 3, 6), (3, 3, 4), (1, 2, 8), (1, 5, 5), (2, 5, 4)$. □

In the next lemmas we consider the remaining cases $c = 0, 1, 2$.

Lemma 5 *Let $c = 0$. Then the solutions of* (3) *are as follows.*

r	5	3	3	2	2	2	1	1	1	1
s	10	11	3	5	5	13	5	9	10	10
a	5	3	1	0	1	2	0	1	1	1
b	6	7	1	3	1	9	2	3	2	5

Proof. For $c = 0$, the equation (4) becomes

$$abx + aby + (a+b-1)(xy-1) = 3ab + ay + bx. \tag{5}$$

With no loss of generality we assume that $x \leq y$ (if we find a solution such that $r > s$, we should interchange the roles of r and s). Note that $y \neq 0$. First suppose that $x = 0$. Then $(b-1)ay = a+b-1+3ab$. Since $r \geq 1$, we have $a \geq 1$. Also $b \geq 2$, since otherwise we get $y = 0$ or $a = 0$, a contradiction. We now conclude that $4a$ is congruent to 0 modulo $b-1$ and $b-1$ is congruent to 0 modulo a. Therefore, $b-1 = a, 2a$ or $4a$. First let $b-1 = a$. Then $ay = 5+3a$ which gives the solutions $(r, s, a, b) = (1, 10, 1, 2), (5, 10, 5, 6)$. Next let $b-1 = 2a$. Then $ay = 3+3a$ which gives the solutions $(r, s, a, b) = (1, 9, 1, 3), (3, 11, 3, 7)$. Finally, let $b-1 = 4a$. Then from $ay = 3a+2$ we find the solutions $(r, s, a, b) = (1, 10, 1, 5), (2, 13, 2, 9)$.

Now we assume that $x > 0$. We claim that $x > 2$ is impossible. On the contrary, suppose that $x > 2$. From (5), it can easily been seen that $ab \neq 0$. Since we have assumed $y \geq x$, (5) yields $3(ab-b)+3(ab-a)+9(a+b-1)+1 \leq a+b+3ab$ which in turn gives $3ab+5a+5b \leq 8$, a contradiction. Therefore, $x \leq 2$. First let $x = 1$. Then (5) yields $(ab+b-1)(y-2) = a+1$. This gives $y \geq 3$. Now $a+1$ is congruent to 0 modulo $y-2$ and $y-2$ is congruent to 0 modulo $a+1$. Therefore, $y-2 = a+1$ and so $ab+b = 2$ which gives the solutions $(r,s,a,b) = (1,5,0,2),(2,5,1,1)$. Next let $x = 2$. From (5) we find $(ab+a)y+(2b-2)y+1 = a+ab+3b$. If $b = 0$, then $(a-2)y = a-1$ and we find the solution $(r,s,a,b) = (5,2,3,0)$. If $a = 0$, then $(2b-2)y = 3b-1$ and we have the solution $(r,s,a,b) = (2,5,0,3)$. Hence, let $a,b \neq 0$. Then we have $ab+a+b \leq 3$ and so the solution $(r,s,a,b) = (3,3,1,1)$ is obtained. □

Lemma 6 *Let $c = 1$. Then the solutions of (3) are as follows:*

(i) *r,s arbitrary and $a = b = 0$.*

(ii) $(r,s,a,b) = (2,5,2,2),(1,5,1,1)$.

(iii) *r,s arbitrary and $a = r-1, b = s-1$.*

Proof. With $c = 1$ the equation (4) becomes

$$ab(x+y)+(a+b)xy = a+b+2ab. \tag{6}$$

If $a,b = 0$, then obviously x,y are arbitrary and hence (i) holds. So let $a+b \neq 0$. If $x = 0$, then $ab(y-2) = a+b$ which means that $a = b$ and so we find the solutions $(r,s,a,b) = (2,5,2,2),(1,5,1,1)$. For $y = 0$, the same solutions are found with the roles of r and s interchanged. Now let $xy \neq 0$. We have $x,y < 2$, since otherwise from (6) we have $ab(2+y)+2(a+b)y \leq a+b+2ab$ which is a contradiction. Therefore, $x = y = 1$ and (iii) holds. □

Lemma 7 *Let $c = 2$. Then in (3) we have $r = a = 1$ and $b = s-2$.*

Proof. Letting $c = 2$ in (4) we have

$$ab(x+y)+(a+b+1)xy+ay+bx = a+b+1+ab. \tag{7}$$

If $a, b = 0$, then we find $x = y = 1$ and hence $r = s = 1$, a contradiction. So let $a + b \neq 0$. It is seen from (7) that $xy = 0$. If $x = 0$, then (7) yields $a(b + 1)(y - 1) = b + 1$ and thus $a = 1$ and $y = 2$. The case $y = 0$ is similar with the roles of r and s interchanged. $\square$

We summarize the results of the previous lemmas in the following Theorem.

Theorem 2 *The graph $H(a, b, c)$ is of one of the following forms.*

#	H	(a, b, c)
1	$K_{1,2} + tK_1$	$(0, 2, 3), (1, 1, 4), (1, 2, 8), (0, 0, 1), (0, 1, 1), (1, 0, 2)$
2	$K_{1,3} + tK_1$	$(1, 3, 6), (0, 0, 1), (0, 2, 1), (1, 1, 2)$
3	$K_{1,5} + tK_1$	$(0, 2, 0), (0, 0, 1), (1, 1, 1), (0, 4, 1), (1, 3, 2), (1, 5, 5)$
4	$K_{1,9} + tK_1$	$(1, 3, 0), (0, 0, 1), (0, 8, 1), (1, 7, 2)$
5	$K_{1,10} + tK_1$	$(1, 2, 0), (1, 5, 0), (0, 0, 1), (0, 9, 1), (1, 8, 2)$
6	$K_{2,2} + tK_1$	$(2, 2, 5), (0, 0, 1), (1, 1, 1)$
7	$K_{2,5} + tK_1$	$(1, 1, 0), (0, 3, 0), (0, 0, 1), (2, 2, 1), (1, 4, 1), (2, 5, 4)$
8	$K_{2,13} + tK_1$	$(2, 9, 0), (0, 0, 1), (1, 12, 1)$
9	$K_{3,3} + tK_1$	$(1, 1, 0), (0, 0, 1), (2, 2, 1), (3, 3, 4)$
10	$K_{3,11} + tK_1$	$(3, 7, 0), (0, 0, 1), (2, 10, 1)$
11	$K_{5,10} + tK_1$	$(5, 6, 0), (0, 0, 1), (4, 9, 1)$
12	$K_{1,s} + tK_1$ *(none of the above)*	$(0, 0, 1), (0, s - 1, 1), (1, s - 2, 2)$
13	$K_{r,s} + tK_1$ *(none of the above)*	$(0, 0, 1), (r - 1, s - 1, 1)$

3 Maximal graphs

When H is one of the cases #1 to #12 in Theorem 2, there are different types of vertices in the star set which makes it a tedious task to find all maximal graphs with H as a star complement. However, in the case #13 there are only two types of vertices and it seems tractable. Hence, in this section we investigate the maximal extensions G of $H = K_{r,s} + tK_1$ when H is the case #13 in Theorem 2. Note that $t \geq 1$ and there are two types of vertices in the star set X. Let $u \in X$ and $H + u = H(a, b, c)$. We say that u is of type 1 (2) if $(a, b, c) = (0, 0, 1)$ ($(a, b, c) = (r - 1, s - 1, 1)$). Then (1) and (2) show that any vertex of type 1 lies in a component of G which is K_2. We can ignore such vertices for the following reason: If G is a maximal graph for $H = K_{r,s} + tK_1$ containing r vertices of type 1, then G' obtained from G by removing r components K_2 is a maximal graph for $H' = K_{r,s} + (t-r)K_1$. Therefore, we may assume that G has no vertices of type 1. We index a vertex of type 2 by (i, j, k) if it is not adjacent (not adjacent, adjacent) to u_i (v_j, w_k) in U (V, W). Therefore, the vertices of the compatibility graph are indexed by the triples (i, j, k), where $1 \leq i \leq r, 1 \leq j \leq s$ and $1 \leq k \leq t$.

Let $u, v \in X$ be two distinct vertices of type 2. Then by (2),

$$< \mathbf{b}_u, \mathbf{b}_v >= \rho - r - s + 2, \tag{8}$$

where ρ denotes the number of common neighbors of u and v in H. Since $< \mathbf{b}_u, \mathbf{b}_v >= -1, 0$, we conclude that at least one and at most two of U-, V- and W-neighborhoods of u and v must coincide. Moreover, u is joined to v in G if and only they coincide for exactly one of these neighborhoods. In the compatibility graph, (i_1, j_1, k_1) is joined to (i_2, j_2, k_2) if and only if they coincide in at least one coordinate and at most two. We now find the maximal cliques in the compatibility graph.

Theorem 3 *A maximal clique in the compatibility graph, up to isomorphism, is of one of the following forms:*

(i) $M_l = \{(i_1, i_2, i_3) \mid i_l = 1\}$, $1 \leq l \leq 3$.

(ii) $\{(i_1, i_2, i_3) \mid$ *at least two of* i_1, i_2, i_3 *are* $1\}$, *if* $t > 1$.

(iii) $\{(1, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1)\}$, *if* $t > 1$.

Proof. Let M be a maximal clique. If $t = 1$, then obviously we have the case (i). Therefore, let $t > 1$. First suppose that M has two vertices which have the same entries in two coordinates. With no loss of generality, we let $(1, 1, 1), (1, 1, 2) \in M$. Then the remaining vertices in M are of the form (1) $(1, j, k)$ or (2) $(i, 1, k)$. If all vertices are of type (1) or all are of type (2), then we conclude that M is of the form (i). Otherwise, M is of the form (ii). Now assume that no two vertices in M coincide in two coordinates. With no loss of generality, let $(1, 1, 1), (1, 2, 2) \in M$. Then the remaining vertices in M are of the form (1) $(1, j, k)$, (2) $(i, 1, 2)$ or (3) $(i, 2, 1)$. If M has vertices of type (2) or (3), then clearly M is of the from (iii). Otherwise, we find that M is not maximal, a contradiction. □

The theorem above along with the preceding paragraph describe all possible maximal graphs G up to isomorphism.

4 Regular graphs

In this section we identify regular graphs which have H as a star complement for eigenvalue 1. Suppose that G is a k-regular extension of H with star set X. Let $u \in X$. Then it is well known that $< \mathbf{b}_u, \mathbf{j} >= -1$. Therefore, if $H + u = H(a, b, c)$, then by (2), we have

$$a(s + 1) + b(r + 1) + (c + 1)(1 - rs) = 0.$$

Using Theorem 2, we find the solutions of this equation. The results are given in the following Theorem.

Theorem 4 *For a regular extension G, the graph $H(a, b, c)$ is of one of the following forms.*

#	H	(a, b, c)
1	$K_{1,2} + tK_1$	$(0, 2, 3), (1, 1, 4), (0, 1, 1), (1, 0, 2)$
2	$K_{1,5} + tK_1$	$(0, 2, 0), (1, 1, 1), (0, 4, 1), (1, 3, 2)$
3	$K_{2,5} + tK_1$	$(1, 1, 0), (0, 3, 0), (2, 2, 1), (1, 4, 1)$
4	$K_{3,3} + tK_1$	$(1, 1, 0), (2, 2, 1)$
5	$K_{1,s} + tK_1$ *(none of the above)*	$(0, s - 1, 1), (1, s - 2, 2)$
6	$K_{r,s} + tK_1$ *(none of the above)*	$(r - 1, s - 1, 1)$

First we consider the case #6. Here, we have $(a, b, c) = (r-1, s-1, 1)$. Suppose that X has p vertices. Then by the regularity of G, we have $r(k-s) = p(r-1), s(k-r) = p(s-1)$ and $kt = p$. From the first two equations, we have $k(r-s) = p(r-s)$. This implies $k = p$, since if $r = s$, then from the second and third equations, we have $sk + kt = kst + rs$ which gives $t = 1$ and so $k = p$. From $k = p$, we have $t = 1$ and $p = rs$. We index the vertices of G as follows. The vertices of X are indexed by (i, j), where $1 \leq i \leq r$ and $1 \leq j \leq s$. The vertices of U and V are indexed by $(i, 0)$ $(1 \leq i \leq r)$ and $(0, j)$ $(1 \leq j \leq s)$, respectively. The vertex of W is indexed by $(0, 0)$. Then by the results of the previous section it is seen that (i, j) is joined to (i', j') in G if and only if $i \neq i'$ and $j \neq j'$. Therefore, G is the complement of the line graph of $K_{r+1,s+1}$.

The next case is #5. Here, we have $(a, b, c) = (0, s-1, 1), (1, s-2, 2)$, where $s \neq 1, 2, 5$. Suppose that X has p vertices of type $(0, s-1, 1)$ and q vertices of type $(1, s-2, 2)$. By the regularity of G, we have $k - s = q$, $s(k-1) = p(s-1) + q(s-2)$ and $kt = p + 2q$. These equations give $(s^2 - s)(t-1) + tq(s-1) = 2qs$ which in turn yields $t \leq 3$. If $t = 3$, then $s \leq 2$, a contradiction. If $t = 1$, then $q = 0$ which has been dealt with in the preceding paragraph and hence G is the complement of the line graph of $K_{2,s+1}$. Now suppose that $t = 2$. Then $q = \binom{s}{2}$ and $p = 2s$. Therefore, G is of order $\binom{s+3}{2}$. Now it follows that G is the complement of the line graph of $K_{2,s+3}$ (also known as the Kneser graph $\mathrm{KG}(s+3, 2)$) since it has a star complement $K_{1,s} + 2K_1$ for eigenvalue 1 (see also [10]).

Next consider the case #1. Suppose that X has p_1, p_2, p_3, p_4 vertices of type $(0, 2, 3)$, $(1, 1, 4)$, $(0, 1, 1)$, $(1, 0, 2)$, respectively. By the regularity of G, we have

$$\begin{cases} k - 2 = p_2 + p_4, \\ 2(k-1) = 2p_1 + p_2 + p_3, \\ tk = 3p_1 + 4p_2 + p_3 + 2p_4. \end{cases}$$

These equations yield $t \leq 5$. Let $t = 5$. Then $p_i \leq 10$ for $1 \leq i \leq 4$. We have $p_2 = k - p_4 - 2$ and $p_3 = k - p_4 - 16$. Therefore, $k - p_4 \geq 16$ and so $p_2 \geq 14$, a contradiction. Hence, $1 \leq t \leq 4$. If $t = 1$, then $p_i = 0$ for $i = 1, 2, 4$ and $k = p_3 = 2$ and we have $G = C_6$. If $t = 2$, then the case coincides with the case #5 and hence G is the Petersen graph ($\mathrm{KG}(5, 2)$). Let $t = 3$. Then $p_2 = 0$, $p_1 \leq 1$, $p_3 \leq 6$ and $p_4 \leq 3$. Also we have $p_1 = 6 - k$ which means $k = 5, 6$. Now from $p_4 = k - 2 \leq 3$, it is obtained that $k = 5$ and hence $p_1 = 1, p_3 = 6$ and $p_4 = 3$. The unique graph we obtain is a $\mathrm{srg}(16, 5, 0, 2)$. But there is a unique strongly regular graph with these parameter which is the complement of the Clebsch graph [6]. Its eigenvalues are $5^1, 1^{10}, -3^5$. Finally, let $t = 4$. By taking all possibilities for vertices of any type, we obtain a $\mathrm{srg}(27, 10, 1, 5)$. There is a unique strongly regular graph with these parameter which is the complement of the Schläfli graph [6]. Its eigenvalues are $10^1, 1^{20}, -5^6$. We summarize our results in the following theorem.

Theorem 5 *Let $K_{1,2} + tK_1$ be a star complement for eigenvalue 1 in a regular graph G. Then $1 \le t \le 4$. Moreover,*

(*i*) *If $t = 1$, then G is the cycle C_6.*

(*ii*) *If $t = 2$, then G is the Petersen graph.*

(*iii*) *If $t = 3$, then G is the complement of Clebsch graph.*

(*iv*) *If $t = 4$, then G is a regular induced subgraph of the complement of the Schläfli graph.*

We now study the case #2. Suppose that X has p_1, p_2, p_3, p_4 vertices of type $(0, 2, 0)$, $(1, 1, 1)$, $(0, 4, 1)$, $(1, 3, 2)$, respectively. By the regularity of G, we have

$$\begin{cases} k - 5 = p_2 + p_4, \\ 5(k-1) = 2p_1 + p_2 + 4p_3 + 3p_4, \\ tk = p_2 + p_3 + 2p_4. \end{cases}$$

These equations yield $t \le 2$. Let $t = 0$. Then $p_i = 0$ for $2 \le i \le 4$, $k = 5$ and $p_1 = 10$. The unique graph we obtain is a srg$(16, 5, 0, 2)$, i.e. the complement of the Clebsch graph. If $t = 1$, then $p_4 = 0$ and if we take all possibilities for vertices of other types, then we find a srg$(27, 10, 1, 5)$, i.e. the complement of the Schläfli graph. Finally, let $t = 2$. Then $p_i \le 10$ for $1 \le i \le 4$. Since $p_3 = p_2 + 10$, we have $p_3 = 10$, $p_2 = 0$, $p_1 = k - 15$ and $p_4 = k - 5$. This implies $p_4 = 10$ and $p_1 = 0$. Therefore, in this situation the case coincides with the case #5 and G is KG$(8, 2)$. Here is a summary of the results.

Theorem 6 *Let $K_{1,5} + tK_1$ be a star complement for eigenvalue 1 in a regular graph G. Then $0 \le t \le 2$. Moreover,*

(*i*) *If $t = 0$, then G is the complement of the Clebsch graph.*

(*ii*) *If $t = 1$, then G is a regular induced subgraph of the complement of the Schläfli graph.*

(*iii*) *If $t = 2$, then G is the complement of the line graph of K_8*

Now we deal with the case #4. Suppose that X has p_1 and p_2 vertices of type $(1,1,0)$ and $(2,2,1)$, respectively. By the regularity of G, we have

$$\begin{cases} 3(k-3) = p_1 + 2p_2, \\ tk = p_2. \end{cases}$$

These equations yield $t \leq 1$. If $t = 0$, then $p_2 = 0$ and if we take all possibilities for vertices of type 1, then we find a srg$(15,6,1,3)$. There is a unique strongly regular graph with these parameters which is KG$(6,2)$ [6]. Its eigenvalues are $6^1, 1^9, -3^5$. Now let $t = 1$. Then $p_1, p_2 \leq 9$. We have $p_1 = k - 9$ and $p_2 = k$ which yield $p_1 = 0$ and $p_2 = 9$. Thus we have the case #6 and G is the complement of the line graph of $K_{4,4}$. We summarize the above results in the following theorem.

Theorem 7 *Let $K_{3,3} + tK_1$ be a star complement for eigenvalue 1 in a regular graph G. Then $0 \leq t \leq 1$. Moreover,*

(*i*) *If $t = 0$, then G is a regular induced subgraph of the line graph of K_6.*

(*iii*) *If $t = 1$, then G is the complement of the line graph of $K_{4,4}$.*

It remains to consider the case #3 which is somewhat different from the other cases. Suppose that X has p_1, p_2, p_3, p_4 vertices of type $(1,1,0)$, $(0,3,0)$, $(2,2,1)$, $(1,4,1)$, respectively. By the regularity of G, we have

$$\begin{cases} 2(k-5) = p_1 + 2p_3 + p_4, \\ 5(k-2) = p_1 + 3p_2 + 2p_3 + 4p_4, \\ tk = p_3 + p_4. \end{cases}$$

These equations yield $t \leq 1$. Let $t = 0$. Then $p_3 = p_4 = 0$. If we take all possibilities for vertices of types 1 and 2, then we find a srg$(27,10,1,5)$, i.e. the complement of the Schläfli graph. Now assume that $t = 1$. Note that $p_i \leq 10$ for $1 \leq i \leq 4$. We have $p_4 = p_1 + 10$ which yields $p_4 = 10$, $p_1 = 0$ and $p_2 = p_3 = k - 10$. It implies any regular graph containing $K_{2,5} + K_1$ as star complement for 1, has a 10-regular induced subgraph F, that is, the subgraph induced on 10 vertices of type 4. We index a vertex of type 2 by the triple $\{i,j,k\}$ if it is adjacent to v_i, v_j, v_k in V. Similarly, we index a vertex of type 3 by the pair $\{i,j\}$ if it is adjacent to v_i, v_j in V. From (2), it is seen that any two vertices of X, one of type 2 and the other of type 3 must have intersecting neighborhoods in V. This implies that $p_2, p_3 \leq 5$ and so $k \leq 15$. Suppose that $k = 15$. Then $p_2 = p_3 = 5$. Our goal is to find a 15-regular graph, say G, on 28 vertices containing F as an induced subgraph and having 5 vertices for each of types 2 and 3. By (2), we see that a vertex of type 2 (3) is adjacent to a vertex of type 4 in the compatibility graph if and only if they have 2 (1) or 3 (2) common neighbors in V and moreover a vertex of type 2 (3) is adjacent to a vertex of type 4 in G if and only if they have 2 (1) common neighbors in V. Now an easy analysis shows that up to isomorphism the following cases may occur for the vertices of type 2 and 3 in G:

1. type 2: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$,
   type 3: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$;
2. type 2: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$ $\{2,3,4\}$,
   type 3: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$ $\{2,4\}$;
3. type 2: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,5\}$ $\{2,3,4\}$,
   type 3: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$ $\{2,5\}$;
4. type 2: $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$ $\{2,3,4\}$,
   type 3: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$ $\{4,5\}$;
5. type 2: $\{1,2,4\}$, $\{1,2,5\}$, $\{1,4,5\}$, $\{1,3,5\}$ $\{2,3,4\}$,
   type 3: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,5\}$ $\{4,5\}$;
6. type 2: $\{1,2,4\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{2,3,5\}$ $\{2,4,5\}$,
   type 3: $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{4,5\}$ $\{1,5\}$.

Since the vertices of types 2 and 3 in G induces a 6-regular graph, only the case 6 can hold. The graph we obtain is a srg$(28,15,6,10)$. There are four strongly regular graphs with these parameters, one is the Kneser graph KG$(8,2)$ and the other three are the complements of the Chang graphs (see [7, page 258] and [6]). Since KG$(8,2)$ has no induced subgraph $K_{2,5}+K_1$, G must be the complement of a Chang graph. Similarly, we deal with the other values of k and we find that there are solutions for $k = 10, 12, 13$ and they are induced subgraphs of the one with $k = 15$. In the following we demonstrate the choices for vertices of type 2 and 3 in each case.

- $k = 10$, $p_2 = p_3 = 0$.
- $k = 12$, type 2: $\{1,3,5\}, \{2,4,5\}$, type 3: $\{1,2\}, \{3,4\}$.
- $k = 13$, type 2: $\{1,2,4\}, \{1,3,5\}, \{2,3,5\}$, type 3: $\{1,2\}, \{3,4\}, \{4,5\}$.

We summarize the above in the following theorem.

Theorem 8 *Let $K_{2,5} + tK_1$ be a star complement for eigenvalue 1 in a regular graph G. Then $0 \leq t \leq 1$. Moreover,*

(i) If $t = 0$, then G is a regular induced subgraph of the complement of the Schläfli graph.

(ii) If $t = 1$, then G is a regular induced subgraph of the complement of a Chang graph.

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